Apparatus-dependent contributions to $g-2$ and other phenomena

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We consider the modification of the photon propagator for a system confined between two conducting plates. After a detailed discussion of the boundary conditions that apply in such a case, we calculate the change Δg in the anomalous magnetic moment g of the electron due to the plates. For both scalar and vector photons this is of order $\Delta g \approx \alpha \ln(2a\Lambda)/ma$, where α is the fine-structure constant, m is the electron mass, a is the plate spacing, and $\Lambda \approx 1$ eV is the cutoff frequency above which the plates become "transparent" to photons. This correction to g is near the threshold of what can be detected experimentally, and may suggest some difficulties in continuing attempts to compare higher-order predictions of QED with experiment. We also apply our results to discuss other novel phenomena, such as the regeneration of coherent kaons in the empty space between the plates.

I. INTRODUCTION

In 1948 Casimir[1,2] pointed out the existence of a new effect which arises from a modification of the vacuum due to the introduction of a pair of parallel conducting plates. In the usual vacuum the zero-point energy, although infinite, is usually discarded because it can be reabsorbed by a redefinition of the zero of energy. What Casimir noted was that the presence of the conducting plates modifies the zero-point energy of the vacuum in such a way that the *difference* of the zero-point energies with and without the plates becomes measurable. This comes about if some fields (such as the electromagnetic field) satisfy boundary conditions at the plates, in which case the spectrum of these fields will be different when the plates are present, and may even be discrete. Since the vacuum energy density between the plates is different from what it is outside, the plates are subject to a residual pressure given by

$$P = -\frac{\pi^2}{240a^4} = -\frac{0.013}{a^4}\,\mathrm{dyn\,cm^{-2}}\,, \tag{1.1}$$

where a is the plate spacing (in units of 10^{-6} m). The dependence of P on $1/a^4$, which can be inferred on dimensional grounds, is partially responsible for the smallness of P. Despite the evident difficulty in detecting so weak a force, Eq. (1.1) has been verified experimentally, at least at the qualitative level.[3]

There can, of course, be other manifestations of the boundary conditions that arise in the presence of conducting plates. Among these, effects which vanish more slowly than $1/a^4$ in the limit $a \to \infty$ would be particularly interesting. These would evidently require the introduction of additional dimensional factors along with a, and hence would represent phenomena that are fundamentally different from the Casimir effect. The purpose of this paper is to point out that there exists in fact just such an effect, namely, an additional contribution to the electron anomalous magnetic moment, or $(g-2)$, due to the modification of the virtual photon field in the presence of the plates. This effect is interesting for theoretical reasons because it arises in a nonrelativistic system and because the correction, presumably $O((1/ma)^n)$, can be maximized for the electron, which is the lightest charged particle. It is also interesting experimentally because the expected size of the effects suggests that they, although small, are at the threshold of what can be detected by current techniques. This in turn may have far-ranging implications for the comparison of theory and experiment in quantum electrodynamics (QED): It suggests that there is an extrinsic (i.e., apparatus-dependent) contribution to $(g-2)$, in addition to the usual intrinsic contribution of QED. This, and similar effects in other processes, may represent a barrier of sorts, both in principle and in practice, to further refinements in tests of QED.

It is evident that both the Casimir effect and the modification of $(g-2)$ to be discussed below depend in a crucial way on the boundary conditions that apply at the plates. For example, the result quoted in Eq. (1.1) is based on the highly idealized assumption that all Fourier components of the electromagnetic field vanish at the surface of the plates. However, difficulties can arise from an uncritical application of such boundary conditions, since global conditions may run into conflict with the requirements of causality as demanded by special relativity. At the same time the proper use of boundary conditions does lead to experimentally reasonable predictions, as in the Casimir effect,[1,2] and in the description of atomic decays in a cavity.[4] Hence in the process of discussing the modification of $(g-2)$ in the presence of the plates, we also wish to clarify the circumstances under which the use of such boundary conditions is appropriate.

As will be discussed in greater detail in Sec. II, the use of the boundary conditions can be justified for essentially nonrelativistic systems in the following way: A global condition, such as a pair of boundary conditions, requires that information about a field configuration at one point should be transmitted to another point *instantaneously*. This requirement is never satisfied exactly, since information cannot travel with infinite velocity. Although this observation seems trivial, it suggests where boundary conditions can be meaningful, at least as an approximation. Specifically, the photon field can be viewed as transmitting information on the boundary conditions "instantaneously" in a system where all of the other particles involved are nonrelativistic, so that photons can be regarded as traveling almost infinitely rapidly. In the Casimir effect, for example, the constituents of the metal (which are predominantly free electrons) are in fact nonrelativistic. To ensure the validity of our approximations, the energies of the photons in question should also be sufficiently small so that even the recoiling (virtual) particles remain nonrelativistic. Again, in the Casimir effect, this second requirement is satisfied as follows: For any real conductor there is a high-frequency cutoff, denoted by Λ, such that photons for which $\omega \geq \Lambda$ can penetrate the plates, thus giving no contribution to the Casimir effect. This Λ may be estimated from the plasma frequency to be ≈ 1 eV, which is much smaller than the electron mass m, thus ensuring that the electrons remain nonrelativistic. For the case of an atom in a cavity, the requirement that the system be nonrelativistic is obviously met.

In the remainder of this section we will present a heuristic derivation of our results, but before doing so we define the various physical quantities of interest. The matrix element of the electromagnetic current $J_\mu(x)$ of the electron is given by[5]

$$\langle p' | J_\mu(0) | p \rangle = \bar{u}(p') \left[G_1(k^2)\gamma_\mu - G_2(k^2)\frac{1}{2m}\sigma_{\mu\nu}k_\nu \right] u(p) , \quad (1.2)$$

where $k \equiv p'-p$. In the absence of radiative corrections $G_2(0)=0$, while $G_1(0)=1$ is always satisfied because of current conservation. From Eq. (1.2) the nonrelativistic effective Hamiltonian can be obtained as

$$\begin{aligned} \overline{H}_{\text{eff}} &= -(e/2m)(2\vec{p}\cdot\vec{A} + g\vec{s}\cdot\vec{B}) , \\ g &= 2[G_1(0)+G_2(0)] = 2[1+G_2(0)] , \end{aligned} \quad (1.3)$$

where $\vec{A}$ is the vector potential, $\vec{B}$ is the magnetic field, and $\vec{s}$ is the electron spin. In the absence of the plates $G_2(0)$ is given, to the lowest order in the fine-structure constant α, by

$$G_2(0) = \frac{\alpha}{2\pi} , \quad (1.4)$$

which is the contribution of the vertex correction shown in Fig. 1. In the presence of the plates, g gets modified to g_{2p}, where the subscript denotes the value of g for an electron moving between two parallel plates, and we define $\Delta g = g_{2p} - g$. This Δg differs from zero in the presence of the plates (located at $z=0$ and $z=a$), due to the fact that virtual photons "feel" the assumed boundary conditions on the electromagnetic field at $z=0,a$. As a consequence of this, the photon spectrum is modified, and it is this difference which gives rise to Δg. It should be emphasized that although we will restrict our discussion to the case of an electron moving between two plates, we expect the results obtained to apply to the geometry of the Penning trap in which g is measured,[6] up to possible numerical factors of order unity.

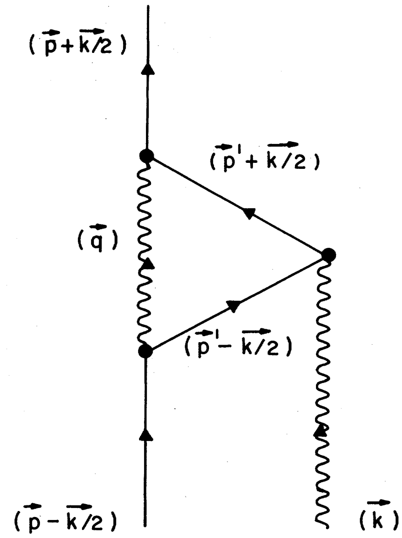

FIG. 1. Contribution to the electron anomalous magnetic moment in lowest-order covariant perturbation theory. The solid lines denote the electron, the wavy lines the photon, and the momentum associated with each line is shown in parentheses.

The difference Δg can be roughly estimated in the following way. We note to start with that the maximum wavelength $\lambda_{\max}$ that can propagate in the space between the plates is $\lambda_{\max}=2a$, and hence there exists a minimum energy $\mu=\pi/a$ for these photons. In the space between the plates, the photon thus has a discrete spectrum with energies $n\mu$ ($n=1,2,\ldots$) for photon energies in the range

$$\omega \leq \Lambda , \quad (1.5)$$

where Λ is the energy cutoff given previously. Thus Δg arises from the difference between the contributions from the discrete and continuous photon spectra restricted in the range (1.5). The calculation will be limited to the one-loop correction shown in Fig. 1, since the formulation in terms of boundary conditions ceases to be valid for higher-order contributions for which recoil effects are important. For $a=1$ cm, which is the characteristic size of the Penning trap used in Ref. 6, $\mu=6.2\times10^{-5}$ eV.

To estimate Δg we carry out the usual calculation of $(g-2)$, but with a photon of "mass" Λ. We find

$$\begin{aligned} \frac{g-2}{2} &= \frac{\alpha}{\pi}\int_0^1 dx \left[1-x+\frac{\Lambda^2}{m^2}\frac{2x-1}{x^2+(\Lambda^2/m^2)(1-x)} + O\left[\frac{\Lambda^2}{m^2}\right]\right] \\ &= \frac{\alpha}{2\pi}\left[1-\frac{2\Lambda}{m}\arctan\frac{m}{\Lambda}+O\left[\frac{\Lambda^2}{m^2}\right]\right] \\ &= \frac{\alpha}{2\pi}\left[1-\frac{\pi\Lambda}{m}+O\left[\frac{\Lambda^2}{m^2}\right]\right] . \end{aligned} \quad (1.6)$$

For $\Lambda \neq 0$ there is no contribution from the photons in the range (1.5), and hence by comparing this result to the usual one (1.4), we can find the contribution to g from the soft photons in the range (1.5),

$$\tfrac{1}{2} g_{\text{soft}} = \frac{\alpha}{2} \frac{\Lambda}{m} . \tag{1.7}$$

When the modification of g_{soft} due to the plates is taken into account, the plate spacing will enter through dimensionless factors such as $(1/ma)$ and $(1/\Lambda a)$. Since $m \gg \Lambda$, the dominant contribution to Δg from the energy range (1.5) is given by

$$\frac{\Delta g}{2} \approx \frac{\alpha}{2} \frac{\Lambda}{m} \frac{1}{a\Lambda} = \frac{\alpha}{2} \frac{1}{ma} . \tag{1.8}$$

In fact the detailed calculations presented in Secs. III and IV below indicate that Δg is enhanced relative to the naive estimate in (1.8) by a factor of order $\ln(2\Lambda a)$. For $\Lambda = 1$ eV and $a = 1$ cm this gives numerically

$$\frac{\Delta g}{2} \approx \frac{\alpha}{2} \frac{1}{ma} \ln(2\Lambda a) \approx 1.63 \times 10^{-12} . \tag{1.9}$$

We note that an effect of this magnitude is near the threshold of what can be detected experimentally by current techniques. This can be seen by comparing (1.9) to the error on $(g/2)$:[6,7]

$$\begin{aligned} g/2 &= 1.001\,159\,652\,209 \pm 0.000\,000\,000\,031 , \\ \Delta g/2 &= 0.000\,000\,000\,001\,6 . \end{aligned} \tag{1.10}$$

It is interesting to observe the way in which the linear dependence of g_{soft} on Λ/m arises, namely, through an arctangent. That such a linear dependence should emerge from $(g-2)$ is not at all obvious, in view of the fact that Λ and m enter *quadratically* at the outset in Eq. (1.6). If g_{soft} were to depend on $(\Lambda/m)^2$ as naively expected, the numerical result in (1.9) would have been further reduced by a factor of (Λ/m), and would be undetectable by present means. If $(g-2)$ had a logarithmic divergence, then we might have expected Δg to depend on $\ln(m/\Lambda)$. However, this is not the case since $(g-2)$ is both infrared and ultraviolet finite. We note in passing that the analogous correction to $(g-2)$ for the muon would be 207 times smaller, and hence correspondingly more difficult to measure.

The outline of this paper is as follows. In Sec. II, we discuss the boundary conditions at the plates in greater detail, focusing our attention on the justification for their use. In Sec. III, we present a one-loop calculation of Δg for the case of a hypothetical scalar photon field $\varphi(x)$, the soft components of which vanish at $z=0$ and $z=a$. Since the electron is always nonrelativistic, old-fashioned perturbation theory and a $(1/m)$ expansion are extensively used. This calculation demonstrates how the characteristic a dependence in (1.9) comes about. Section IV describes the analogous calculation for the usual electromagnetic field on the assumption that the soft components of the vector potential $\vec{A}$, instead of the fields themselves, vanish similarly. This somewhat unusual set of boundary conditions is discussed and justified in Sec. II. The modification of the photon propagator due to the plates can, of course, lead to a host of other effects, one of which is discussed in Sec. V. We note that one consequence of the modified photon propagator, is that a transition between the free-space eigenfunctions K_L and K_S of the K^0-$\bar{K}^0$ system can occur between the plates. This leads to the phenomenon of the regeneration of coherent kaons in empty space, which is, however, probably too small to detect. Our conclusions are summarized in Sec. VI. In the Appendix, we derive the expression for the modified "scalar-photon" propagator for the sake of completeness, based on the idealized assumption that the boundary conditions, $\varphi(z=0)=\varphi(z=a)=0$, would be valid for the whole spectrum of φ.

II. DISCUSSION OF BOUNDARY CONDITIONS

Following the ideas outlined in the Introduction, we proceed to consider the electron anomalous magnetic moment, or $(g-2)$, at the one-loop order (Fig. 1). As noted previously, we have chosen to study g because of the great precision to which it can be measured, and because the expected effects are larger for g than for any other system. As can be seen from Ref. 6, such experiments are actually performed in a Penning trap, which is a metal cavity. Since the photon field satisfies certain boundary conditions at the walls of the cavity, the eigenstates of the photons will be modified inside it. If they are modified at all, then the resulting value of $(g-2)$ will be somewhat different from the ordinary one, thus giving rise to a nonzero value of $\Delta g = g_{2p} - g$. We wish to present several model calculations in the subsequent sections to estimate Δg. Before doing so, we will analyze the physical meaning of the boundary condition in the remainder of this section. The purpose of this analysis is to specify the procedure we use to obtain our results, and at the same time to discuss several unsolved problems which arise in the course of the derivation.

To facilitate our analysis we will consider a simplified set-up where two plates are located, parallel to the xy-plane, at $z=0$ and $z=a$ (Fig. 2). We also introduce initially a massless scalar field φ ("scalar photon") as shown

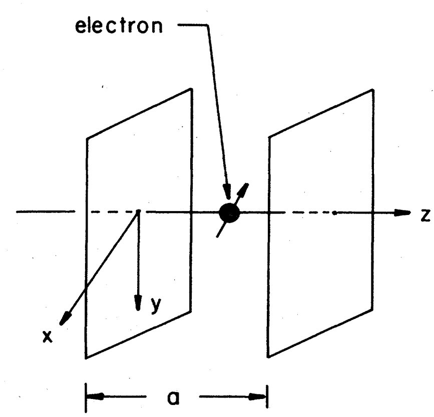

FIG. 2. Propagation of an electron between parallel conducting plates, located at $z=0$ and $z=a$.

in Fig. 3. We assume that the field φ satisfies the boundary conditions

$$\varphi(z=0)=\varphi(z=a)=0\ , \tag{2.1}$$

and therefore that $\varphi(z)$ can be expanded in terms of the eigenstates ("standing waves"),

$$\sin q_z z,\ \ q_z=(\pi/a)n_z\ \ (n_z=1,2,\ldots)\ . \tag{2.2a}$$

These replace the ordinary plane-wave solutions,

$$e^{iq_z z},\ \ q_z=(2\pi/L)n_z\ \ (n_z=0,\pm1,\pm2,\ldots)\ , \tag{2.2b}$$

where L is the length of a usual quantization volume ($L\rightarrow\infty$). Naturally, in the x and y directions, the plane-wave solutions are always used:

$$e^{iq_T x_T},\ \ q_T=(2\pi/L)(n_x,n_y)$$

$$(n_x,n_y=0,\pm1,\pm2,\ldots)\ . \tag{2.3}$$

Accordingly, the usual φ propagator

$$D(x-x')=(-i)\int\frac{d^4q}{(2\pi)^4}e^{iq(x-x')}\frac{1}{q^2-i0} \tag{2.4}$$

is modified to

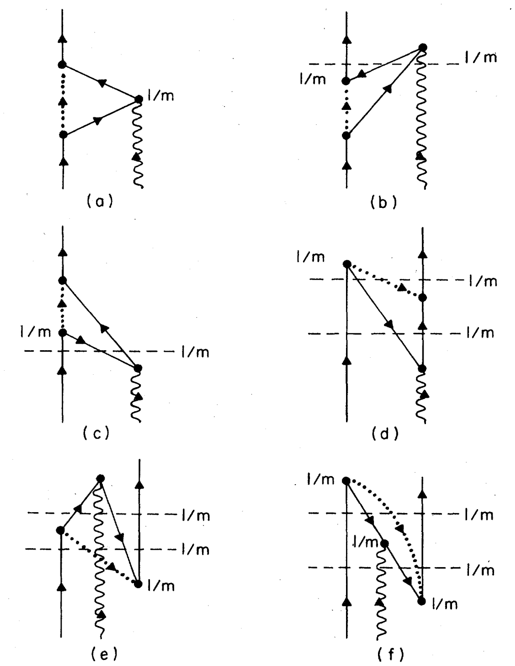

FIG. 3. Diagrams contributing to the electron anomalous magnetic moment in noncovariant perturbation theory due to intermediate scalar photons. The dotted lines denote the virtual scalar photon φ whose wave function vanishes at the plates, located at $z=0$ and $z=a$. Also shown are the $1/m$ suppression factors which arise both from some of the vertices and from energy denominators (indicated by the horizontal dashed lines). It should be emphasized that counting powers of $1/m$ merely gives the superficial number of powers: In fact the leading contributions to Δg from diagrams (a)—(c) all vanish. Diagrams (d)—(f) are of too high an order in $1/m$ to be of interest. See text for further details.

$$\widetilde{D}(x,x')=(-i)\int\frac{d^3\underline{q}}{(2\pi)^3}\frac{2}{a}\sum_{n=1}^{\infty}e^{i\underline{q}\cdot(\underline{x}-\underline{x}')}\sin(q_z z)\sin(q_z z')\frac{1}{q^2-i0}\ , \tag{2.5}$$

where $\underline{q}=(q_0,q_x,q_y)$ and $q_z=(n\pi/a)$. We will show in the Appendix that the sum in (2.5) can be performed explicitly to yield a closed expression for $\widetilde{D}$ in both the coordinate and the momentum representations. In principle the diagrams in Fig. 3 can be calculated in two ways, using D and $\widetilde{D}$, respectively, to find their difference Δg. In practice, however, this approach leads to several problems, such as unmanageable divergences, and must be abandoned. The reason for this is that there are two problems associated with the boundary conditions in (2.1): One is that they explicitly destroy all of the indispensable properties of the usual field theory, such as causality and Lorentz covariance, which are necessary for a consistent formulation of a relativistic quantum field theory. The second objection against (2.1) is that it does not physically make sense for higher-frequency modes of φ since γ rays, for example, can easily penetrate a real metal wall. The first objection is particularly serious because it essentially implies that there is no consistent formulation of boundary conditions in the true sense (unless $a\rightarrow\infty$). We should instead consider the walls to be a collection of their constituent particles (predominantly electrons), and a boundary condition at a wall would then arise as a manifestation of a coherent interaction of a photon field with a large number of electrons in the wall. This procedure would yield correct results, but would be virtually impossible to carry out explicitly.

Although of limited validity, the use of boundary conditions is nonetheless convenient and legitimate in certain circumstances. It is known, for example, that there are experimentally verified phenomena, such as the Casimir effect,[1,2] which can be understood as consequences of boundary conditions. It thus appears that there may be particular situations where invoking boundary conditions may be as valid, at least approximately, as the correct but tedious procedure outlined above. In the following, we wish to show that $(g-2)$ is actually one of these examples. Let us take seriously the second point raised in the previous paragraph, and assume that the boundary conditions (2.1) are not always valid except for lower-frequency modes. Accordingly, a physical cutoff Λ will be introduced such that the standing waves (2.2a) will be used in place of the plane waves (2.2b) only for eigenfrequencies ω in the range

$$\omega \leq \Lambda \ . \tag{2.6}$$

Without a loss of generality, $(a\Lambda/\pi)=N$ can be set equal to an integer, and then n_z in (2.2a) becomes limited to the region

$$1 \leq n_z \leq N \ . \tag{2.7}$$

Obviously, when the difference Δg is considered, the intermediate photon in Figs. 1 or 3 can be restricted to such soft photons. Numerically, Λ will be of order 1 eV, which is determined by typical values of the plasma frequency for metals. Thus, interestingly enough, we find a set of inequalities,

$$(\pi/a) \ll \Lambda \ll m \ , \tag{2.8}$$

where $(\pi/a)\approx(\pi/1\ \text{cm})=6\times10^{-5}$ eV, and $m=0.5$ MeV is the electron mass. It then follows that, in the one-loop diagrams, the electron will always be nonrelativistic. Since the velocity of the electron is small, the photons can be approximately considered to travel with infinite velocity. From the preceding discussion it follows that global boundary conditions become a meaningful approximation. In summary, we will employ boundary conditions only on the low-frequency modes of photon fields, (2.6) or (2.7). We will then calculate the diagrams in Fig. 3 by old-fashioned perturbation theory, with a high-frequency cutoff Λ for intermediate photons, and with nonrelativistic kinematics for electrons. In this way, we can avoid the difficulties with causality and relativistic covariance, insofar as we limit ourselves to the above one-loop corrections.

As was explained in the Introduction, we can roughly estimate Δg by starting with the soft-photon contribution to g

$$\tfrac{1}{2}g_{\text{soft}} \approx \frac{\alpha}{2}\frac{\Lambda}{m} \ , \tag{2.9}$$

and then multiplying by the plate contribution $(1/a\Lambda)$, which gives

$$\frac{\Delta g}{2} \approx \frac{\alpha}{2}\frac{\Lambda}{m}\frac{1}{a\Lambda} = \frac{\alpha}{2}\frac{1}{ma} \ . \tag{2.10}$$

However, in this heuristic method we cannot exclude the possibility that the plate contribution may also give rise to additional factors of $\ln(2\Lambda a)$. Thus, we may find in place of (2.10),

$$\frac{\Delta g}{2} \approx \frac{\alpha}{2}\frac{1}{ma}\ln(2\Lambda a) \ , \tag{2.11}$$

for example, in which case the resulting value of Δg would be larger than (2.10) by a factor ≈ 10. In order to clarify this point, it is necessary to perform more detailed calculations. We will show, in fact, that the dependence of Δg on m, Λ, and a as given in (2.11) is what emerges from more detailed nonrelativistic calculations.

In subsequent sections, Δg will be calculated explicitly as discussed above. In Sec. III the hypothetical scalar photon will be used, and for this field the nature of the boundary condition has already been clarified. In Sec. IV, on the other hand, a similar set of boundary conditions will be applied to the true photon field, or more specifically, to the vector potential $\vec{A}$ rather than to the field-strengths $\vec{E}$ and $\vec{B}$. Since this is an unusual set of boundary conditions, we add some further discussion. In elementary electromagnetism, it is well known that the tangential components of the electric field $\vec{E}$, and the normal component of the magnetic field $\vec{B}$, should vanish at the surface of a perfect conductor. In this idealized case, boundary conditions on the vector potential $\vec{A}$ will be too involved to be useful in the present calculation because it is the derivatives of $\vec{A}$ (and not $\vec{A}$ itself) that satisfy simple boundary conditions.[8] However, we argued previously that such boundary conditions are simply a phenomenological method which we use in place of a real and complicated calculation. Therefore, a simple generalization of (2.1) to $\vec{A}$ in the form

$$\vec{A}(z=0)=\vec{A}(z=a)=0 \ , \tag{2.12}$$

will presumably be acceptable as an approximation. We thus replace the plane waves in (2.2b) by the standing waves in (2.2a) in the region (2.6). We note that even in the microscopic picture, boundary conditions for perfect conductors are not necessarily realistic, because a real metal is not a perfect conductor but has a finite electric conductivity, σ. As a result of σ, the components of the electromagnetic field fall off exponentially with a finite depth from the surface, rather than vanishing exactly at the surface. (This is known as the "skin depth effect," and depends on both σ and the frequency of the wave.[9]) Since such an exponential damping factor can be passed on to $\vec{A}$ through derivatives, at least some components of $\vec{A}$ will vanish smoothly. Given the complexity of the microscopic description, it is not unreasonable to suppose that the consequences of (2.12) may somehow reflect the gross features of the true result.

III. CONTRIBUTION TO Δg FROM AN INTERMEDIATE SCALAR PHOTON

In this section, Δg will be derived when the intermediate state is the hypothetical scalar-photon field φ. Following the discussion in the Introduction and Sec. II, the one-loop correction will be obtained (Fig. 1), with the photon momentum cutoff Λ, and nonrelativistic electrons. In order to take the noncovariance of the system into account, we use old-fashioned perturbation theory. Although there are a larger number of diagrams to calculate [Figs. 3(a)—3(f)], the actual manipulations are not as involved owing to the fact that the electron is nonrelativistic. The expression for each diagram can be expanded in powers of $(1/m)$, or eventually (Λ/m), and only the first one or two lowest-order terms are significant.

We start by studying the effective Hamiltonian density

$$H_{\text{eff}} = -\frac{e}{2m}\left\{\Psi^\dagger(-i)\overleftrightarrow{\partial}\Psi\cdot\vec{A} + g\Psi^\dagger\tfrac{1}{2}\vec{\sigma}\Psi\cdot\vec{B}\right\} \ , \tag{3.1}$$

where Ψ is the nonrelativistic electron field, which is the second-quantized version of Eq. (1.3). Of the two form factors in Eq. (1.2), G_1 requires a subtraction, while G_2 does not. At the same time, in the nonrelativistic limit, $G_1(0)$ contributes to both terms in (3.1), but $G_2(0)$ contributes only to g. Therefore, as a direct consequence of perturbative calculations (without a subtraction), Eq. (3.1) will read

$$H_{\text{eff}}=-\frac{e}{2m}\left[\zeta\Psi^\dagger(-i)\overleftrightarrow{\partial}\Psi\cdot\vec{A}+\left[\zeta+\frac{(g-2)}{2}\right]\Psi^\dagger(\vec{\partial}\times\vec{A})\cdot\vec{\sigma}\Psi\right], \quad (3.2)$$

where $\zeta\equiv G_1^{\text{unsub}}(0)$. In the presence of the plates, H_{eff} will deviate from this by

$$\Delta H_{\text{eff}}=-\frac{e}{2m}\left[\sum_{\alpha,\beta}\Delta\zeta_{\alpha\beta}\Psi^\dagger(-i)\overleftrightarrow{\partial}_\alpha\Psi A_\beta+\sum_{\alpha,\beta}\left[\Delta\zeta+\frac{\Delta g}{2}\right]_{\alpha\beta}\Psi^\dagger\epsilon_{\alpha ij}(\partial_\beta A_i)\sigma_j\Psi\right], \quad (3.3)$$

due to the intermediate field φ satisfying the boundary conditions (2.1). One unavoidable consequence of (2.1) in (3.3) is its noninvariance under rotations, which is not a problem by itself. A more serious problem, though, is that (3.3) consequently becomes gauge noninvariant. This is another manifestation of the fact that boundary conditions are an incomplete formulation of the problem. We can avoid this difficulty by specifying that our results are valid up to possible gauge-noninvariant terms. Correspondingly, we will neglect other possible contributions to ΔH_{eff} such as

$$\sum_{\alpha,\beta}\zeta^{(1)}_{\alpha\beta}\Psi^\dagger\Psi i\partial_\alpha A_\beta,\quad \sum_{\alpha,\beta}\zeta^{(2)}_{\alpha\beta}\epsilon_{\alpha ij}\Psi^\dagger\overleftrightarrow{\partial}_\beta\sigma_i\Psi A_j\ ,$$

and obtain Δg only, which is *defined* in (3.3). Since Δg in (3.3) deviates from zero only between the plates (i.e., in the region $0\le z\le a$), the (integrated) effective Hamiltonian $\overline{H}_{\text{eff}}$ is given by

$$\Delta\overline{H}_{\text{eff}}=\int_{L^2a}d^3x\,\Delta H_{\text{eff}}\ . \quad (3.4)$$

For the purposes of calculating ΔH_{eff} and $\Delta\overline{H}_{\text{eff}}$, it is sufficient to introduce a field φ which effectively exists only between the plates. Correspondingly, the interaction Hamiltonian of φ reads

$$V_{\bar\psi\psi\varphi}=-g\int_{L^2a}d^3x\,\bar\psi\psi\varphi\ , \quad (3.5)$$

compared to the true photon interaction,

$$V_{\bar\psi\psi A}=-e\int_{L^3}d^3x\,\bar\psi i\vec{\gamma}\psi\cdot\vec{A}\ . \quad (3.6)$$

When we view φ in this way, we should always take the difference between the plate and the vacuum results. Specifically, the sum over the φ spectrum means that for any function $f(\vec{q})$

$$\sum_{\vec{q}}f(\vec{q})\to\sum_{\vec{q}}{}'f(\vec{q})\equiv\sum_{n_x,n_y=-\infty}^{\infty}\sum_{n_z=1}^{\infty}f(\vec{q})-\frac{L^2a}{(2\pi)^2\pi}\int_{-\infty}^{\infty}d^2q_T\int_0^\infty dq_zf(\vec{q})$$
$$=\frac{L^2}{(2\pi)^2}\int d^2q_T\left[\sum_{n_z=1}^{\infty}f(\vec{q})-\frac{a}{\pi}\int_0^\infty dq_zf(\vec{q})\right]. \quad (3.7)$$

Moreover, these sums and/or integrations are limited by the cutoff Λ since $\vec{q}^{\,2}\le\Lambda^2$. (See Sec. II.) Furthermore, we will choose the following normalizations for the states:

$$1\text{ particle}/L^3,\ \text{ for } e \text{ and } \gamma\ ,$$
$$1\text{ particle}/L^2a,\ \text{ for } \varphi\ , \quad (3.8)$$

so that the corresponding matrix elements become

$$\langle 0|\vec{A}(\vec{x})|(\vec{k},\vec{\epsilon})\rangle=L^{-3/2}e^{i\vec{k}\cdot\vec{x}}(\vec{\epsilon}/\sqrt{2\omega})\ , \quad (3.9a)$$

$$\langle 0|\psi(\vec{x})|(\vec{p},s)\rangle=L^{-3/2}e^{i\vec{p}\cdot\vec{x}}u(\vec{p},s)\ , \quad (3.9b)$$

$$\langle 0|\varphi(\vec{x})|\vec{q}\rangle=(2/L^2a)^{1/2}e^{iq_Tx_T}\sin(q_zz)\frac{1}{\sqrt{2q}}\ , \quad (3.9c)$$

where $\vec{\epsilon}=\vec{\epsilon}(\vec{k},\lambda)$ is the polarization vector of a photon with momentum $\vec{k}$ and polarization λ, and $u(\vec{p},s)$ is a Dirac spinor for an electron with momentum $\vec{p}$ and spin s.

In what follows, we will consider a particular matrix element of $\Delta\overline{H}_{\text{eff}}$

$$\langle(\vec{p}+\vec{k}/2,s')|\Delta\overline{H}_{\text{eff}}|(\vec{p}-\vec{k}/2,s),(\vec{k},\vec{\epsilon})\rangle\ , \quad (3.10)$$

which has the structure

$$\frac{L^2a}{L^{9/2}}\frac{-e}{2m}\sum_{\alpha,\beta}\frac{\epsilon_\alpha}{\sqrt{2k}}\chi^\dagger\left[\Delta\zeta_{\alpha\beta}2p_\beta-\sum_\gamma\left[\Delta\zeta+\frac{\Delta g}{2}\right]_{\beta\gamma}ik_\beta\epsilon_{\alpha\gamma j}\sigma_j\right]\chi\ , \quad (3.11)$$

as a consequence of the preceding definitions and normalizations. The general expression in (3.11) can then be compared to the specific results obtained from the diagrams in Fig. 3. We will denote the energies of the electron, photon, and scalar photon with momentum $\vec{p}$ by $(m+E_{\vec{p}})$, $\omega_{\vec{p}}$, and $\epsilon_{\vec{p}}$, respectively, where

$$\{E_{\vec{p}},\omega_{\vec{k}},\epsilon_{\vec{q}}\}=\left\{\frac{\vec{p}^{\,2}}{2m},k,q\right\}. \quad (3.12)$$

We note that among the diagrams in Fig. 3, (a), (b), and (c) are superficially of order $(1/m)$, $(1/m^2)$, and $(1/m^2)$, respectively, while (d)—(f) are at most $O(1/m^3)$, and are therefore negligible. Consider next the diagram of Fig. 3(a) which gives the following contribution to (3.10):

$$\sum_{\vec{q}}{}' \sum_{\vec{p}\,'} \frac{1}{L^{9/2}} \frac{1}{L^6} \frac{2}{L^2 a} L^3 \int_{L^2 a} d^3x\, e^{i(\vec{p}\,'-\vec{p})\cdot\vec{x}} e^{iq_T x_T} \sin q_z z$$
$$\times \int_{L^2 a} d^3x' e^{-i(\vec{p}\,'-\vec{p})\cdot\vec{x}\,'} e^{-iq_T x_T'} \sin q_z z' \left[\frac{-g}{\sqrt{2q}}\right]^2 \left[\frac{-e\epsilon_i}{\sqrt{2k}}\right] \chi^\dagger \frac{2p_i' - ik_j \epsilon_{ijk}\sigma_k}{2m}\chi$$
$$\times \frac{1}{E_{\vec{p}-\vec{k}/2} + \omega_{\vec{k}} - E_{\vec{p}\,'+\vec{k}/2} - \epsilon_{\vec{q}}} \frac{1}{E_{\vec{p}-\vec{k}/2} - E_{\vec{p}\,'-\vec{k}/2} - \epsilon_{\vec{q}}} . \tag{3.13}$$

After performing the integrations in the x and y directions, we find from (3.13) the expression

$$\frac{L^2 a}{(L^{3/2})^3} \frac{-e}{2m} \frac{\epsilon_i}{\sqrt{2k}} \sum_{\vec{q}}{}' \sum_{p_z'} \frac{2}{L^3 a^2} \int_0^a dz\, dz' e^{i(p'-p)_z(z-z')} \sin q_z z \sin q_z z' , \tag{3.14}$$

which is multiplied by

$$\frac{g^2}{2q} \chi^\dagger (2p_i' - ik_j \epsilon_{ijk}\sigma_k) \chi ,$$

and by the same energy denominators as in (3.13). Here p_T' becomes $(p_T - q_T)$. When comparing Eqs. (3.14) and (3.11), we find it convenient to define two functions F and G (after setting $k=0$) as follows:

$$\sum_{\vec{q}}{}' \sum_{p_z'} \frac{2}{L^3 a^2} \int_0^a dz\, dz' e^{i(p'-p)_z(z-z')} \sin q_z z \sin q_z z' \frac{g^2}{2q} \frac{p_\alpha'}{(E_{\vec{p}} - E_{\vec{p}\,'} - \epsilon_{\vec{q}})^2} \equiv \sum_\beta F_{\alpha\beta}(\vec{p}) p_\beta , \tag{3.15}$$

$$\sum_{\vec{q}}{}' \sum_{p_z'} \frac{2}{L^3 a^2} \int_0^a dz\, dz' \cdots \frac{g^2}{2q} \frac{1}{(E_{\vec{p}} - E_{\vec{p}\,'} - \epsilon_{\vec{q}})^2} \equiv G(\vec{p}) . \tag{3.16}$$

In terms of F and G, the contribution to Δg can be expressed as

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}^{(a)} = G(0)\delta_{\alpha\beta} - F_{\alpha\beta}(0) . \tag{3.17}$$

Since we wish to expand F and G in powers of $(1/m)$, we will first take the limit $m \to \infty$ to find the leading terms, in which case the denominators of Eqs. (3.15) and (3.16) are simply q^2. When $\alpha = \{x,y\}$, the integrand of F contains $p_\alpha' = p_\alpha - q_\alpha$, in which q_α gives no contribution because of the symmetric $d^2 q_T$ integrations. We thus find

$$F_{\alpha\beta}^{(0)}(\vec{p}) = \sum_{\vec{q}}{}' \sum_{p_z'} \frac{2}{L^3 a^2} \int_0^a dz\, dz' \cdots \frac{g^2}{2q^3} \delta_{\alpha\beta}$$
$$= \delta_{\alpha\beta} G^{(0)}(\vec{p}) \tag{3.18}$$

for $\{\alpha,\beta\} = \{x,y\}$. To find the result for $\alpha = z$, we need to calculate

$$\sum_{p_z'} \int_0^a dz\, dz' e^{i(p'-p)_z(z-z')} \sin q_z z \sin q_z z' (p_z')^n , \tag{3.19}$$

where $n=1$ in this case. For later purposes, however, we study (3.19) for more general n. We perform the sum first, finding nth derivatives of $\delta(z-z')$, and then carry out the $d^2 z$ integrations. The results are

$$\frac{La}{2} \{ 1, p_z, (p_z{}^2 + q_z{}^2), (p_z{}^2 + 3q_z{}^2) p_z, \cdots \} , \tag{3.20}$$

for $n = 0,1,2,3,\ldots$, respectively. For the present case, i.e., $n=1$, (3.20) tells us that p_z' can be effectively replaced in the integrand by p_z. Consequently, Eq. (3.18) is valid for all α and β, leading to the conclusion that $\Delta g^{(a)} = 0$ in the leading order of $(1/m)$. The next-leading results can be found from the second term of the expansion,

$$\frac{1}{(q + E_{\vec{p}\,'} - E_{\vec{p}})^2} = \frac{1}{q^2} - \frac{1}{m} \frac{\vec{p}\,'^2 - \vec{p}^{\,2}}{q^3} + O((1/m)^3) . \tag{3.21}$$

For $\alpha = \{x,y\}$, we find in the integrand

$$(\vec{p}\,'^2 - \vec{p}^{\,2}) p_\alpha' = (p_z'{}^2 - p_z{}^2 + q_T{}^2 - 2q_T p_T)(p_\alpha - q_\alpha)$$
$$\to (p_z'{}^2 - p_z{}^2 + 2q_T{}^2) p_\alpha , \tag{3.22}$$

where the symmetric $d^2 q_T$ integrations are taken care of beforehand. Combined with Eq. (3.20), this gives

$$F_{\alpha\beta}^{(1)}(\vec{p}) = \delta_{\alpha\beta} \frac{-1}{L^2 a} \frac{g^2}{2m} \sum_{\vec{q}}{}' \frac{(q_z{}^2 + 2q_T{}^2)}{q^4}$$
$$\text{for } (\alpha,\beta) = (x,y) . \tag{3.23}$$

For $\alpha = z$, similar operations lead to

$$(\vec{p}\,'^2-\vec{p}\,^2)p_z' \to (p_z'^2-p_z^2+q_T^2)p_z' \to [(p_z^2+3q_z^2)-p_z^2+q_T^2]p_z \;, \tag{3.24}$$

yielding

$$F_{zz}^{(1)}(\vec{p})=\frac{-1}{L^2a}\frac{g^2}{2m}\sum_{\vec{q}}{}' \frac{(3q_z^2+q_T^2)}{q^4} \;. \tag{3.25}$$

Similarly, we find

$$G^{(1)}(\vec{p})=\frac{-1}{L^2a}\frac{g^2}{2m}\sum_{\vec{q}}{}' \frac{(q_z^2+q_T^2)}{q^4} \;, \tag{3.26}$$

and hence

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}^{(a)}=\begin{cases}\delta_{\alpha\beta}\dfrac{1}{L^2a}\dfrac{g^2}{2m}\displaystyle\sum_{\vec{q}}{}'\frac{q_T^2}{q^4}+O\left[\frac{1}{m^2}\right] & \text{for } (\alpha,\beta)=(x,y),\\[2ex] \dfrac{1}{L^2a}\dfrac{g^2}{2m}\displaystyle\sum_{\vec{q}}{}'\frac{2q_z^2}{q^4}+O\left[\frac{1}{m^2}\right] & \text{for } \alpha=\beta=z,\end{cases} \tag{3.27}$$

with the other components being at most $O((1/m)^2)$.

The remaining two diagrams shown in Figs. 3(b) and 3(c) give the matrix elements

$$(b)=[\text{the same factors as in Eq. (3.14)}]\times\frac{g^2}{2q}\chi^\dagger\vec{\sigma}\cdot(-\vec{p}\,'-\vec{p}-\vec{k})\sigma_i\chi\frac{1}{E_{\vec{p}-\vec{k}/2}-E_{\vec{p}\,'-\vec{k}/2}-\epsilon_{\vec{q}}} \times\frac{1}{E_{\vec{p}-\vec{k}/2}-E_{\vec{p}+\vec{k}/2}-E_{-\vec{p}\,'-\vec{k}/2}-E_{\vec{p}\,'-\vec{k}/2}-2m} \;, \tag{3.28}$$

$$(c)=[\text{the same factors as in Eq. (3.14)}]\times\frac{g^2}{2q}\chi^\dagger\sigma_i\vec{\sigma}\cdot(-\vec{p}\,'-\vec{p}+\vec{k})\chi\frac{1}{E_{\vec{p}-\vec{k}/2}+\omega_{\vec{k}}-E_{\vec{p}\,'+\vec{k}/2}-\epsilon_{\vec{q}}} \times\frac{1}{\omega_{\vec{k}}-E_{-\vec{p}\,'+\vec{k}/2}-E_{\vec{p}\,'+\vec{k}/2}-2m} \;, \tag{3.29}$$

where $p_T'=p_T-q_T$. Since both (3.28) and (3.29) already contain an overall factor $(1/m^2)$, as seen from (3.14), the limit $m\to\infty$ can be taken in the remainder of these expressions. Apart from the kinematical structures, we find an equivalent expression from both (b) and (c), after taking $\vec{k}=0$. The kinematical terms sum to

$$-(p'+p+k)_j\sigma_j\sigma_i-(p'+p-k)_i\sigma_i\sigma_j=-2(p'+p)_i+i\epsilon_{ijk}(2k_j)\sigma_k \;. \tag{3.30}$$

Thus, if we define $\tilde{F}$ and $\tilde{G}$ by

$$\sum_{\vec{q}}{}'\sum_{p_z'}\frac{2}{L^3a^2}\int_0^a dz\,dz'e^{i(p'-p)_z(z-z')}\sin q_z z\,\sin q_z z'\frac{-g^2}{4mq^2}\begin{cases}p_\alpha'=\displaystyle\sum_\beta \tilde{F}_{\alpha\beta}(\vec{p})p_\beta \;,\\[2ex] 1=\tilde{G}(\vec{p}) \;,\end{cases} \tag{3.31}$$

we find

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}^{(b)+(c)}=\tilde{G}(0)\delta_{\alpha\beta}-\tilde{F}_{\alpha\beta}(0) \;. \tag{3.32}$$

The structures of $\tilde{F}$ and $\tilde{G}$ in Eq. (3.31) are quite similar to those of $F^{(0)}$ and $G^{(0)}$ in (3.18). Hence a similar analysis applies to show

$$\tilde{F}_{\alpha\beta}(\vec{p})=\delta_{\alpha\beta}\tilde{G}(\vec{p}) \;, \tag{3.33}$$

which leads to

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}^{(b)+(c)}=O((1/m)^2) \;. \tag{3.34}$$

Finally, we carry out the $\vec{q}$ summations and/or integrations. The previous results show that the single diagram in Fig. 3(a) gives the dominant contribution of order $(1/m)$,

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}=\left[\frac{\Delta g}{2}\right]_{\alpha\beta}^{(a)}+O((1/m)^2) \;, \tag{3.35}$$

where $\Delta g^{(a)}$ is given in Eq. (3.27). The precise form of the "sum" $\sum'$ is given in Eq. (3.7), where

$$f(\vec{q})=\frac{g^2}{2mL^2a}\left[\frac{1}{q^2}-\frac{q_z^2}{q^4}\right], \quad \text{for } (\alpha,\beta)=(x,y) \;,$$
$$f(\vec{q})=\frac{g^2}{2mL^2a}\left[\frac{2q_z^2}{q^4}\right], \quad \text{for } \alpha=\beta=z \;. \tag{3.36}$$

We first carry out the integration over q_T in (3.7) which gives

$$\int_{q^2\le\Lambda^2} d^2q_T \frac{1}{(q^2)^n} = \pi \int_{q_z^2}^{\Lambda^2} d(q^2)\frac{1}{(q^2)^n} = \begin{cases} 2\pi \ln\left[\dfrac{\Lambda}{q_z}\right] & \text{for } n=1 , \\ \pi\left[\dfrac{1}{q_z^2}-\dfrac{1}{\Lambda^2}\right] & \text{for } n=2 . \end{cases} \tag{3.37}$$

Using (3.7) we can now write $\sum' f(\vec{q})$ in the form

$$\sum_{\vec{q}}' f(\vec{q}) = \left[\frac{L}{2\pi}\right]^2 \left[\frac{g^2}{2mL^2a}\right] \times \left[\sum_{n_z=1}^{N} (\cdots) - \frac{a}{\pi}\int_0^{\Lambda} dq_z(\cdots)\right] , \tag{3.38a}$$

where

$$(\cdots) = \begin{cases} \pi\left[2\ln\dfrac{\Lambda}{q_z} - 1 + \dfrac{q_z^2}{\Lambda^2}\right], & \text{for } (\alpha,\beta)=(x,y) , \\ 2\pi\left[1-\dfrac{q_z^2}{\Lambda^2}\right], & \text{for } \alpha=\beta=z . \end{cases} \tag{3.38b}$$

The integrals in (3.38) are elementary, and the sums can be carried out using the following relations:

$$\sum_{n_z=1}^{N}\left[\frac{q_z^2}{\Lambda^2}\right] = \sum_{n_z=1}^{N}\left[\frac{n_z^2}{N^2}\right] = \frac{(N+1)(2N+1)}{6N} , \tag{3.39}$$

$$\sum_{n_z=1}^{N} \ln\left[\frac{\Lambda}{q_z}\right] = \sum_{n_z=1}^{N} \ln\left[\frac{N}{n_z}\right] = N\ln N - \ln N! = N - \tfrac{1}{2}\ln(2\pi N) , \tag{3.40}$$

where in the last step use has been made of Stirling's formula,

$$\ln N! \approx N\ln N - N + \tfrac{1}{2}\ln 2\pi N . \tag{3.41}$$

We can eliminate N in terms of Λ by using (2.7) in the form

$$N = a\Lambda/\pi . \tag{3.42}$$

Collecting the previous results together we can write the final result in the form

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta}\dfrac{-g^2}{4\pi}\dfrac{1}{2ma}\left[\ln 2a\Lambda - \tfrac{1}{2}\right], & \text{for } (\alpha,\beta)=(x,y) , \\ \dfrac{-g^2}{4\pi}\dfrac{1}{2ma}, & \text{for } \alpha=\beta=z . \end{cases} \tag{3.43}$$

The result (3.43) shows two important features: One is the overall factor $(1/ma)$, which is expected on the basis of the heuristic arguments given in the Introduction. The second is that the xx and yy components have an additional factor of $\ln(2a\Lambda)$ which can enhance the effect by approximately one order of magnitude. It should be noted that this logarithmic factor is revealed to exist for the first time by the present explicit calculations.

IV. CONTRIBUTION TO Δg FROM AN INTERMEDIATE VECTOR PHOTON

In this section we generalize the previous formalism for scalar photons to true photons. As discussed in Sec. II, we assume that the boundary conditions (2.12) hold for the low-frequency components of $\vec{A}$, noting that, in any case, this formulation and its consequences make sense only as an approximation. In certain situations, such as when a superconducting cavity is used, the boundary conditions (2.12) may be even more plausible.

Once we assume Eq. (2.12), their application becomes relatively straightforward, except that in this case the photon field itself induces Δg. For convenience, we will separate the photon field explicitly into two parts: One is the normal part which is not new, and the other is a component which exists only between the plates, and which thus represents the difference between the plate and the

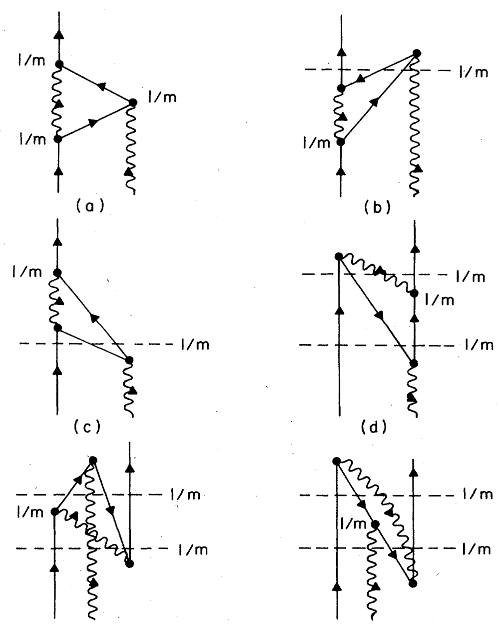

FIG. 4. Diagrams contributing to the electron anomalous magnetic moment due to intermediate vector photons. These are the noncovariant versions of the diagram in Fig. 1, but with an assumption that the wave function of the intermediate photon vanishes at the plates. See text and caption to Fig. 3 for further details.

usual vacua. With this convention, we obviously attach the normal part to the *external* photon vertices of Figs. 1 or 4. Naturally, this component has the ordinary normalization and gives momentum conservation at the respective vertices. On the other hand, since we are interested only in the difference Δg, the *internal* photon lines in those diagrams should evidently correspond to the other part of the photon field. Note that we omit the instantaneous Coulomb interaction, since it gives the same contribution with or without the plates, and hence makes no contribution to Δg. It can be seen that in this way all that is required is to repeat the previous calculations in a manner similar to Sec. III, except for the appropriate changes in the vertices in going from scalar to vector photons.

Among the graphs in Fig. 4, (a), (d), (e), and (f) are all negligible, since they are at most $O(1/m^3)$, and only (b) and (c) can yield $O(1/m^2)$ results. In fact, the last two diagrams give explicitly

$$(b)=[\text{the same factors as in Eq. (3.14)}]$$
$$\times\frac{e^2}{2q}\left[\delta_{jk}-\frac{q_jq_k}{q^2}\right]\chi^\dagger\sigma_j\sigma_i[(p'+p-k)_k-i(p'-p)_l\epsilon_{kln}\sigma_n]\chi$$
$$\times\frac{1}{E_{\vec{p}-\vec{k}/2}-E_{\vec{p}+\vec{k}/2}-E_{-\vec{p}'-\vec{k}/2}-E_{\vec{p}'-\vec{k}/2}-2m}\,\frac{1}{E_{\vec{p}-\vec{k}/2}-E_{\vec{p}'-\vec{k}/2}-\omega_{\vec{q}}}\,, \tag{4.1}$$

$$(c)=[\text{the same factors as in Eq. (3.14)}]$$
$$\times\frac{e^2}{2q}\left[\delta_{jk}-\frac{q_jq_k}{q^2}\right]\chi^\dagger[(p'+p+k)_j-i(p-p')_l\epsilon_{jln}\sigma_n]\sigma_i\sigma_k\chi$$
$$\times\frac{1}{E_{\vec{p}-\vec{k}/2}+\omega_{\vec{k}}-E_{\vec{p}'+\vec{k}/2}-\omega_{\vec{q}}}\,\frac{1}{\omega_{\vec{k}}-E_{-\vec{p}'+\vec{k}/2}-E_{\vec{p}'+\vec{k}/2}-2m}\,, \tag{4.2}$$

where $p'_T=p_T-q_T$. As in Sec. III, we take the limit $m\to\infty$ [except for the overall factor $(1/m^2)$], as well as taking $k=0$ (except in the kinematical structures). We then find an equivalent expression from both (b) and (c), with the exception of the kinematical terms which can be summed to yield

$$4[(\delta_{ij}-q_iq_j/q^2)p_j+(p'-p)_i]+2i\epsilon_{ijl}(\delta_{jk}-q_jq_k/q^2)k_k\sigma_l\,. \tag{4.3}$$

Repeating the previous discussion, we can similarly show that $(p'-p)_i$ gives no contribution. Namely, for $i=\{x,y\}$, $(p'-p)_i$ is equal to $(-q_i)$, which vanishes after the symmetric d^2q_T integrations. In addition (p'_z-p_z) gives no contribution because of (3.20). We thus find that the sum of Eqs. (4.1) and (4.2) is given by

$$\frac{L^2a}{L^{9/2}}\frac{-e}{2m}\sum_{\alpha,\beta}\frac{\epsilon_\alpha}{\sqrt{2k}}\chi^\dagger\left[I_{\alpha\beta}2p_\beta+\sum_\gamma I_{\beta\gamma}i\epsilon_{\alpha\beta i}k_\gamma\sigma_i\right]\chi\,, \tag{4.4}$$

where

$$I_{\alpha\beta}(p)\equiv\frac{2}{L^3a^2}{\sum_{\vec{q}}}'\sum_{p'_z}\int_0^a dz\,dz'e^{i(p'-p)_z(z-z')}\sin q_zz\,\sin q_zz'\frac{e^2}{2mq^2}\left[\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{q^2}\right]. \tag{4.5}$$

The difference Δg is related to $I_{\alpha\beta}$ by

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}=-2I_{\alpha\beta}(0)\,. \tag{4.6}$$

After calculations similar to the previous ones, we finally find

$$\left[\frac{\Delta g}{2}\right]_{\alpha\beta}=\begin{cases}\delta_{\alpha\beta}\dfrac{e^2}{4\pi}\dfrac{1}{2ma}(\ln 2a\Lambda+\tfrac{1}{2}), & \text{for } (\alpha,\beta)=(x,y)\,,\\[2ex] \dfrac{e^2}{4\pi}\dfrac{1}{ma}(\ln 2a\Lambda-\tfrac{1}{2}), & \text{for } \alpha=\beta=z\,.\end{cases} \tag{4.7}$$

Again, the result in (4.7) exhibits the characteristic form discussed in the Introduction.

Before drawing any conclusions from the results in (4.7), we should remind ourselves of one of the difficulties mentioned previously, namely, that the result in (4.7) is clearly not gauge invariant. (See also the discussion in Sec. VI below.) One might suspect that this arises because we impose boundary conditions on the gauge-noninvariant vector potential $\vec{A}$ as in (2.12). This suspicion, however, leaves unanswered the question of why the scalar-photon result in (3.43) shows a similar behavior. This problem is, therefore, most likely a direct consequence of the rotational noncovariance of the boundary conditions (2.1) and (2.12), and hence unavoidable in the present formulation. In the detailed microscopic formulation which we described in Sec. II, every interaction would be gauge invariant, and thus the final result would also have to be gauge invariant. Given these uncertainties in the calculation, we can summarize this discussion by observing that in the presence of the plates we expect an additional contribution to g which is proportional to $1/ma$, where a is the typical dimension of the apparatus and m is the electron mass. The proportionality constant cannot be obtained in a theoretically unambiguous way at the present time, due to the lack of more detailed and reliable calculations, and thus requires experimental determination. Nonetheless, the model calculations given here suggest that Δg may likely contain logarithmic enhancement factors, as shown in (3.43) and (4.7), and hence may be as large as 10^{-12}, which should be a detectable effect in the near future.

V. REGENERATION OF COHERENT KAONS IN EMPTY SPACE

We discuss in this section another consequence of the modified electromagnetic spectrum in the region between two parallel plates. Consider the eigenfunctions of the K^0-$\bar{K}^0$ system in the region $0 \le z \le a$. In the absence of the plates the free-space eigenfunctions are $|K_L\rangle$ and $|K_S\rangle$, which are the linear combinations[10,11]

$$|K_L\rangle = (|p|^2+|q|^2)^{-1/2}(p\,|K^0\rangle + q\,|\bar{K}^0\rangle)\ ,$$
$$|K_S\rangle = (|p|^2+|q|^2)^{-1/2}(p\,|K^0\rangle - q\,|\bar{K}^0\rangle)\ . \tag{5.1}$$

The CP-violating parameter ϵ is given in terms of p and q by

$$\epsilon = 1-q/p,\quad |\epsilon| = (4.548\pm 0.044)\times 10^{-3}\ , \tag{5.2}$$

and the complex K_L-K_S mass difference is

$$i(m_L-m_S)+\tfrac{1}{2}(\Gamma_L-\Gamma_S)\approx \Delta m\,(i-1)=2pq\ , \tag{5.3}$$

where $m_{L,S}$ and $\Gamma_{L,S}$ are the masses and widths of $K_{L,S}$. The mixing of $|K^0\rangle$ and $|\bar{K}^0\rangle$, which leads to the eigenfunctions $|K_L\rangle$ and $|K_S\rangle$, arises from the existence of $|\Delta S|=2$ transitions, such as the one-photon and two-pion contributions shown in Fig. 5.

Consider now what happens in the presence of the plates, located at $z=0$ and $z=a$. From the preceding discussion of $(g-2)$ we note that the photon spectrum, and

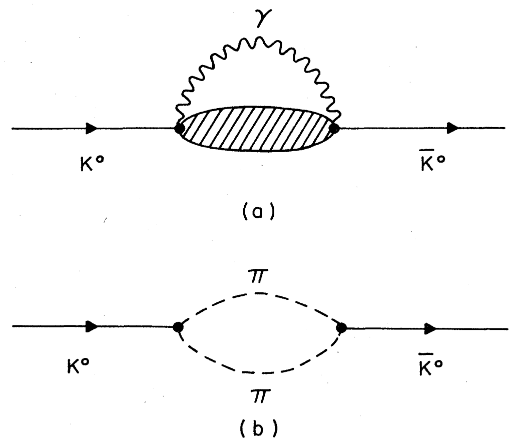

FIG. 5. Contributions to the K^0-$\bar{K}^0$ transition matrix element. (a) The one-photon contribution which is modified by the presence of the plates. (b) The two-pion intermediate state, which is left unchanged by the plates. Because the plates affect some, but not all of the intermediate states, the eigenstates of the K^0-$\bar{K}^0$ system in the vacuum between the plates can be in principle different from those in the plate-free vacuum.

hence the photon propagator, is modified in the region $0 \le z \le a$, which leads in turn to a change in the one-photon contribution in Fig. 5(a). At the same time, however, other contributions to the K^0-$\bar{K}^0$ transition amplitude, such as the 2π contribution shown in Fig. 5(b), are *not* modified since the pion field is presumably not constrained by any boundary conditions at the plates. It follows that, in the presence of the plates, the relative contributions of the various diagrams which determine p and q are different from what they would be in the usual vacuum. Although the additional contribution from the plates is CP-conserving, its effect in the presence of an intrinsic CP violation is to change the relative strengths of the CP-violating and CP-conserving amplitudes. This in turn implies that in general p and q will be modified to some new values p' and q', respectively, and hence that $|K_L\rangle \to |K_{L'}\rangle$ and $|K_S\rangle \to |K_{S'}\rangle$. Thus the physical consequence of the presence of the plates is that the K^0-$\bar{K}^0$ eigenfunctions in the region $0 \le z \le a$ are different from what they would be in the usual vacuum. In a sense, the empty space between the plates acts as if it were a material medium.

Suppose now that a K_L beam is incident from the left on the plate arrangement shown in Fig. 2. Upon leaving the first plate at $z=0$, the kaon wave function $|\Psi(0)\rangle$ can be written in the form[10,11]

$$|\Psi(0)\rangle \approx |K_L\rangle + \rho\,|K_S\rangle\ , \tag{5.4}$$

where ρ is the usual regeneration parameter, which characterizes the regeneration of $|K_S\rangle$ from $|K_L\rangle$ in the left plate via the strong interactions. For a (thin) plate of thickness L, $\rho \propto L$, and $|\rho| \ll 1$. Let us then assume for simplicity that L has been chosen sufficiently small that $\rho=\rho(L)$ is negligible in (5.4). Then

$$|\Psi(0)\rangle \approx |K_L\rangle \approx |K_{L'}\rangle + r'\,|K_{S'}\rangle\ , \tag{5.5}$$

where r' is the plate contribution arising from the modification of the photon propagator. The $|K_{L'}\rangle$ and $|K_{S'}\rangle$ components now propagate from $z=0$ to $z=a$ with their characteristic time dependences, with the result that the wave function of the system at $z=a$ is given by

$$|\Psi(a)\rangle \approx |K_L\rangle - |K_S\rangle r'(1-e^{-l(1/2-i\Delta m/\Gamma_S)}) \equiv |K_L\rangle + \rho' |K_S\rangle , \quad (5.6)$$

$$l = L/\Lambda_S ,$$

where $\Lambda_S = \gamma v \tau_S$ is the kaon mean decay length, and $\tau_S = 1/\Gamma_S$ is the lifetime of K_S. We see from (5.6) that the effect of the modified photon propagator is to regenerate a coherent $|K_S\rangle$ component from $|K_L\rangle$ in the *empty space* between the plates, much as a material medium would in the same region.

We can estimate the magnitude of r' in (5.5), and hence ρ', by arguing in analogy to Eq. (1.7) that a typical soft-photon diagram in Fig. 5(a) will give rise to a contribution to r' of order

$$|r'_{\text{soft}}| \approx \alpha \frac{\Lambda}{m_K} , \quad (5.7)$$

where m_K is the kaon mass. When the modification of r'_{soft} due to the plates is taken into account, the plate spacing a can in principle enter through dimensionless factors such as $1/m_K a$, $1/\Lambda a$, and $1/\Delta m\, a$, where $\Delta m = (3.521 \pm 0.014)\times 10^{-6}$ eV. However, even the largest of these factors, which is $1/\Delta m\, a$, leads to a value of r' which is too small to be detectable by present experiments. To amplify on this we note that

$$1/\Delta m = 5.60 \text{ cm} , \quad (5.8)$$

so that $1/\Delta m a = 5.60$ for $a = 1$ cm. This leads to an estimate of r' which is of order

$$|r'| \approx |r'_{\text{soft}}| \frac{1}{\Delta m a} \approx \alpha \frac{\Lambda}{m_K} \frac{1}{\Delta m a} \approx 8\times 10^{-11} \quad (5.9)$$

for $\Lambda = 1$ eV. The situation is somewhat improved by noting[10] that the K^0-$\bar{K}^0$ system remains coherent for distances of order $\gamma/\Delta m$, where $\gamma = E_K/m_K$ is the usual relativistic factor. If this coherence could be utilized the plate contribution might be of order $\gamma/\Delta m\, a$, and the estimate of $|r'|$ in (5.9) might be correspondingly enhanced. Under these circumstances we might expect $|r'| \approx 2\times 10^{-8}$ for 100-GeV kaons in Ref. 10, and $|r'| \approx 2\times 10^{-6}$ for 10-TeV kaons at the planned Superconducting Super Collider. It should be emphasized, however, that the use of $1/\Delta m\, a$ for the plate factor leads to the most optimistic estimate for $|r'|$. In the more likely case that the plate factor is $1/\Lambda a$, $|r'|$ would be correspondingly reduced.

These estimates of $|r'|$, and hence of the magnitude of the empty-space regeneration effect, have an important bearing on the results of Refs. 10 and 11. Since the plates in this analysis (which represent in a simplified way the experimental apparatus) are fixed in the laboratory, their spacing and their coupling to the virtual photons in the kaons, would lead to energy-dependent effects as seen in the kaon proper frame. If the energy-dependence of Δm, τ_S, and η_{+-} suggested by data of Refs. 10 and 11 is confirmed by other experiments, the empty-space regeneration effect could in principle have been a candidate for the source of these effects. However, what the preceding analysis demonstrates is that the empty-space regeneration is in all likelihood far too small to have shown up in the existing data. It follows that if the energy-dependent effects suggested by the data in Refs. 10 and 11 are real, then their origin lies somewhere else, perhaps in a new long-range force or medium.

VI. CONCLUSIONS

We summarize in this section the main conclusions of our analysis, as well as the open questions which remain to be explored. We have seen that for an electron propagating between two parallel plates, there is an additional contribution Δg to the anomalous magnetic moment g of the electron, whose general magnitude is given by Eq. (1.9). The detailed expression for Δg where the virtual photon is treated as a massless scalar, subject to the boundary conditions (2.1), is given in Eq. (3.43), and the analogous result for vector photons is given by Eq. (4.7). In the latter case we assume the boundary conditions (2.12) for the vector potential, for the reasons discussed in Sec. II. As noted in Eq. (1.10), the magnitude of Δg suggests that such an effect may be at the threshold of being observed.

There remain, however, a number of unanswered questions which relate in one way or another to the assumed boundary conditions. Foremost among these is the fact that the results for Δg in (3.43) and (4.7) are not gauge invariant, at least not in the usual sense. To be more specific, gauge invariance of ΔH_{eff} in (3.3) requires that if we replace the external photon field A_i by the corresponding photon momentum k_i, then ΔH_{eff} should vanish. For the term proportional to $(\Delta g)_{\alpha\beta}$ the substitution $A_i \to k_i$ gives in momentum space

$$\tfrac{1}{2}(\Delta g)_{\alpha\beta}\epsilon_{\alpha ij} k_\beta k_i \Psi^\dagger \sigma_j \Psi , \quad (6.1)$$

which vanishes if $(\Delta g)_{\alpha\beta} \propto \delta_{\alpha\beta}$, but not generally otherwise. As noted in Sec. III, this difficulty is not a consequence of imposing boundary conditions on the gauge-noninvariant vector potential $\vec{A}$, because the same problem arises for the contribution from the scalar photon, for which these boundary conditions would be justified. Presumably the origin of this problem is the fact that the boundary conditions are not rotationally covariant, which leads to the consequence that $(\Delta g)_{\alpha\beta}$ is not isotropic. It is reasonable to suppose that a more detailed microscopic treatment of the boundary conditions, along the lines proposed in Sec. II, would lead to a fully gauge-invariant result. This remains to be demonstrated explicitly, but is a separate problem which is beyond the scope of the present paper.

We have calculated Δg for the simplified case of an electron propagating between two parallel plates, and have argued that this should qualitatively simulate the behavior of the electron in the Penning-trap geometry. It remains, of course, to actually calculate Δg for the experimental geometry to confirm this suggestion, and to obtain at the same time the correct numerical coefficients in the analogs of (3.43) and (4.7). Although the quantitative results for the two-plate geometry suggest that Δg may be too small to bear on the comparison of theory and experiment at the current level of precision, one cannot exclude the possibility that additional numerical factors may arise in the calculation of Δg for the Penning-trap geometry which would change the picture. This is an interesting possibility to consider in light of the fact that both the world-average value[7] of $g/2$, and the most recent determination by Dehmelt and co-workers,[6]

$$g/2=1.001\,159\,652\,200\pm0.000\,000\,000\,040\ , \tag{6.2}$$

differ from the latest theoretical result of Kinoshita and Lindquist[12] (KL),

$$g/2=1.001\,159\,652\,460\pm0.000\,000\,000\,127 \pm0.000\,000\,000\,075\ , \tag{6.3}$$

by approximately 2 standard deviations. In (6.3) the first error is due to the experimental uncertainty in the fine-structure constant α, and the second is due to various theoretical uncertainties. It is interesting to compare $\Delta g/2$ in (1.9) to the various small contributions which are relevant at the current level of precision in the comparison of theory and experiment. In terms of $a_e \equiv (g-2)/2$, KL quote the following results:[12]

$$a_e(\text{muon})=2.8\times10^{-12}\ , \tag{6.4a}$$

$$a_e(\tau)=0.1\times10^{-12}\ , \tag{6.4b}$$

$$a_e(\text{hadronic})=1.6(2)\times10^{-12}\ , \tag{6.4c}$$

$$a_e(\text{weak})\approx0.05\times10^{-12}\ , \tag{6.4d}$$

$$a_e(\alpha^4)=-23(73)\times10^{-12}\ , \tag{6.4e}$$

which compare to

$$|\Delta g/2|=1.63\times10^{-12}\ ,$$

as given in (1.9). In Eqs. (6.4a) and (6.4b) the indicated contributions are from the muon and τ loops. We thus see that the apparatus-dependent contribution to $\Delta g/2$ is in fact comparable to the intrinsic contributions in (6.4). It is important to emphasize that the heuristic arguments leading to (1.9) do not fix the sign of $\Delta g/2$. In fact the scalar-photon contributions in (3.43) and the vector-photon contributions in (4.7) have opposite signs. Thus, there is a possibility that a complete calculation of $\Delta g/2$ for the Penning geometry could yield a result whose sign and magnitude would help improve the agreement between theory and experiment.

Thus far we have focused our attention exclusively on the anomalous moment g of the electron, except for noting that the corresponding correction $\Delta g(\mu)$ for the muon moment $g(\mu)$ would be 207 times smaller. In addition the high-precision determinations of $g(\mu)$ come from accelerator experiments where the characteristic scale a of the experiments is much larger than 1 cm. These observations, coupled with the fact that $g(\mu)$ itself is not as precisely determined as g (Ref. 13) would seem to rule out any direct observation of an apparatus-dependent contribution to $g(\mu)$. On the other hand, Δg is necessarily energy-dependent, since the geometry of the apparatus (e.g., the plate spacing a) which is fixed in the laboratory, will appear energy- (or velocity-) dependent in the proper frame of the muon where $g(\mu)$ is defined. The calculation of $\Delta g(\mu)$ for a relativistic particle is beyond our present capabilities for the reasons discussed, having to do mainly with the consistency of the boundary conditions in such a circumstance, as discussed in Sec. II. However, if it were to develop that $\Delta g(\mu)$ was proportional to the relativistic factor γ ($=E_\mu/m_\mu$), or perhaps γ^2, then it could perhaps become large enough to be detected in a future high-precision experiment. For this and other reasons it will be important in the future to formulate a consistent relativistic treatment of Δg.

Note added. After completing this work we learned of an earlier paper by Barton and Grotch[14,15] (BG) which calculates the change in the magnetic interaction of an electron in the vicinity of a perfectly conducting surface. BG do not present a result involving a logarithmic dependence on a cutoff parameter Λ since they integrate the photon frequency to infinity, and, moreover, in their analysis the contribution from any presumed upper limit (proportional to $\ln\Lambda$) multiplies a surface contact term $\delta'(2z)$. It is therefore discarded since BG only retain magnetic corrections for $z>0$. In our calculation there are two plates present, separated by a distance a, and thus the geometry is different from that of BG. For $\Lambda=1$ eV and $a=1$ cm we find an enhancement factor in the magnetic-moment-correction calculation of $\ln(2\Lambda a)=12$, and hence the contribution from the logarithmic factor is important both experimentally and theoretically. Since our geometry and our goal in computing Δg is different from that employed in Refs. 14 and 15, it is not surprising that our results differ. We are currently comparing our work, in which Λ has the physical value of $\simeq1$ eV, with Refs. 14 and 15 which take $\Lambda\to\infty$. We note in passing that for $\Lambda\gg m$ our results in (3.43) and (4.7) are no longer valid, since they were derived under the assumption that $\Lambda\ll m$. Likewise for $\Lambda\to0$ the use of Stirling's formula in (3.41) would also not be valid: In that case the sums in (3.39) and (3.40) can be done directly and lead to $\Delta g=0$ as expected. These and other more recent results will be presented elsewhere.

Note added in proof. At the Ninth International Conference on Atomic Physics (ICAP-IX, Seattle, 1984) a new result for a_e was announced by R. Van Dyck:

$$a_e=1\,159\,652\,193(4)\times10^{-12}\ . \tag{6.5}$$

Comparison of (6.5) and (1.9) indicates that the plate-dependent contribution $\Delta g/2$ in (1.9) is now essentially the same as the nominal precision of the latest experiments, which is 4×10^{-12}. Furthermore Van Dyck also announced the existence of a new Penning trap approximately three times smaller than the present one. For such a trap, the nominal estimate of $\Delta g/2$ would be $\sim5\times10^{-12}$, and hence the plate-dependent contribution Δg may be accessible experimentally in the very near future.

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APPENDIX

In this appendix we derive the closed expression for the modified scalar-photon propagator in (2.5). First, the location of the plates is shifted from $(0,a)$ to $(-a/2, a/2)$ for convenience. Taking the Fourier transformation of (2.5) in the x and y directions, we find

$$\tilde{D}=\frac{1}{a}\sum_{n=-\infty}^{\infty}\sin\frac{n\pi}{a}\left[z+\frac{a}{2}\right]\sin\frac{n\pi}{a}\left[z'+\frac{a}{2}\right]\times\frac{i}{\underline{q}^2-(n\pi/a)^2-\Lambda^2+i0}\,, \tag{A1}$$

where a mass Λ is introduced for convenience. We then perform a Wick rotation, and choose (π/a) as a unit of energy. Equation (A1) thus becomes

$$\begin{aligned}\tilde{D}&=\frac{-i}{\pi}\sum\sin\left[n\left[z+\frac{\pi}{2}\right]\right]\sin\left[n\left[z'+\frac{\pi}{2}\right]\right][n^2+\underline{q}^2+\Lambda^2]^{-1}\\&=\frac{-i}{2\pi}\sum_{n=-\infty}^{\infty}[e^{in(z-z')}-(-1)^n e^{in(z+z')}]\int_0^\infty d\xi\, e^{-\xi(n^2+\underline{q}^2+\Lambda^2)}\,,\end{aligned} \tag{A2}$$

where the denominator has been exponentiated. The summations in (A2) can be carried out by utilizing the well-known elliptic ϑ functions of the zeroth and the third kinds:[16]

$$\vartheta_0(\nu,\tau)=\sum_{n=-\infty}^{\infty}(-1)^n e^{i2\pi\nu n+i\pi\tau n^2},\quad \vartheta_3(\nu,\tau)=\sum_{n=-\infty}^{\infty}e^{i2\pi\nu n+i\pi\tau n^2}\,. \tag{A3}$$

Other than their obvious periodicity in ν, ϑ_0, and ϑ_3 have several important properties, one of which is that their Laplace transforms with respect to $t\equiv -i\tau/\pi$ are elementary functions:

$$\vartheta_0(\nu,i\pi t)\rightarrow\frac{\cosh 2\nu\sqrt{p}}{\sqrt{p}\sinh\sqrt{p}}\quad\text{for } -\tfrac{1}{2}\le\nu\le\tfrac{1}{2},\quad \vartheta_3(\nu,i\pi t)\rightarrow\frac{\cosh(2\nu-1)\sqrt{p}}{\sqrt{p}\sinh\sqrt{p}}\quad\text{for } 0\le\nu\le 1\,. \tag{A4}$$

Consequently, the first term in the square bracket of (A2) becomes

$$\int_0^\infty d\xi\sum_{n=-\infty}^{\infty}e^{in(z-z')-\xi(n^2+A^2)}=\int_0^\infty d\xi\, e^{-\xi A^2}\vartheta_3\left[\frac{z-z'}{2\pi},\frac{i\xi}{\pi}\right]=\pi^2\frac{\cosh[(|z-z'|-\pi)A]}{\pi A\sinh\pi A}\,, \tag{A5}$$

where $A^2\equiv\underline{q}^2+\Lambda^2$. The second term similarly yields

$$-\int_0^\infty d\xi\, e^{-\xi A^2}\vartheta_0\left[\frac{z+z'}{2\pi},\frac{i\xi}{\pi}\right]=-\pi^2\frac{\cosh[(z+z')A]}{\pi A\sinh\pi A}\,. \tag{A6}$$

Combining these results, and reinserting the correct dimensional factors, we find

$$\tilde{D}=(-i)\frac{\cosh[(|z-z'|-a)(\underline{q}^2+\Lambda^2)^{1/2}]-\cosh[(z+z')(\underline{q}^2+\Lambda^2)^{1/2}]}{2(\underline{q}^2+\Lambda^2)^{1/2}\sinh[a(\underline{q}^2+\Lambda^2)^{1/2}]} \tag{A7}$$

in the mixed representation. To return to the Minkowski metric, we simply make the replacement,

$$(\underline{q}^2+\Lambda^2)^{1/2}\rightarrow(\Lambda^2+q_T{}^2-q_0{}^2-i0)^{1/2}\,. \tag{A8}$$

When $a\rightarrow\infty$, (A7) reduces to

$$\tilde{D}=(-i)\frac{1}{2(\underline{q}^2+\Lambda^2)^{1/2}}\exp[-|z-z'|(\underline{q}^2+\Lambda^2)^{1/2}] \tag{A9}$$

as expected, which is the ordinary propagator in the mixed representation.

Next, we will transform (A7) to the momentum representation by carrying out the following operation on (A7):

$$\int_{-a/2}^{a/2}dz\,dz'e^{iq_z z-iq_z' z'}\times(\text{A7})\,. \tag{A10}$$

The first term in the numerator of (A7) is a function of $\zeta\equiv z-z'$ only and hence, after an appropriate change of variables, we find an expression such as

$$\tilde{D}=\int_{-a/2}^{a/2}dz\, e^{i(q_z-q_z')z}\int_{-a}^{a}d\zeta\, e^{i(q_z+q_z')\zeta/2}F(|\zeta|)\,. \tag{A11}$$

Thus, we will define for convenience

$$\delta_a(q)\equiv\frac{1}{2\pi}\int_{-a/2}^{a/2}dz\, e^{iqz}\equiv\frac{\sin(qa/2)}{\pi q}\underset{a\rightarrow\infty}{\longrightarrow}\delta(q)\,. \tag{A12}$$

Here, $\delta_a(q)$ is approximately a δ function and yet is slightly different from $\delta(q)$, due to its dependence on a. Equation (A11) can then be written explicitly as

$$\tilde{D}=2\pi\delta_a(q_z-q_z')2\int_0^a d\zeta\cos[(q_z+q_z')\zeta/2]\cosh(\zeta-a)(\Lambda^2-\underline{q}^2-i0)^{1/2}$$
$$\times\frac{-i}{2(\Lambda^2-\underline{q}^2-i0)^{1/2}\sinh[a(\Lambda^2-\underline{q}^2-i0)^{1/2}]}\ . \tag{A13}$$

The $d\zeta$ integration is elementary using

$$2\cos P\cosh Q=\cosh(Q+iP)+\cosh(Q-iP)\ . \tag{A14}$$

We find for the first term in (A7)

$$2\pi\delta_a(q_z-q_z')\frac{-i}{\Lambda^2-\underline{q}^2+(q_z+q_z')^2/4-i0}\left[1+\frac{q_z+q_z'}{2(\Lambda^2-\underline{q}^2-i0)^{1/2}}\frac{\sin[(q_z+q_z')a/2]}{\sinh[a(\Lambda^2-\underline{q}^2-i0)^{1/2}]}\right]. \tag{A15}$$

Similarly, the second term in (A7) leads to

$$-2\pi\delta_a(q_z+q_z')\frac{-i}{\Lambda^2-\underline{q}^2+(q_z-q_z')^2/4-i0}$$
$$\times\left[\cos\left[\frac{q_z-q_z'}{2}a\right]+\frac{q_z-q_z'}{2(\Lambda^2-\underline{q}^2-i0)^{1/2}}\coth[a(\Lambda^2-\underline{q}^2-i0)^{1/2}]\sin\left[\frac{q_z-q_z'}{2}a\right]\right], \tag{A16}$$

and the sum of (A15) and (A16) is the final expression for the modified propagator $\tilde{D}$ in the momentum representation. When $a\to\infty$, (A15) reduces to

$$2\pi\delta(q_z-q_z')\frac{-i}{\Lambda^2-q^2-i0} \tag{A17}$$

as it should, and (A16) vanishes in the following sense: (A16) depends on a through factors such as $\sin(aX)$ *or* $\cos(aX)$, and when they are included in the Feynman integrand, their contributions vanish in the limit $a\to\infty$ by virtue of the Riemann-Lebesgue theorem.

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