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Uniqueness theorem for anti–de Sitter spacetime

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It is shown that anti–de Sitter spacetime is the unique solution to $R_{ab}=\Lambda g_{ab}$ with $\Lambda<0$ which is strictly stationary and asymptotically anti–de Sitter. Thus, in the absence of horizons, there are no soliton solutions to Einstein's equation with negative cosmological constant. The analogous statement for de Sitter spacetime ($\Lambda>0$) is discussed and some preliminary results are obtained.

I. INTRODUCTION

There has been considerable interest recently in the effects of a nonvanishing cosmological term in Einstein's equation. For example, in the maximally supersymmetric phase of gauged extended supergravity theories[1] a negative cosmological constant Λ arises which is equal to

$$\Lambda=-3e^2/4\pi G\ , \tag{1.1}$$

where e is the gauge coupling constant. In inflationary cosmological models,[2] a positive cosmological constant of magnitude

$$\Lambda=8\pi GV \tag{1.2}$$

arises, where V is the value of the effective potential for the scalar fields at the origin. It is therefore of interest to determine the ground states of these theories and ask whether or not there exist additional regular finite-energy, time-independent solutions of the classical equations of motion, i.e., the analog of "solitons" in flat-space theories.

If $\Lambda=0$, then the answers to these questions are by now well known. The ground state is Minkowski spacetime: It has the lowest energy among all asymptotically flat vacuum solutions.[3] Furthermore, there are no solitons that are strictly stationary in the sense that they admit an everywhere timelike Killing field, regardless of the topology of the manifold.[4] Minkowski spacetime is the unique strictly stationary, asymptotically flat vacuum spacetime. If one allows the Killing field to become spacelike in the interior, the black-hole uniqueness theorems show that the only solutions are the Kerr family.

If $\Lambda<0$, then it has recently been shown that the lowest-energy solution to $R_{ab}=\Lambda g_{ab}$ is anti–de Sitter spacetime.[5] In this paper we show that again there are no solitons without black holes. More precisely, we prove that anti–de Sitter spacetime is the unique strictly stationary, asymptotically anti–de Sitter solution to $R_{ab}=\Lambda g_{ab}$. It turns out that although the proof of the positive-energy theorem required relatively minor modifications to be extended from $\Lambda=0$ to $\Lambda<0$, the same is *not* true for the soliton result. A completely different technique must be used.

To see why, recall the proof that there are no strictly stationary solitons when $\Lambda=0$. There are, in fact, at least three separate arguments. The original proof[4] consisted of first showing that a strictly stationary spacetime must be static, that is, possess a further time-reversal invariance. One then writes the field equation in terms of the norm of the Killing field $-V^2$ and the induced metric h_{ab} on the three-surface orthogonal to the Killing field. One component of the field equation states that V must satisfy Laplace's equation. Asymptotic flatness implies that V is constant, and the remaining components of the field equation then require h_{ab} to be flat.

Using the positive-energy theorem, it is now possible to give two additional proofs of this result. Perhaps the simplest is to use Komar's expression for the total energy in terms of a surface integral at infinity of the derivative of an asymptotically timelike Killing field.[6] Converting the surface integral to a volume integral by Stokes's theorem, one sees immediately that the energy vanishes if $R_{ab}=0$. The positive-energy theorem then implies that the spacetime is flat. Another proof of the absence of solitons uses a scaling argument which was discussed in the context of flat-space field theories by Derrick,[7] and in the context of curved space by Schutz and Sorkin.[8] The idea is to first show that any stationary solution must be an extremum of the total energy. One then notes that rescaling the metric g_{ab} by a constant λ^2 rescales the energy by λ. Hence the energy must vanish, and the spacetime is flat. [Strictly speaking, rescaling the metric by λ^2 violates the boundary condition at infinity, since the energy is extremized only for perturbations which vanish asymptotically like r^{-1}. However, this is easily corrected by a coordinate transformation. To be more explicit, if x^μ are asymptotically Cartesian coordinates, then $g_{ab}(\lambda x^\mu)$ satisfies the appropriate boundary condition and rescales the total energy.]

This scaling argument can in fact be extended to the case of gravity coupled to certain matter fields. These include Abelian gauge fields and massless scalars (but *not* non-Abelian gauge fields). In particular, the scaling argument rules out the existence of solitons in ungauged extended supergravity theories. Here the Lagrangian density is

$$\mathscr{L} = \frac{R}{16\pi G}\sqrt{-g} - \frac{1}{8\pi G} G_{AB}(\varphi)\nabla_a\varphi^A\nabla^a\varphi^B\sqrt{-g} - \tfrac{1}{4}M_{ij}(\varphi)F^i_{ab}F^{jab}\sqrt{-g} + N_{ij}(\varphi)F^i_{ab}F^j_{cd}\epsilon^{abcd}\,, \quad (1.3)$$

where the dimensionless scalars φ^A take their values in a Riemannian manifold with metric $G_{AB}(\varphi)$, the F^i_{ab} are the field strengths of the Abelian vector fields, and M_{ij} and N_{ij} are functions of φ. Since G_{AB} and M_{ij} are positive definite, these models satisfy the dominant energy condition and therefore the positive-energy theorem applies. Rescaling the fields by $\tilde{g}_{ab}=\lambda^2 g_{ab}$, $\tilde{F}^i_{ab}=\lambda F^i_{ab}$, $\tilde{\varphi}^A=\varphi^A$ takes solutions into solutions. However, the energy of the new solution is λ times the energy of the old solution. Thus, by the above argument, stationary solutions must have zero energy and hence must be flat.

It is easy to see why the above proofs do not generalize to include $\Lambda<0$. In the first case the equation for V is changed, so that no direct information about h_{ab} can be obtained. In the second case, the direct analog of the Komar formula diverges at infinity when $\Lambda<0$ and is not related to the total energy. In the last case, the cosmological constant breaks scale invariance. Nevertheless, we will show in Sec. II that one can combine a direct analysis of the field equation with the positive-energy theorem to rule out the existence of solitons.

The case of positive cosmological constant $\Lambda>0$ is qualitatively different from $\Lambda<0$ due to the presence of cosmological event horizons. In this case the natural ground state is the maximally symmetric de Sitter spacetime. It has been conjectured[9] that the de Sitter solution is the unique stationary solution with a single horizon. In Sec. III, we discuss this conjecture and obtain some partial results including an upper bound on the area of the horizon and a proof that any solution which is stationary inside the horizon must be static.

II. THE ANTI–DE SITTER CASE

The anti–de Sitter metric may be written in the manifestly static form

$$ds^2 = -(1+r^2/a^2)dt^2+(1+r^2/a^2)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\,, \quad (2.1)$$

where $\Lambda=-3/a^2$. Since we shall work on the covering space, the coordinate t takes its full range $-\infty<t<\infty$. The Killing field $\partial/\partial t$ is clearly timelike everywhere.

An example of a spacetime which asymptotically approaches anti–de Sitter spacetime is the Schwarzschild–anti–de Sitter metric:[10]

$$ds_0{}^2 = -\left[1-\frac{2M}{r}+\frac{r^2}{a^2}\right]dt^2 + \left[1-\frac{2M}{r}+\frac{r^2}{a^2}\right]^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \quad (2.2)$$

This spacetime has a Killing field $\partial/\partial t$ which is timelike for $r>r_0$, where r_0 is the unique positive root of the equation $g_{00}=0$. It is null at $r=r_0$ which represents the horizon of a black hole, and spacelike for $r<r_0$. The quantity M is the total Abbot-Deser mass for the spacetime.[5]

We define a spacetime to be *asymptotically anti–de Sitter* if there exists a chart t,r,θ,φ defined outside of a spatially compact world tube such that the metric has the following asymptotic behavior:

$$ds^2 = ds_0{}^2+O(r^{-2})dt^2+O(r^{-6})dr^2 + O(r)(\text{remaining differentials not involving } dr) + O(r^{-1})(\text{remaining differentials involving } dr)\,, \quad (2.3)$$

where $ds_0{}^2$ is the metric in (2.2). This definition is fairly weak in the sense that known exact solutions, e.g., the Kerr anti–de Sitter metric, approach $ds_0{}^2$ faster than that required above. However, the above falloff is all that will be needed to prove our main result. Another definition of asymptotically anti–de Sitter spacetimes has been given[11] in terms of a conformal completion. We believe that any spacetime satisfying this condition will also satisfy (2.3), but this has not yet been proven.

We now restrict ourselves to strictly stationary spacetimes, that is, those admitting an everywhere timelike Killing field which approaches $\partial/\partial t$ asymptotically. Our main result is the following.

Theorem: The only strictly stationary asymptotically anti–de Sitter solution to $R_{ab}=\Lambda g_{ab}$ is anti–de Sitter spacetime.

To prove this, we first establish the following lemma.

Lichnerowicz lemma:[12] A strictly stationary asymptotically anti–de Sitter solution to $R_{ab}=\Lambda g_{ab}$ must be static.

Proof: Define the norm and twist of the Killing field K^a by

$$-V^2=K^aK_a\,, \quad (2.4)$$

$$\omega_a=\tfrac{1}{2}\epsilon_{abcd}K^b\nabla^c K^d\,, \quad (2.5)$$

where ϵ_{abcd} is the alternating tensor. The field equation and Ricci identity yield

$$\nabla_{[a}\omega_{b]}=0\,, \quad (2.6)$$

whence

$$\omega_a = \nabla_a U \ . \tag{2.7}$$

(One can ensure that U will be globally well defined by working on the universal covering space of the manifold.) Equation (2.5) implies that

$$\nabla_a \left[\frac{\omega^a}{V^4} \right] = 0 \ , \tag{2.8}$$

so that

$$\nabla_a \left[\frac{U\omega^{[a}K^{b]}}{V^4} \right] = \frac{\omega^a \omega_a}{2V^4} K^b \ . \tag{2.9}$$

One now integrates (2.9) over a nonsingular spacelike hypersurface Σ in M whose boundary is a large two-sphere S. By Stokes's theorem

$$\oint_S \frac{U\omega^a K^b}{V^4} dS_{ab} = \int_\Sigma \frac{\omega^a \omega_a}{V^4} K^b d\Sigma_b \ . \tag{2.10}$$

It is easy to verify that for any metric satisfying (2.3), the left-hand side vanishes as S goes to infinity. Since $K^a \omega_a = 0$, ω_a is spacelike everywhere. The vanishing of the right-hand side therefore implies that $\omega_a = 0$ on Σ. But Σ is arbitrary, so $\omega_a = 0$ on M and the spacetime is static.

We now proceed to the proof of the theorem. Let h_{ab} be the induced metric on a static slice Σ. The field equation $R_{ab} = \Lambda g_{ab}$ is equivalent to

$$D^2 V = -\Lambda V \ , \tag{2.11}$$

$$^3R_{ab} = V^{-1} D_a D_b V + \Lambda h_{ab} \ , \tag{2.12}$$

where D_a and $^3R_{ab}$ are the covariant derivative and Ricci tensor of the metric h_{ab}. We now use an identity due to Lindblom[13] which follows from (2.11) and (2.12):

$$\begin{aligned} D^a[V^{-1}D_a(W-W_0)] &= \tfrac{1}{4} V^3 W^{-1} R_{abc} R^{abc} \\ &\quad + \tfrac{3}{4} V^{-1} W^{-1} D_a(W-W_0) D^a(W-W_0) \ , \end{aligned} \tag{2.13}$$

where

$$W = D_a V D^a V \ , \tag{2.14}$$

$$W_0 = \frac{\Lambda}{3}(1-V^2) \ , \tag{2.15}$$

and

$$R_{abc} = 2D_{[c}{}^3R_{b]a} + \tfrac{1}{2} h_{a[c} D_{b]}{}^3R_m^m \ . \tag{2.16}$$

The tensor R_{abc} is the analog of the Weyl tensor in three dimensions: It vanishes if and only if h_{ab} is conformally flat. The function $W - W_0$ vanishes for anti–de Sitter spacetime. Thus, the right-hand side of (2.13) is non-negative and vanishes for the anti–de Sitter solution. Integrating over Σ we obtain

$$\oint_S V^{-1} n^a D_a (W - W_0) dS \geq 0 \ , \tag{2.17}$$

where S is the two-sphere at infinity and n^a is the unit normal to S in Σ. Since our boundary condition requires all metrics to approach the Schwarzschild–anti–de Sitter metric asymptotically, we may evaluate $W - W_0$ in this surface integral using (2.2). This yields

$$W - W_0 = \frac{4M}{a^2 r} + O(r^{-2}) \ . \tag{2.18}$$

Since $V^{-1} n^a D_a = \partial/\partial r$, we obtain

$$-\frac{16\pi M}{a^2} \geq 0 \ . \tag{2.19}$$

This contradicts the positive-energy theorem for asymptotically anti–de Sitter spacetimes[5] unless $M = 0$ and the spacetime is exactly anti–de Sitter. This completes the proof of the theorem.

It is natural to conjecture that if one allows horizons, then the only stationary solution which is nonsingular outside the horizon, is the Kerr anti–de Sitter spacetime.

III. THE DE SITTER CASE

It seems natural to assume that solutions to $R_{ab} = \Lambda g_{ab}$ with $\Lambda > 0$ will be spatially compact, i.e., diffeomorphic to $\Sigma \times R$, where Σ is a compact three-manifold without boundary. If this is the case, then it follows by integrating Eq. (2.11) that the spacetime cannot be globally static. There must be a surface where the Killing field becomes null. For example, in de Sitter spacetime—obtained by setting $\alpha^2 = -a^2$ in (2.1)—this null surface is the cosmological event horizon. Intuitively, the horizon occurs because of the rapid expansion of space caused by the Λ term. If matter is present, then the cosmological repulsion may be balanced by gravitational attraction resulting in an (unstable) globally static solution. An obvious example is the Einstein static universe. However, in the absence of matter, it appears that horizons are a generic property of stationary solutions. Therefore, in attempting to formulate a uniqueness theorem for de Sitter spacetime analogous to the one proved in Sec. II, we must incorporate the horizon explicitly into our boundary conditions.

A general solution to $R_{ab} = \Lambda g_{ab}$ may have several horizons. In this section, we shall restrict consideration to the region of spacetime inside a single horizon. Thus, we consider spacetimes with boundary such that (i) $H \cong \partial M$ is a smooth, connected, null three-surface and (ii) there is a Killing field K^a which is timelike in M and tangent to H. We will call these stationary "single-horizon" spacetimes. Notice that condition (i) excludes the presence of black holes inside a cosmological horizon. Since there is no asymptotic region, there is no preferred scaling for the Killing field K^a.

The analog of the uniqueness theorem proved in the previous section is the following.

Conjecture ("cosmic no hair"): The only stationary single-horizon solution to $R_{ab} = \Lambda g_{ab}$ is de Sitter spacetime.

Some evidence for this conjecture was given in Ref. 14 and a possible new approach is discussed in Ref. 15. There is also evidence which indicates that generically any solution will approach the de Sitter metric within the horizon of every timelike world line reaching infinity.[14,16] We have been unable to establish this conjecture, but we have some partial results which may help in this direction.

To begin, we prove the following lemma.

Lichnerowicz-Hawking lemma: A stationary single-horizon solution to $R_{ab}=\Lambda g_{ab}$ must be static.

Proof: The only difference to the $\Lambda<0$ case is that the integral at infinity on the left-hand side of (2.10) is replaced by one over (a cross section of) the horizon H. Since K^a is tangent to the horizon, ω^a must vanish there. Thus U may be chosen to vanish on H, whence U/V^4 has a finite limit (if the horizon is nondegenerate, i.e., $\nabla_a V^2 \neq 0$). Therefore, the left-hand side of (2.10) vanishes and so $\omega^a=0$ everywhere. This argument is essentially identical to the standard black-hole argument given in Ref. 12.

Since the field equation is identical to the one in Sec. II (except for the sign of Λ), a static solution still satisfies Eqs. (2.11), (2.12), and therefore the Lindblom identity (2.13). Evaluating this identity in the present case, one finds that area A of the horizon must satisfy the inequality

$$A \leq 12\pi/\Lambda \tag{3.1}$$

with equality if and only if the metric is de Sitter. We shall not show this in detail because we will soon see that this result can be obtained in several ways, some of which are rather simpler and perhaps more illuminating. Obviously, in order to prove the conjecture, one must establish either equality in (3.1) or another inequality going in the opposite direction, just as Israel did in his proof of the uniqueness of the Schwarzschild solution.[17] Unfortunately, we were unable to do so. However, the inequality (3.1) seems to have independent physical interest which we shall discuss below.

Perhaps the simplest way to derive inequality (3.1) is by reformulating the problem in terms of four-dimensional Riemannian geometry. Consider a static single-horizon solution to $R_{ab}=\Lambda g_{ab}$,

$$ds^2=-V^2dt^2+h_{ab}dx^a dx^b \ . \tag{3.2}$$

At the horizon $V=0$, h_{ab} is regular, and $\kappa \equiv n^a D_a V$ is a positive constant (where n^a is the unit inward normal). Integrating Eq. (2.11) over a static ball B yields

$$\kappa A = \Lambda \int_B V \ . \tag{3.3}$$

We now analytically continue $t=iT$ in the metric (3.2). A conical singularity at $V=0$ can be avoided by identifying T with period $2\pi/\kappa$. We thus obtain a smooth positive-definite metric on a compact four-manifold N. Since (3.2) satisfies $R_{ab}=\Lambda g_{ab}$, the same will be true of the metric on N. For example, starting with the de Sitter metric, one obtains the standard round metric on S^4.

In fact, we now show that the Euler number χ and Hirzebruch signature τ of N are always equal to their values for S^4. The Euler number of N is equal to that of ∂B since the horizon is a "bolt" in the terminology of Ref. 18. Since ∂B is an orientable two-manifold, χ can be 2, 0, $-2, \ldots$. But by the Gauss-Bonnet theorem for spaces satisfying $R_{ab}=\Lambda g_{ab}$,

$$\chi=\frac{1}{32\pi^2}\int_N (C_{abcd}C^{abcd}+\tfrac{8}{3}\Lambda^2) \ , \tag{3.4}$$

χ must be positive. Hence $\chi=2$ and we have established the following lemma.

Lemma ("spherical topology"): The horizon of a stationary single-horizon solution to $R_{ab}=\Lambda g_{ab}$ has topology $S^2\times R$.

The signature τ of N is

$$\tau=\frac{1}{96\pi^2}\int_N R_{abcd}\epsilon^{cd}{}_{mn}R^{abmn} \ . \tag{3.5}$$

Time reversal $T\rightarrow -T$ is an orientation reversing isometry of N which changes the sign of the integrand in (3.5). Hence $\tau=0$. If N is a simply connected spin manifold, then ($\chi=2$, $\tau=0$) imply that it is homeomorphic to S^4.

The inequality (3.1) is now an immediate consequence of the Gauss-Bonnet theorem (with $\chi=2$) and Eq. (3.3). From (3.4) we obtain[19]

$$V_4\Lambda^2 \leq 24\pi^2 \ , \tag{3.6}$$

where V_4 is the four-volume of N, with equality if and only if g_{ab} is a metric of constant curvature. Using (3.3) we can evaluate V_4,

$$V_4=\int_N \sqrt{g}\, d^4x=\frac{2\pi}{\kappa}\int_B V\sqrt{h}\, d^3x=\frac{2\pi A}{\Lambda} \ . \tag{3.7}$$

Combining (3.6) and (3.7) we recover the inequality $A \leq 12\pi/\Lambda$. Notice that by reformulating the problem in terms of Riemannian geometry, we see that the cosmic-no-hair conjecture is a special case of the conjecture that there is only one Einstein metric on S^4.

To understand the physical significance of (3.1) it is useful to consider time-dependent solutions to $R_{ab}=\Lambda g_{ab}$. In this case it can be shown that the area of the cosmological horizon is nondecreasing.[9] This result also applies to solutions with matter providing T_{ab} satisfies the weak energy condition. Thus if a solution eventually settles down to the static de Sitter metric inside the horizon, then the area of the intersection of the horizon with any spacelike hypersurface must be less than $12\pi/\Lambda$. In other words, the area of the horizon should satisfy $A \leq 12\pi/\Lambda$ independently of the stationarity assumption.

There is a close analogy between this inequality and Penrose's inequality[20] for black holes in asymptotically flat spacetimes:

$$A \leq 16\pi M^2 \ , \tag{3.8}$$

where A is the area of a black-hole event horizon and M is the total Arnowitt-Deser-Misner mass. In fact the above argument motivating inequality (3.1) for time-dependent spacetimes is virtually identical to the one given for (3.8). The only difference is that one considers spacetimes that settle down to the Schwarzschild solution rather than the de Sitter solution. We emphasize that neither of the above inequalities has been proven rigorously. Jang and Wald[21] have given an argument in support of (3.8). We now show how this argument can be modified to support (3.1).

Consider a time-symmetric surface B in a single-horizon spacetime. Let s label a sequence of nested topological spheres centered on a point p. Set

$$f(s)=\int \left[2\widetilde{R}-\widetilde{p}^{2}-\frac{4\Lambda}{3}\right]dA \ , \qquad (3.9)$$

where $\widetilde{R}$ and $\widetilde{p}$ are the scalar curvature and trace of the extrinsic curvature of the spheres. Define ϕ by $\phi n^a D_a s=1$, where n^a is the unit outward normal to the spheres. Suppose one can choose the spheres such that $\phi\widetilde{p}=1$, and $s=s_0$ is the horizon ∂B. Then, differentiating (3.9) with respect to s, one can show[21]

$$\frac{df}{ds}=-\tfrac{1}{2}f+\int ({}^3R-2\Lambda+\widetilde{p}^{ab}\widetilde{p}_{ab}-\tfrac{1}{2}\widetilde{p}^{2} +2\phi^{-2}\widetilde{D}_a\phi\widetilde{D}^a\phi)dA \ . \qquad (3.10)$$

Since the integral on the right-hand side is non-negative we find that $fe^{s/2}$ is an increasing function of s. Since $f\rightarrow 0$ as the spheres shrink to a point, we conclude that $f\geq 0$. Evaluating f on the horizon ($\widetilde{p}=0$), we obtain $A\leq 12\pi/\Lambda$. This argument can also be applied to spacetimes with black-hole horizons in both the $\Lambda>0$ and $\Lambda<0$ cases.

Finally, we comment on the possibility of stationary solutions to $R_{ab}=\Lambda g_{ab}$ with more than one horizon. The only example known to us is the Kerr—de Sitter metric which also includes the Nariai metric[22] as a limiting case. This solution has two horizons and corresponds intuitively to two black holes placed at antipodal points on a three-sphere. In the Nariai case, the horizons are symmetrical and can be viewed as two black holes whose gravitational attraction is balanced by the cosmic repulsion. It seems plausible that other more exotic examples could be constructed by placing a number of black holes at the vertices of a regular polytope in S^3, in a manner similar to that described by Lindquist and Wheeler.[23] If stationary solutions could be constructed in this way, then unlike Kerr, and perhaps Kerr—anti—de Sitter, the Kerr—de Sitter solution would *not* be unique.

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