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New Euclidean solutions of SU(2) gauge theory

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A new time-dependent solution to the Euclidean SU(2) gauge theory is constructed by making use of a solution recently discovered by Arodz. The solution approaches a Wu-Yang monopole configuration as $t \rightarrow \pm\infty$. Solutions, with analogous properties, for the self-duality equation are also discussed.

In a recent paper[1] Arodz obtained a new time-dependent solution to SU(2) Yang-Mills theory in Minkowski space. The solution is of infinite energy and its significance in quantum theory is not very clear. In this Comment we point out that a section of the above-mentioned solution can be used to construct a new solution, with some interesting properties, in Euclidean space. Solutions with similar properties are also found for the self-duality equations.

The ansatz

$$A_l^a = \frac{1}{g}\epsilon_{aln}\frac{r_n}{r^2}[1-H(r,t)], \quad A_0^a = 0 \tag{1}$$

reduces the Yang-Mills equations of motion,

$$D_\alpha F^{\alpha\beta a} = 0 \ , \tag{2}$$

to

$$r^2(H_{rr}-H_{tt}) + H - H^3 = 0 \ . \tag{3}$$

In Ref. 1, this equation was reduced to the form

$$(2+\tau)\tau H_{\tau\tau} + 2(1+\tau)H_\tau + H - H^3 = 0 \tag{4}$$

by means of the independent-variable transformation

$$\tau = \frac{t-t_0}{r} - 1 \ . \tag{5}$$

The fields were considered as evolving from an initial time $t=t_0$ to $t=\infty$. The domain of the τ variable was thus fixed as $-1 \leq \tau < \infty$. A regular solution in this domain was obtained in Ref. 1 with the following properties:

$$H \rightarrow 0 \ \text{as}\ \tau \rightarrow \infty \ , \qquad 0 < |H| < 1 \ \text{as}\ \tau \rightarrow -1 \ . \tag{6}$$

We now observe that Eq. (4) can be obtained by a more general transformation. For convenience we introduce a new variable

$$x = 1 + \tau \tag{7}$$

in terms of which (5) becomes

$$(x^2-1)H_{xx} + 2xH_x + H - H^3 = 0 \ . \tag{8}$$

We note that (8) can be obtained from (3) by a general independent-variable transformation[2]

$$x = \frac{A + Bt + C(r^2-t^2)}{2r} \ , \qquad B^2 + 4AC = 1 \ , \tag{9}$$

where A, B, and C are constants. Transformation (9), however, sets the domain of the variable $\tau = x-1$ to be $-\alpha < \tau < \infty$. This change is not in any way advantageous because no regular solution exists for $\tau < -1$. However, the situation is different if we consider the Euclidean version of Eq. (3) obtained by the substitution $t \rightarrow -it$:

$$r^2(H_{rr}+H_{tt}) + H - H^3 = 0 \ . \tag{10}$$

In this case the transformation

$$x = \frac{A + Bt + C(r^2+t^2)}{2r} \ , \qquad B^2 - 4AC = -1 \tag{11}$$

brings the Euclidean equation to the form (8). The linear dependence of t in (11) can be removed since (10) is invariant under time translations $t \rightarrow t+\beta$. If we take $\beta = -B/2C$, then

$$x = \frac{1 + 4C^2(r^2+t^2)}{4Cr} \ .$$

Further (10) is invariant under scale transformations $r \rightarrow \lambda r$, $t \rightarrow \lambda t$. Choosing $\lambda = 1/2C$ we find

$$x = \frac{1 + r^2 + t^2}{2r} \ ,$$

or

$$\tau = \frac{1 + r^2 + t^2}{2r} - 1 \ , \tag{12}$$

the domain of variables now being

$$1 \leq x < \infty, \quad 0 \leq \tau < \infty \ . \tag{13}$$

The only difference compared with Eq. (4) is the change in the domain of τ as given by (13). Hence to construct a solution in Euclidean space we need take only a section (by excluding the domain $-1 \leq \tau < 0$) of Arodz's solution.

As this is a Euclidean solution, the first thing to consider would be the evaluation of the action. That the action of the solution is infinite can be seen in the following way. We have the action

$$S=\frac{4\pi}{g^2}\int_{-\infty}^{\infty} dt \int_0^{\infty} dr \left[(H_r{}^2+H_t{}^2)+\frac{1}{2r^2}(H^2-1)^2 \right] .$$

Since we are considering an H which depends on τ given in (12) this can be rewritten as[3]

$$S=\frac{4\pi}{g^2}\int_0^{2\pi} d\tau' \int_0^1 d\rho \left[(1-\rho^2)H_\rho{}^2+\frac{1}{2\rho^2}(H^2-1)^2 \right] ,$$

where

$$\rho=\frac{1}{1+\tau} ,$$

$$\tau'=\tan^{-1}2t/(1-t^2-r^2) .$$

Since H is independent of τ',

$$S=\frac{8\pi^2}{g^2}\int_0^1 d\rho \left[(1-\rho^2)H_\rho{}^2+\frac{1}{2\rho^2}(H^2-1)^2 \right] . \tag{14}$$

The integral of the first term in the square brackets is evidently convergent. However the second integral is singular at $\rho=0$ and is finite only if $H\to 1$ as $\rho\to 0$. The limit $\rho\to 0$ corresponds to the limit $\tau\to\infty$, and in Ref. 1 it was shown that the limiting value of regular solutions is zero as $\tau\to\infty$. Hence the Euclidean counterpart of Arodz's solution given through relation (12) is an infinite-action solution. This is, however, not an unexpected result; all the known non-self-dual solutions are of infinite action. [It is easy to see that the self-dual solutions within the ansatz (1) are the trivial solutions ± 1, of (10) and all other solutions are non-self-dual.] However, as has been shown by Boutaleb-Joutei, Chakrabarti, and Comtet[3] complex solutions can be found for (10) with finite complex actions.

Comparing the above-obtained solution with other known solutions in the existing literature, we find it is neither an instanton nor a meron. Even though merons are infinite-action non-self-dual Euclidean solutions they carry half-unit topological charge. In the present case, since $H(r,t)$ is regular the topological charge density is zero everywhere and the topological charge of the solution is zero. Furthermore, the Euclidean time evolution of this solution is quite different from that of the meron. Meron solutions start from (at $t=-\infty$) a vacuum configuration and evolve through a Wu-Yang monopole configuration (at $t=0$), and finally end up (at $t=\infty$) in a vacuum configuration.[4] In contrast, the present solution assumes the form of a Wu-Yang monopole configuration as $t\to\pm\infty(\tau\to\infty)$ because $H(\tau)\to 0$ at these limits. In this regard the present solution looks similar to the bounce solutions[5] of scalar field theories which start from a vacuum at $t=-\infty$ and return to the same vacuum at $t=\infty$. However, because of its infinite action, the implications of the newly obtained solution for the quantized theory can be understood only by going beyond semiclassical approximations.

Finally we note that it is possible to obtain self-dual solutions possessing properties analogous to those of the solution presented above. Considering a special case of Witten's ansatz

$$\begin{aligned} gA_i^a &= \epsilon_{ain}\frac{r_n}{r^2}[1-H(r,t)] \mp \delta_{ia}\frac{H(r,t)}{r} , \\ gA_0^a &= -\frac{r_a}{r^2}[1-H(r,t)] , \end{aligned} \tag{15}$$

it may be verified that the self-duality equation

$$F_{\alpha\beta}^a=\tilde{F}_{\alpha\beta}^a$$

can be satisfied if

$$\begin{aligned} rH_r &= H(1-H) , \\ rH_t &= \mp H^2 . \end{aligned} \tag{16}$$

The solution to (16) is given by

$$H=\frac{r}{r\pm t+\beta} , \tag{17}$$

where β is an arbitrary constant. It may be noted that the solution corresponding to (17) may also be easily obtained within the ϕ^4 ansatz.[4] The solutions have singularities at $r=0$ as well as on a hypersurface $r\pm t+\beta=0$. As Euclidean time $t\to\pm\infty$, the gauge potentials become that of a point dyon configuration:

$$gA_i^a=\epsilon_{ain}r_n/r^2 ,$$

$$gA_0^a=-r^a/r^2 .$$

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