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Positive-energy theorem for $R + R^2$ gravity

Andrew Strominger
The Institute for Advanced Study, Princeton, New Jersey 08540
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We prove that asymptotically flat extrema of the action $S=\int[R+(1/2\beta^2)R^2]$ have non-negative energy, provided there exists a spacelike hypersurface on which $R > -\beta^2$. Flat space is shown to be the unique topologically Minkowskian stationary point of the energy. This result leads to a heuristic functional variational argument for positivity that does not involve restrictions on R. We also prove that flat space is semiclassically stable. Possible extensions of the theorems and relevance to quantum gravity are briefly discussed.

I. INTRODUCTION

Higher-derivative theories of gravity are of interest because, unlike general relativity, they are renormalizable when quantized.[1] Yet relatively little is known about these theories at the classical level. Of particular interest are the questions of positivity of energy and stability. There are reasons to suspect that the general higher-derivative theory is unstable, and this could translate into disastrous unitary violations in the quantum version.[1,2]

In this Rapid Communication we investigate a restricted class of higher-derivative theories described by the action

$$S=\int\left(R+\frac{1}{2\beta^2}R^2\right) . \tag{1}$$

(We use units in which $16\pi G=1$ and conventions $[\nabla_a,\nabla_b]\lambda_c=R_{abc}{}^d\lambda_d$; $R_{ac}=R_{abc}{}^b$ and signature $-+++$.) We prove that all asymptotically flat extrema of S have non-negative energy provided that there exists a spacelike hypersurface Σ on which $R > -\beta^2$ everywhere. The key element in the proof is explicit construction of a conformally related metric with positive stress energy. This result can be extended to show that flat space is a local mininum of the energy. As argued by Brill and co-workers[3] in the context of general relativity, the existence of a negative-energy solution that could be generated by a continuous deformation of flat space would suggest a saddle point of the energy functional. We prove that there are no such saddle points. We also prove that the theory is semiclassically stable in the sense that there are no asymptotically Euclidean instantons. We conclude with a few comments on the relevance of the action (1) and our results to quantum gravity.

II. POSITIVITY OF ENERGY

The equation of motion derived from (1) is

$$G^{ab}=\beta^{-2}(\nabla^a\nabla^b R-g^{ab}\Box R-R^{ab}R+\tfrac{1}{4}R^2g^{ab})\equiv T^{ab} . \tag{2}$$

Because the dynamics of large distances are governed by the lower-derivative Einstein term, the conserved energy of an asymptotically flat space-time is given by the usual Arnowitt-Deser-Misner (ADM) expression[4]

$$E=\int d^2S_i(g_{ij,j}-g_{jj,i}) . \tag{3}$$

Because the right-hand side of (2), regarded as an effective stress energy arising from the R^2 term, does not obey the dominant energy condition, one cannot immediately conclude that the energy is positive. It has long been known that the linearized theory does have positive energy.[1] This, in itself, does not imply very much—the ϕ^3 scalar theory also has positive energy at the linearized level. Nevertheless, we will show that the energy is positive for a wide class of space-times.

To demonstrate positivity of energy, consider a spacelike surface Σ in a space-time (M,g_{ab}) that is asymptotically flat. In general relativity, the weakest definition of asymptotic flatness permitting a proof of the positive-energy theorem is[5]

$$\begin{aligned}&{}^3g_{ij}\to\delta_{ij}+O(r^{-1/2-\epsilon}) ,\\&K_{ij}\to O(r^{-3/2-\epsilon}) ,\end{aligned}\tag{4a}$$

and we will require the same falloff behavior here. As we are considering a higher-derivative theory, however, additional falloff rates must be imposed for the extra canonical variables.[4] These variables are the four-dimensional scalar curvature R and its conjugate momentum which is linear in the time derivative of R. We require

$$\begin{aligned}&R\to O(r^{-3/2-\epsilon}) ,\\&t^a\nabla_aR\to O(r^{-3/2-\epsilon}) ,\end{aligned}\tag{4b}$$

where t^a is the unit normal to Σ. As R is linear in the time derivative of the extrinsic curvature, (4b) are the weakest boundary conditions that insure that (4a) is maintained under time evolution. In addition, we require that spatial derivatives on R fall off faster than r^{-2}. This excludes oscillatory behavior at spatial infinity. [Without this requirement there would be an additional contribution to the energy expression (3).]

The proofs will be phrased in terms of four-dimensional quantities rather than the three-dimensional Cauchy data because it is notationally simpler and emphasizes the relation to an Einstein-scalar theory. The correspondence between the three- and four-dimensional formulations, i.e., the well-posedness of the Cauchy problem, has been demonstrated with the restriction $R > -\beta^2$ (Ref. 6).

We now state and prove the following:

Theorem I. Let Σ be an asymptotically flat, nonsingular spacelike surface in a space-time (M,g_{ab}). If

$(g,K,R,t^a\nabla_a R)$ satisfy $t_a G^{ab}=t_a T^{ab}$ and $R > -\beta^2$ on Σ, then the energy is non-negative.

Proof. Define a conformally related four-metric

$$\tilde{g}_{ab}=(1+\beta^{-2}R)g_{ab} \ . \tag{5}$$

Because of the falloff rate of R, $E[\tilde{g}]=E[g]$. So we now need only show that $\tilde{g}$ has positive energy. To do so, simply compute

$$\tilde{G}^{ab}=\tfrac{3}{2}(\tilde{\nabla}^a\phi\tilde{\nabla}^b\phi-\tfrac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi)-\tfrac{1}{2}\tilde{g}^{ab}V(\phi) \ ,$$

$$\phi\equiv\ln(1+\beta^{-2}R) \ , \tag{6}$$

$$V(\phi)\equiv\beta^2(1-e^{-\phi})^2/2 \ .$$

(These equations may be derived from the action

$$\tilde{S}=\int[\tilde{R}-\tfrac{3}{2}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi-V(\phi)] \tag{7}$$

by varying with respect to $\tilde{g}$ and ϕ.) It is easily checked that $\tilde{G}$ obeys the dominant energy condition, implying $E[\tilde{g}]\geqslant 0$. Hence, $E[g]\geqslant 0$. Q.E.D.

We have shown that the energy is non-negative in a neighborhood of flat space and it is easily shown that flat space is the only zero-energy solution in this neighborhood and is therefore a local minimum of the energy. As argued in Ref. 3, if there did exist a negative-energy solution connected by a continuous path in solution space to flat space, then one expects a saddle point of the energy "in between" these two solutions. These authors further point out that such arguments cannot be made rigorous without detailed knowledge of the geometry of the space of solutions. The nonexistence of nontrivial extrema of the energy is nevertheless suggestive of positivity. We therefore prove the following:

Theorem II. Flat space is the unique topologically Minkowskian extrema of both the action S and the energy E.

Proof. The proof follows readily from the work of Brill and co-workers,[3] who show that such extremizing space-times are static. We first show that there are no asymptotically flat static solutions containing a region in which $R < -\beta^2$. Consider the boundary of this region where $R=-\beta^2$. On the boundary, the equations of motion imply

$$12\beta^{-4}\nabla^a\nabla^b R=-g^{ab} \ , \tag{8}$$

which implies the space-time is not static. Hence, $R > -\beta^2$ for static space-times.

Since all static solutions have $R > -\beta^2$, one can transform to the variables $\tilde{g}$ and ϕ and look for static extrema of (7). The ϕ equation of motion is, for time-independent ϕ,

$$3^3g^{ij}\nabla_i\nabla_j\phi=\frac{\partial V(\phi)}{\partial\phi} \ . \tag{9}$$

Multiplying by ϕ and integrating by parts we obtain

$$-3\int_\Sigma \nabla_i\phi\nabla^i\phi=\int_\Sigma\phi\frac{\partial V}{\partial\phi} \ . \tag{10}$$

Both integrands are positive definite so the only solution is $\phi=0$. The remaining equation $\partial\tilde{S}/\partial\tilde{g}=0$ reduces to the vacuum Einstein equation when $\phi=0$. It is well known that the only topologically Minkowskian, static solution to this equation is flat space.[3] Q.E.D.

The above proofs could be readily extended to prove that, under the above conditions, flat space is the only zero-energy solution and that other solutions have future-directed, non-null four-momentum. It seems likely that one could also prove positivity of the Bondi[7] and Abbott-Deser[8] (with the addition of a cosmological constant) energies. Much less obvious is whether the theorems involving black holes[9] can be extended to $R+R^2$ gravity.

We emphasize that we have not proven, under the weakest physically reasonable assumptions, the positivity of energy of nontrivial classical extrema of S. In particular, the existence of solitonlike solutions with $R < -\beta^2$ somewhere that cannot be reached by continuous deformation of flat space has not been ruled out. [Because the supersymmetric extension of (1) has positive-norm fermions, it might be possible to obtain a more complete proof using spinors.]

Even if such solutions do exist, however, it appears that the dynamics of the sector of the theory for which $R > -\beta^2$ is, modulo the usual problems with singularities, in some sense complete. It appears from inspection of the action $\tilde{S}$ that initial data satisfying this bound will continue to satisfy it under time evolution. In order to violate it, the field ϕ would have to climb over the exponentially high potential past minus infinity. We now give a semiclassical argument indicating that it may be possible to restrict oneself to this sector of the theory quantum mechanically as well.

III. SEMICLASSICAL STABILITY

To demonstrate the semiclassical stability of flat space, we look for asymptotically Euclidean (AE) extrema of action (1). The traced equation of motion is

$$\Box R=\beta^2R/3 \ . \tag{11}$$

Multiplying by R and integrating by parts, we obtain

$$-\int\nabla_a R\nabla^a R=\beta^2\int R^2/3 \ . \tag{12}$$

For a Euclidean metric this implies $R=0$. The equation of motion is then equivalent to the usual vacuum Einstein equation, for which it is known that there are now AE solutions.[10] This implies the semiclassical stability of flat space, and is certainly consistent with the nonexistence of nontrivial zero-energy classical solutions.

IV. DISCUSSION

We conclude with comments on the relevance of this result to quantum gravity. S is not a candidate for a fundamental quantum gravitational action because it is not renormalizable without the addition of the cosmological constant and the squared Weyl tensor. Nevertheless, there are many examples in field theory where one can consistently impose some symmetry requirement that restricts one to a subsector of the theory. In this subsector, the omitted terms may not be relevant. Possible examples are the requirements that the metric be conformally flat or that it be spherically symmetric. The former case is relevant to cosmological mini-superspace models[11] and the second is relevant to the study of black-hole evaporation.[12] In the former case the Weyl term is probably not relevant because it vanishes for conformally flat metrics. In the latter case the Weyl term might be consistently omitted because its dynamical content consists of nonspherically symmetric, spin-two radiation. The positivity of energy of classical extrema of S then suggests that, if S describes a consistent quantum theory for these special cases, there will not be difficulties with stability and unitarity. The quantum theory derived from S could provide an interesting theoretical laboratory for understanding quantum gravity.

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