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Spherical self-dual monopoles with maximal embedding in subgroups

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Solutions for a spherically symmetric self-dual monopole are derived for the case where the SU(2) subgroup which implements the spherical symmetry is maximal in a simple subgroup of the overall gauge group. For these solutions the relation between the asymptotic $\vec{B}$ field and the asymptotic Higgs field is discussed.

For many interesting cases of a spherically symmetric self-dual monopole, the spherical symmetry is implemented by an SU(2) group which is maximal in a simple subgroup, G', of the overall gauge group G. Since the maximally imbedded (in G) spherical self-dual monopole solutions have been derived by Ganoulis, Goddard, and Olive[1] (GGO) perhaps it is not surprising that solutions for the maximal imbedding in G' can be obtained. Some solutions for $G=\mathrm{SU}(N)$ are given in Ref. 2. This note derives such solutions for a large class of $G'\subset G$.

I consider a gauge group G, a subgroup $G'\subset G$, and a maximal (in the sense of GGO) SU(2) subgroup of G. The subgroup G' is restricted to be one for which there exists a Cartan-Weyl basis of G in which the simple roots of G are a subset of the simple roots of G'. (Thus the Dynkin diagram for G' is obtained from that of G by removing dots.[3]) This implies that there exists a Cartan-Weyl basis and a choice of simple roots such that the Cartan subalgebra set (H_i') and the simple root ladder operators (E_{α_i}) are subsets of (H_i) and (E_{α_i}).

For self-dual monopoles of a gauge group G, if one defines

$$\underline{\Psi}=(ne\underline{\Phi}-T_3/r) \ , \tag{1}$$

where $\underline{\Phi}$ is the Higgs field, the spherical symmetry conditions become

$$[T_3,\underline{\Psi}]=0 \ , \tag{2}$$

$$[T_3,N^{\pm}]=N^{\pm} \ . \tag{3}$$

$\vec{T}$ are the generators of an SU(2) subgroup, $\eta=+1$ (-1) if the Bogomolny equations are self-dual (anti-self-dual) and the gauge field is given by

$$e\vec{W}=\vec{r}\times(\vec{T}-\eta r\vec{N})/r^2 \ . \tag{4}$$

The Bogomolny equations become

$$\frac{\partial\Psi}{\partial r}=\tfrac{1}{2}[N^+,N^-] \ , \tag{5}$$

$$\frac{\partial N^+}{\partial r}=\pm[\underline{\Psi},N^{\pm}] \ . \tag{6}$$

The radial magnetic field on the z axis is

$$e\underline{B}_r=e\eta\underline{\Phi}'=\underline{\Psi}'-T_3/r^2 \ . \tag{7}$$

For maximal imbedding

$$T_3=2\delta^{\nu}\cdot H \ , \tag{8}$$

with

$$\delta^{\nu}=\tfrac{1}{2}\sum_{\alpha\in\Phi^+}\alpha/\alpha^2 \ ,$$

where Φ^+ are the set of positive roots of G. It turns out that $2\delta^{\nu}\cdot\alpha_j=1$ for a simple root in G. For this maximal imbedding the most general $N^{\pm}$ and $\underline{\Psi}$ satisfying Eqs. (2) and (3) are given by

$$N^{\pm}=\sum_{i=1}^{l}C_{\pm i}(r)E_{\pm\alpha_i} \ , \tag{9}$$

$$\underline{\Psi}=\sum_{i=1}^{l}\Psi_i(r)H_i \ . \tag{10}$$

Here l is the rank of G, $E_{\pm\alpha_i}$ are the step operators corresponding to the simple root α_i, and $H=2\alpha_i H/\alpha_i^2$. In turn the Bogomolny equations become

$$\theta_j''=\exp\left(\sum_{i=1}^{l}K_{ji}\theta_i\right) \ , \tag{11}$$

where

$$\theta_i'=2\Psi_i,\ \ln|C_i|^2=\sum_{i=1}^{l}K_{ji}\theta_i,$$

and K_{ji} is the Cartan matrix of G. GGO exhibited the solution to Eq. (11) and a quite explicit calculational scheme has been developed for G.[4]

I will now discuss how these considerations are modified if T are generators of an SU(2) which is maximal in G' rather than G. T is now given by

$$T_3=2\delta^{\nu'}\cdot H \ ,$$

with

$$\delta^{\nu'}=\tfrac{1}{2}\sum_{\alpha\in\Phi^{+'}}\alpha^2/\alpha \ ,$$

where $\Phi^{+'}$ are the positive roots of G'. One can satisfy the spherical-symmetry conditions with the ansatz

$$N^{\pm}(r)=\sum_{i=1}^{l'}C_{\pm i}(r)E_{\pm\alpha_i} \ , \tag{12}$$

$$\underline{\Psi}(r)=\sum_{i=1}^{l}\Psi_i(r)H_i \ . \tag{13}$$

I have introduced the convention that α_i, $i=1,\ldots,l'$ are the simple roots of G', and α_i, $i\neq1,\ldots,l$ are the remaining simple roots of G. l' is the rank of G'. Notice that though the sum involved in $N^{\pm}$ involves only the raising operators of the simple roots of G', the sum involved in

$\underline{\Psi}$ is over the full Cartan subalgebra of G. Since $\underline{\Psi}(\infty)$ is the asymptotic Higgs field one wants to be quite general so that the unbroken subgroup H can be chosen as general as possible. However, Eqs. (12) and (13) are not the most general form for operators which satisfy Eqs. (2) and (3). Generally there are other raising and lowering operators which satisfy Eq. (2) and there are operators other than those in the Cartan subalgebra which satisfy Eq. (3), and thus one would not expect that the ansatz represented by Eqs. (12) and (13) gives the most general solution for this imbedding. However with this ansatz we can proceed in a manner which is similar to the procedure of GGO for the maximal imbedding in G.

The Bogomolny equations become

$$C'_{\pm j} = \sum_{i=1}^{l} \Psi_i K_{ji} C_{\pm j}, \quad j=1,\ldots,l' , \tag{14}$$

$$\Psi_i' = \tfrac{1}{2}|C_i(r)|^2, \quad i=1,\ldots,l' , \tag{15}$$

$$\Psi_j' = 0, \quad j \neq 1,\ldots,l' . \tag{16}$$

If one defines

$$\theta_i' = 2\Psi_i, \quad i=1,\ldots,l , \tag{17}$$

Eqs. (14)–(16) become

$$\theta_j'' = \exp\left[\sum_{i=1}^{l} K_{ji}\theta_i\right], \quad j=1,\ldots,l' \tag{18}$$

and

$$\theta_i = 2\Psi_i r, \quad i \neq 1,\ldots,l' . \tag{19}$$

By use of Eq. (19), Eq. (18) becomes

$$\theta_i'' = \exp\left[\sum_{i=1}^{l'} K_{ji}\theta_i + \sum_{n \neq 1,\ldots,l'} K_{jn}\Psi_n(r)\right] . \tag{20}$$

A change of variables

$$\bar{\theta}_j = \theta_i + \omega_j r, \quad j=1,\ldots,l' ,$$

with

$$\omega_i = \sum_{j=1}^{l'} \bar{K}^{-1}{}_{ij} \sum_{n \neq 1,\ldots,l'} K_{jn}\Psi_n ,$$

where $\bar{K}^{-1}$ is the matrix inverse of K_{ij} $(i,j=1,\ldots,l')$, gives

$$\bar{\theta}_j'' = \exp\left[\sum_{i=1}^{l'} K_{ji}\bar{\theta}_i\right], \quad j=1,\ldots,l' . \tag{21}$$

This is the equation solved by GGO. The boundary condition at $r=0$ implied by the finite-energy requirement is the same expressed in terms of $\bar{\theta}_i$ as θ_i since Ψ_i' is involved.

The GGO solution for a maximal imbedding in G is characterized by a vector q in the weight space of G restricted so that its Dynkin components, $\underline{g}_i$, are all positive. However the limit of this solution exists in which all but one Dynkin component are zero, i.e., $\underline{g}_i = \underline{g}\delta_{ik}$. In this latter case the unbroken symmetry group, H, determined by the asymptotic Higgs field is $U(1)\times K$ where K is semisimple, and the asymptotic radial magnetic field is the same direction as the Higgs field.[1,4] If one defines a vector Q in weight space such that

$$\lim_{r\to\infty} e\underline{B}_r = \frac{Q\cdot H}{r^2} , \tag{22}$$

the Dynkin components of Q, $\underline{Q}_i$, are given by

$$\underline{Q}_i = \underline{Q}_{ik} ,$$

and those of Ψ by

$$\underline{\Psi}_i \propto \underline{q}\,\delta_{ik} .$$

Similiar results obtain for a maximal imbedding in G'. Of course Eq. (7) implies that the radial $\vec{B}$ field, $\underline{B}_r$, and thus the asymptotic B field lies in the Lie algebra of G'. If one again defines a Q by Eq. (22), Q has nonvanishing dual components, Q_i, only for $i=1,\ldots,l'$ but generally Q_i is not proportional to $\Psi_i(\infty)$ for $i=1,\ldots,l'$ and of course Ψ_i can be chosen nonzero for $i\neq 1,\ldots,l'$. If one solves Eq. (21) with $\underline{\bar{q}}_i = \underline{\bar{q}}\delta_{ik}$ an analysis similar to that in Ref. 4 implies the Dynkin components of Q are given by

$$\underline{Q}_i = \underline{Q}\delta_{ik}, \quad i=1,\ldots,l' , \tag{23}$$

$$\underline{Q}_n = \frac{2}{\alpha_n^2}\sum_{i=1}^{l'} K_{ni}\bar{K}^{-1}{}_{ik}\frac{\alpha_k^2}{2}\underline{Q}, \quad n\neq 1,\ldots,l' , \tag{24}$$

with no sum on k; and the Dynkin components of the asymptotic Higgs field are

$$\underline{\Psi}_i(\infty) = \underline{\bar{q}}\delta_{ik}, \quad i=1,\ldots,l' , \tag{25}$$

$$\underline{\Psi}_n(\infty) = \frac{2}{\alpha_n^2}\sum_{j=1}^{l'} K_{nj}\left[\frac{\bar{\theta}_j'(\infty)}{2} - \omega_i\right] + \frac{2}{\alpha_n^2}\sum_{m\neq 1,\ldots,l'} K_{nm}\Psi_m, \quad n\neq 1,\ldots,l' . \tag{26}$$

Since there are $l-l'$ arbitrary constants, Ψ_m, generally one can fix $\underline{\Psi}_n(\infty)$ arbitrarily. From Eq. (25) it is clear that the exact symmetry group H contains the K' subgroup of G'. The complete H is determined by the values of Ψ_m or $\underline{\Psi}_m(\infty)$.

These results have an obvious generalization if $G' = G''\times G'''$ where G'' and G''' are simple and if the SU(2) subgroup which implements the spherical symmetry is maximally imbedded in both, that is, $\vec{T} = \vec{T}'' + \vec{T}'''$, where $\vec{T}''$ ($\vec{T}'''$) are the generators of the maximal SU(2) imbedding in G'' (G''').

Note added. After this work was submitted for publication, I was made aware of two papers[5] which bear on the results of this work. These papers develop techniques for solving the self-dual monopole equation with nonmaximal imbeddings. The boundary conditions at the origin and the relation between the asymptotic $\vec{B}$ field and Higgs field are not discussed.

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