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Force between static charges and universality in lattice QCD

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We present Monte Carlo results for the force between static quarks on a large ($16^3 \times 32$) lattice and with high statistics, obtained by using an action which combines terms in the fundamental and adjoint representations of SU(3). We discuss universality with respect to the choice of the action, which appears to be well verified, and propose a phenomenological formula for the force as a function of separation.

I. INTRODUCTION

In a recent publication[1] we presented numerical results on the force between static quarks, obtained by a high-statistics Monte Carlo (MC) simulation on a $16^3 \times 32$ lattice. Wilson's form of the action was assumed: the action of a plaquette is a linear function of the character of the plaquette transporter in the fundamental representation. We explored the domain of values for the coupling parameter β where the transition from the strong-coupling regime to the weak-coupling scaling regime takes place. We found that at all the values of β considered the lattice spacing could be normalized so as to give consistent results for the force as a function of separation in physical units. At the highest values of β the normalization of the lattice spacing was found in agreement with the expected asymptotic scaling formula, an indication that the lattice regularization approximates well the continuum quantum theory.

The continuum limit should be universal, i.e., largely independent from the regularization adopted, provided some general requirements are met. In particular, it should be possible to define the same continuum physics starting from a wide class of lattice actions and by letting the coupling parameter(s) tend to suitable limits. One of the most straightforward extensions of the Wilson action consists in adding to it a term proportional to the character of the plaquette transporter in the adjoint representation. This gives rise to a two-parameter class of lattice actions, which we shall call mixed fundamental-adjoint actions, or FA actions for short. Lattice systems with FA actions have already formed the object of several investigations.[2] Here we wish to report the results of a numerical calculation of the force between static quarks comparable in scope to the calculation detailed in Ref. 1, but for a system defined with the FA action. We performed this second computation with the purpose of verifying the universality of the continuum limit and of determining whether, in so far as scaling is concerned, the mixed-action system behaved in any remarkably different way from the system with Wilson's action.

A numerical analysis extended to a two-dimensional domain in the FA plane would have been too demanding on computational resources, thus we concentrated on a definite trajectory toward the continuum limit. As criteria in the selection of this trajectory we adopted the facts that it should move farther away than the fundamental axis from the known singularities in the FA plane[2] (this in order possibly to achieve a smoother transition to the continuum limit), yet that it should lie above the line where perturbation theory ceases to make sense (so that theoretical and numerically determined ratios of scales may be compared). We also wanted a trajectory that would be contradistinct by some well-defined theoretical property, rather than by parameters empirically adjusted, so we investigated the line where the term of the fourth order in the field strength, in a perturbative expansion of the action, vanishes. This is the line where the lattice action resembles most closely, from a perturbative point of view, the continuum expression.

We shall give the precise definition of the action together with an account of our computation and of our results in Sec. II. Section III will be devoted instead to a comparison of the results obtained with the FA action and our previous results on the system with Wilson's action. A parametrization of the force between static quarks based both on the numerical results and on theoretical considerations will also be offered.

This paper is meant to be self-contained, in the sense of not requiring knowledge of the results in Ref. 1 for its comprehension. Nevertheless, there are elements of the present analysis, such as the procedure followed to extrapolate the Wilson loop factors to infinite separation in time or the evaluation of statistical errors, which are the same as in Ref. 1. To repeat them here would constitute a useless duplication so we shall be rather brief about those points. References 1, or 5 for the computer codes, should be consulted by the reader who wishes to obtain more information about all computational details.

II. THE FORCE BETWEEN STATIC QUARKS IN A SYSTEM WITH MIXED ACTION

We denote by integer-valued coordinates x^μ the points of the lattice, by a the lattice spacing, by $\hat{\mu}$ a unit vector in the direction μ. The dynamical variables are SU(3) matrices U_x^μ defined over the oriented links, from x to $x+\hat{\mu}$, of the lattice. We denote by $U_{x,ij}^{\mu\nu}$ the transport operator along a rectangular path of size i,j in the (μ,ν) plane,

$$U_{x,ij}^{\mu\nu}=U_x^{\nu\dagger}\cdots U_{x+(j-1)\hat{\nu}}^{\nu\dagger}U_{x+j\hat{\nu}}^{\mu\dagger}\cdots U_{x+(i-1)\hat{\mu}+j\hat{\nu}}^{\mu\dagger}\times U_{x+i\hat{\mu}+(j-1)\hat{\nu}}^{\nu}\cdots U_{x+i\hat{\mu}}^{\nu}U_{x+(i-1)\hat{\mu}}^{\mu}\cdots U_x^{\mu}\,. \tag{2.1}$$

The plaquette transporters $U_{x,11}^{\mu\nu}$ will also be indicated simply by $U_x^{\mu\nu}$. The Wilson loop factors are defined as

$$W_{x,ij}^{\mu\nu}=\tfrac{1}{3}\mathrm{Re\,Tr}U_{x,ij}^{\mu\nu}\,. \tag{2.2}$$

The action combining terms in the fundamental and adjoint representations is given by

$$S_{\mathrm{FA}}=\sum_{x,\mu\le\nu}[\beta_F(1-\tfrac{1}{3}\mathrm{Re\,Tr}U_x^{\mu\nu})+\beta_A(1-\tfrac{1}{9}\,|\,\mathrm{Tr}U_x^{\mu\nu}\,|^2)]\,. \tag{2.3}$$

When $\beta_A=0$ it reduces to the so-called Wilson action.

If one expresses $U_x^{\mu\nu}$ in exponentiated form

$$U_x^{\mu\nu}=\exp(igF_x^{\mu\nu}a^2)\,, \tag{2.4}$$

where g is the bare coupling constant and $F_x^{\mu\nu}$ is a Hermitian matrix in the Lie algebra of SU(3), upon expansion into powers of g one finds

$$S_{\mathrm{FA}}=\sum_{x,\mu\le\nu}\left[\left(\frac{\beta_F}{6}+\frac{\beta_A}{3}\right)g^2a^4\mathrm{Tr}(F_x^{\mu\nu})^2-\left(\frac{\beta_F}{144}+\frac{\beta_A}{24}\right)g^4a^8[\mathrm{Tr}(F_x^{\mu\nu})^2]^2+O(g^6)\right] \tag{2.5}$$

(we made use of the identity, valid for SU(2) and SU(3), $\mathrm{Tr}(F_x^{\mu\nu})^4=\frac{1}{2}[\mathrm{Tr}(F_x^{\mu\nu})^2]^2$). Following the motivation outlined in the Introduction, we set

$$\beta_A=-\frac{\beta_F}{6}\,, \tag{2.6}$$

which makes the $O(F^4)$ term in the expansion of S_{FA} vanish. Equations (2.3) and (2.6) define the system we consider. Henceforth we shall use the notation β for β_F. From the expression of the $O(F^2)$ term we see that coupling parameter β and bare coupling constant g are related by

$$\beta=\frac{9}{g^2}\,. \tag{2.7}$$

The asymptotic form of the relation between lattice spacing a and β becomes then

$$a\approx\frac{1}{\Lambda}\left[\frac{16\pi^2\beta}{99}\right]^{51/121}\exp\left[-\frac{8\pi^2\beta}{99}\right], \tag{2.8}$$

Λ being the scale parameter for the present computation.

The force between static sources in the $\underline{3}$ and $\underline{\bar{3}}$ representations of SU(3), in the pure gauge quantum theory, is obtained from the calculation of expectation values of Wilson loop factors. These have been determined by means of a Monte Carlo simulation. We proceeded as follows. From previous analytical and numerical considerations on the lines of "constant physics" in the (β_F,β_A) plane[3,4] we selected a range of values for β, namely, the interval $6.6\le\beta\le8.6$, which was likely to correspond to the domain where the transition from the strong-coupling regime to the weak-coupling regime takes place. We performed MC simulations at six values of β uniformly spaced in the above interval. 600 MC iterations were performed to equilibrate the $16^3\times32$ lattice at $\beta=6.6$, starting from an equilibrated configuration corresponding to the Wilson action at $\beta=5.6$. The resulting configuration was used to start additional 1000 MC iterations where all spatial loops of size up to 8×8 were measured over 100 configurations, with intervals of 10 MC iterations between measurements. The computations for the other five β values (7.0,7.4,7.8,8.2,8.6) went as follows: the last configuration of the previous β value was used as a starting configuration for 400 equilibrating MC iterations. This was followed by the 1000 MC iterations with measurement of spatial loops as explained above. We shall denote by W_{ij} the averages of the spatial Wilson loop factors as determined in the course of the simulation.

The whole calculation was done on a CDC CYBER 205, exploiting as much as possible its vector processing capabilities. Most of the computation was made in 32-bit precision, but selected parts, where more accuracy is required, made use of the full 64-bit precision. We refer the reader to our previous communications (Refs. 1 and 5) for details of the algorithm and for such technical questions as why only spatial loop factors have been measured. Here we mention only that the inclusion of the term in the adjoint representation, although apparently trivial from a computational point of view, actually increases substantially the length of the simulation. To see why, we recall that the upgrading of an individual link variable U_x^μ proceeds through two major steps. First, one evaluates the transport operators (or forces) $F_x^{\mu(k)}$, $k=1,\ldots,6$, along the three sides distinct from x, $x+\hat{\mu}$ of the six plaquettes containing the link under consideration.

The corresponding contributions to the action are

$$S_F'=\beta_F\sum_k(1-\tfrac{1}{3}\mathrm{Re\,Tr}F_x^{\mu(k)\dagger}U_x^\mu) \tag{2.9}$$

and

$$S_A'=\beta_A\sum_k(1-\tfrac{1}{9}\,|\,\mathrm{Tr}F_x^{\mu(k)\dagger}U_x^\mu\,|^2)\,, \tag{2.10}$$

for the two terms in the fundamental and adjoint representation, respectively. In the second step one evaluates the change of action induced by the attempted upgrading $U_x^\mu\to\widetilde{U}_x^\mu$ and accepts or rejects the change accordingly.

This second step is repeated n_H (number of hits) times before going to another U_x^μ, for better efficiency. Now it is crucial that U_x^μ enters linearly in Eq. (2.9) [but not in Eq. (2.10)]. Thus, in so far as the change in S_F' is concerned, the six forces $F_x^{\mu(k)}$ can be added up into a total force F_x^μ and the evaluation of S_F' reduces to a single long vector product between the elements of the matrices F_x^μ and U_x^μ. But for the variation of S_A' the contributions from the six plaquettes, and indeed both the real and the imaginary parts of the trace, must be computed separately at each hit. The number of arithmetic operations in the second stage of the upgrading (the loop over hits) increases very substantially and so does the CP time required. As a matter of fact some of the loss is recovered, because the corresponding part of the code vectorizes very well and thus the calculation proceeds at a faster rate. In any event, to achieve a better balance between the time spent in the calculation of the forces and the time spent in the loop over hits, we performed this calculation with $n_H=8$ hits per upgrading (versus $n_H=10$ in Ref. 1). Timing data are as follows: total upgrade time per link 57.5 μs, of which 20.0 μs in the calculation of the forces and 37.5 μs in the loop over hits, with sustained performance rate of 130 Mflops. (Corresponding values for the calculations with the fundamental term only and $n_H=8$ are

36.3 μs, 17.9 μs, 18.4 μs ,

and

101 Mflops ,

respectively.)

From the average values W_{ij} of Wilson loop factors the force between static charges is evaluated as follows. One defines

$$X_{ij}=-\ln\left[\frac{W_{ij}W_{i-1j-1}}{W_{i-1j}W_{ij-1}}\right] . \qquad (2.11)$$

Then $\lim_{i\to\infty}X_{ij}$ equals the force, in lattice units, at some separation r' between $(j-1)a$ and ja. It is convenient to assign a definite value to r', namely,

$$r'=[j(j-1)]^{1/2}a ,$$

on the basis of an interpolation which becomes exact if the potential is the superposition of a linear and a Coulombic term. Thus, one arrives at the equation

$$F([j(j-1)]^{1/2}a)=\frac{1}{a^2}\lim_{i\to\infty}X_{ij} . \qquad (2.12)$$

The values found for the Wilson loop factors W_{ij} and the quantities X_{ij} are reproduced in the tables. The statistical errors we quote include the effects of cross correlations between different loop factors, but assume a Gaussian distribution of the W_{ij}'s around the correct quantum expectation values and neglect possible correlations between values obtained in subsequent measurements. Also, a linear formula has been used to propagate the errors from W_{ij} to X_{ij}; this clearly cannot be trusted when the error in X_{ij} is comparable in magnitude to X_{ij} itself and thus such errors have only indicative value (see Ref. 1 for a more detailed discussion of the error analysis). In Tables I—VI we report also the mean values for the "internal energies"

$$E_F=1-\tfrac{1}{3}\mathrm{Re\,Tr}U_x^{\mu\nu} \qquad (2.13)$$

and

$$E_A=1-\tfrac{1}{9}\,|\,\mathrm{Tr}U_x^{\mu\nu}\,|^2 . \qquad (2.14)$$

These numbers have been averaged over the whole set of 1000 final configurations in each run and over all plaquettes, including those with one side along the time axis, and are therefore affected by smaller statistical errors than W_{11}.

While in principle the force is given by the limit for $i\to\infty$ of X_{ij}, in practice a straightforward limiting procedure cannot be used because the error in the determination of X_{ij} increases as i becomes larger. It is therefore necessary to derive a scheme which permits the utilization, for the determination of F, of the X_{ij} with small i as well. This we have done as follows. All values X_{ij} at fixed j are plotted against $1/[i(i-1)]$ and fit by a straight line. The intercept b_j of the line is taken as the limiting value of X_{ij} for $i\to\infty$. In other words, one fits all X_{ij} at fixed j by a formula

$$X_{ij}\sim b_j+\frac{c_j}{i(i-1)} . \qquad (2.15)$$

Such an expression is motivated by considerations of the expected short-distance behavior and has been found to interpolate well all numerical data (see Ref. 1 for a more extensive discussion). From Eqs. (2.12) and (2.15) one deduces

$$a^2F([j(j-1)]^{1/2}a)=b_j . \qquad (2.16)$$

An analogous fit in terms of a constant plus Coulombic term

$$a^2F([j(j-1)]^{1/2}a)=a^2\sigma+\frac{\alpha}{j(j-1)} \qquad (2.17)$$

has been used to obtain the string tension σ, i.e., the limiting value of the force for infinite separation.

Our results for σ are displayed in Fig. 1. The upper part of the figure gives σ in lattice units; the lower part of the figure gives σ in terms of the scale parameter Λ, assuming the relationship between a and β to be given exactly by the asymptotic formula of Eq. (2.8). The errors in Fig. 1, as well as in all subsequent figures, are purely statistical and do not reflect systematic biases induced by the extrapolations.

From Fig. 1 it is apparent that a few of the values for σ, namely, those in the lower to mid range of the domain of β values, are compatible with an asymptotic scaling behavior. Assuming the value of σ at $\beta=7.0$ to set the scale, we determine

$$\Lambda=3.70\times10^{-2}\sqrt{\sigma} . \qquad (2.18)$$

The above placing of the asymptotic scaling curve is to some extent arbitrary and has been motivated by consistency with the procedure followed in Ref. 1, where the

TABLE I. Results for expectation values of both terms in the action, Wilson loop factors, and X factors for $\beta=6.6$.

$E_F=0.462\,490$ (13), $E_A=0.664\,297$ (12)

$10^6 W_{ij}$

$i \backslash j$	1	2	3	4	5	6	7	8
1	537 457 (53)	310 561 (68)	182 312 (65)	107 404 (57)	63 307 (50)	37 327 (47)	22 007 (46)	12 946 (43)
2	399 467 (329)	120 355 (69)	49 723 (47)	20 956 (35)	8895 (31)	3751 (31)	1599 (29)	619 (30)
3	351 309 (503)	287 160 (2565)	15 415 (51)	5087 (31)	1712 (29)	549 (28)	239 (27)	68 (27)
4	334 934 (1114)	224 693 (5057)	242 730 (28 267)	1317 (39)	345 (29)	108 (26)	−16 (25)	45 (30)
5	328 325 (2617)	231 907 (14 944)	250 296 (80 518)	86 045 (417 404)	83 (37)	−39 (27)	−76 (24)	−9 (28)
6	335 109 (6526)	273 817 (45 998)	22 218 (224 627)			−18 (35)	31 (25)	−24 (28)
7	324 455 (14 654)	−20 979 (108 664)					14 (37)	−2 (28)
8	418 973 (40 279)	302 151 (371 811)						34 (35)
i / j	2	3	4	5	6	7	8	

$10^6 X_{ij}$

asymptotic curve was also placed in between the lowest points (in rescaled units). If one used the lowest point for σ to set the scale, one would find instead

$$\Lambda=3.85\times10^{-2}\sqrt{\sigma}\ . \tag{2.19}$$

The discrepancy between the values in Eqs. (2.18) and (2.19) can be considered as an estimate for the uncertainty in the determination of $\Lambda/\sqrt{\sigma}$. We shall return to this point in Sec. III.

The data for σ depart from the asymptotic expression both at the lowest and at the upper values of β. We interpret the deviation from asymptotic scaling at $\beta=6.6$ as genuine, while we attribute the deviations at $\beta\geq7.8$ to an overestimate of the string tension, due to the fact that the maximum separations achieved in the determination of $F(r)$ are not large enough. Such interpretation finds support if all the results for the force are plotted in physical units. This is done in Fig. 2. The values for F and for r are expressed in units of σ and $(\sigma)^{-1/2}$, respectively, assuming for $\beta\geq7$ the relationship between a and β given by the asymptotic formula [Eq. (2.8)], with the scale parameter as in Eq. (2.18). The data for $\beta=6.6$ are instead rescaled assuming the relation between a and $\sqrt{\sigma}$ as given by the MC computation. All the points appear to lie on a

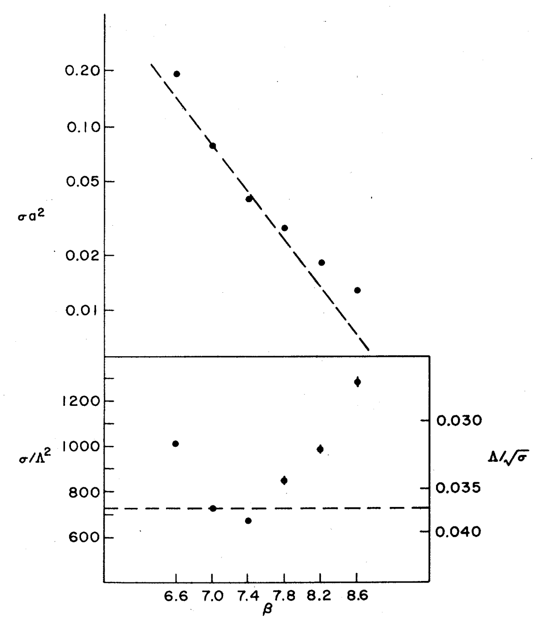

FIG. 1. Results for the string tension, in lattice units (upper diagram) and in rescaled units (lower diagram).

TABLE II. Same as in Table I, but for $\beta=7.0$.

$E_F=0.421\,942$ (10), $E_A=0.626\,764$ (10)

$10^6 W_{ij}$ (upper triangle, including diagonal; columns j = 1–8 above) / $10^6 X_{ij}$ (lower triangle; columns j = 2–8 below)

$i \backslash j$	1	2	3	4	5	6	7	8
1	578 027 (41)	363 685 (56)	233 210 (62)	150 235 (57)	96 871 (54)	62 502 (52)	40 311 (47)	26 005 (41)
2	291 487 (217)	170 967 (68)	86 515 (58)	44 760 (50)	23 323 (41)	12 169 (36)	6359 (33)	3334 (31)
3	236 800 (321)	163 773 (942)	37 166 (57)	16 721 (39)	7661 (30)	3537 (27)	1572 (29)	682 (29)
4	219 271 (576)	139 738 (1471)	101 721 (5703)	6795 (46)	2811 (33)	1185 (29)	519 (24)	226 (24)
5	213 051 (1025)	128 642 (3035)	102 209 (9424)	58 197 (33 537)	1097 (39)	442 (28)	201 (26)	83 (27)
6	212 340 (2014)	122 417 (6619)	90 506 (22 004)	44 509 (59 790)	406 083 (310 755)	119 (37)	48 (25)	30 (29)
7	210 533 (3653)	161 640 (16 860)	15 524 (49 304)	−35 179 (124 883)	111 468 (609 169)	−628 767 (1 343 018)	37 (35)	19 (25)
8	207 436 (7550)	189 985 (39 534)	−3688 (119 914)	54 614 (364 666)	−404 655 (1 059 112)	191 159 (1 810 703)	130 267 (5 062 448)	8 (37)
i / j	2	3	4	5	6	7	8	

universal curve, and this we consider as evidence that the results at the largest values of β are in agreement with the expected asymptotic scaling behavior.

III. DISCUSSION OF UNIVERSALITY

In a previous analysis of the theory with Wilson action[1] we found

$$\Lambda_W=9.63\times10^{-3}\sqrt{\sigma}\ . \tag{3.1}$$

(We shall use, in this section and in the figures, subscripts W and M to characterize results obtained with Wilson's action and with the mixed fundamental-adjoint action, respectively.) Together with the result in Eq. (2.18) this gives

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\mathrm{MC}}=3.84 \tag{3.2}$$

for the numerically determined ratio of scales. We could also choose to fix Λ_M by the result at $\beta=7.4$ [see Eq. (2.19)]. But then consistency would require that the point with the lowest σ in rescaled units be used to set the scale also in the calculation with Wilson's action. This point

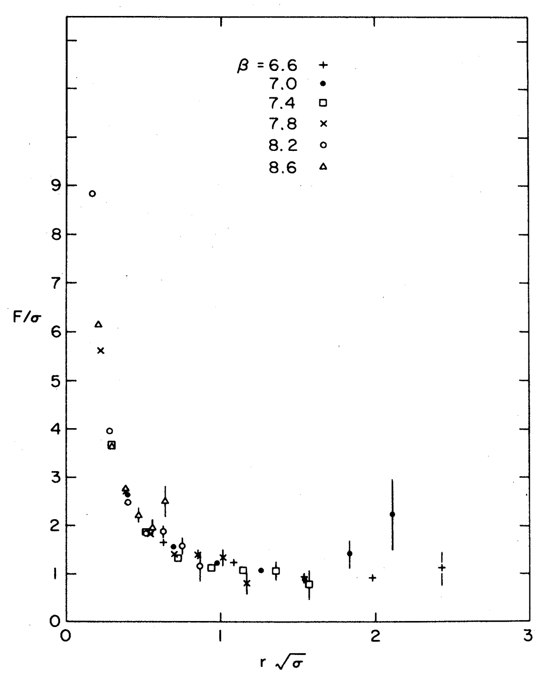

FIG. 2. Force versus separation in rescaled physical units.

TABLE III. Same as in Table I, but for $\beta=7.4$.

	$E_F=0.394\,072$ (9), $E_A=0.598\,583$ (10)							
	$10^6 W_{ij}$							
$i \backslash j$	1	2	3	4	5	6	7	8
1	605 924 (35)	400 404 (53)	269 783 (61)	182 641 (59)	123 780 (57)	83 922 (54)	56 919 (50)	38 559 (47)
2	242 182 (185)	207 682 (68)	115 799 (62)	65 880 (52)	37 731 (44)	21 659 (37)	12 468 (35)	7117 (32)
3	189 294 (239)	120 255 (632)	57 252 (62)	29 531 (46)	15 457 (41)	8137 (30)	4286 (28)	2255 (26)
4	173 933 (393)	97 982 (797)	72 689 (2384)	14 165 (46)	6948 (36)	3406 (36)	1708 (29)	855 (28)
5	168 326 (663)	90 043 (1753)	64 873 (4058)	55 751 (12 525)	3224 (38)	1466 (31)	755 (32)	373 (27)
6	166 431 (966)	86 640 (2739)	71 181 (8689)	74 832 (21 879)	−26 988 (62 726)	685 (40)	288 (27)	132 (26)
7	164 004 (1792)	88 739 (6131)	49 523 (15 230)	−26 397 (41 437)	202 746 (102 378)	−9688 (310 532)	122 (35)	77 (30)
8	171 289 (3169)	81 493 (10 021)	49 555 (33 208)	13 451 (74 717)	78 406 (214 596)	−317 823 (496 455)		−5 (33)
i / j	2	3	4	5	6	7	8	
				$10^6 X_{ij}$				

(see Ref. 1) is the point at $\beta=6.2$ and correspondingly one finds

$$\Lambda=9.84\times10^{-3}\sqrt{\sigma}\ . \tag{3.3}$$

Comparing with Eq. (2.19) one obtains

$$\left.\frac{\Lambda_M}{\Lambda_W}\right| =3.91\ . \tag{3.4}$$

The difference between the two numbers should be considered as a lower bound on the error with which one can reliably estimate the ratio of scales.

A one-loop perturbative calculation can be used to determine the exact value for the above ratio of scales,[6] which is given by

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\rm TH} =4.46\ . \tag{3.5}$$

We thus find

$$\frac{\left.\dfrac{\Lambda_M}{\Lambda_W}\right|_{\rm MC}}{\left.\dfrac{\Lambda_M}{\Lambda_W}\right|_{\rm TH}}=0.869\ , \tag{3.6}$$

using the central value of Eqs. (3.2) and (3.4). The agreement between the numerical results and the theoretical expectation is rather good. The discrepancy, of about 13%, can be easily explained by the fact that the numerical calculation is performed at a finite value of the bare coupling constant, while the theoretical result follows in the limit $g\to0$. Recently, Ellis and Martinelli[7] have published a complete two-loop calculation of the $O(g^2)$ corrections to the ratio of scales (a partial, but already reasonably accurate calculation had been published in Ref. 8). Their computation leads to an estimate

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\rm MC} \approx(1-g^2\delta)\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\rm TH}\ , \tag{3.7}$$

where δ is given by

$$\delta=-0.126r+1.37r^2\ , \tag{3.8}$$

$$r=-\frac{2\beta_A}{\beta_F+2\beta_A}\ . \tag{3.9}$$

In our case $r=\frac{1}{2}$, $\delta=0.28$, $g^2=9/\beta\approx1.18$ in the middle of the range considered, so that Eq. (3.7) predicts

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\rm MC} \approx0.670\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\rm TH}\ . \tag{3.10}$$

TABLE IV. Same as in Table I, but for $\beta=7.8$.

$E_F=0.371\,263$ (8), $E_A=0.574\,109$ (9)

$10^6 W_{ij}$

$i \backslash j$	1	2	3	4	5	6	7	8
1	628 708 (37)	430 698 (53)	300 694 (60)	210 895 (61)	148 067 (61)	103 976 (58)	72 999 (54)	51 258 (53)
2	210 846 (162)	238 961 (73)	141 901 (68)	85 932 (60)	52 328 (51)	31 932 (44)	19 450 (38)	11 870 (39)
3	161 858 (178)	99 794 (418)	76 261 (64)	42 661 (55)	24 190 (45)	13 741 (40)	7802 (34)	4492 (31)
4	146 844 (282)	79 299 (599)	58 547 (1763)	22 508 (54)	12 114 (41)	6572 (36)	3541 (29)	1939 (32)
5	142 330 (477)	71 314 (1075)	52 133 (2018)	38 711 (6293)	6273 (44)	3258 (34)	1720 (30)	924 (27)
6	140 404 (709)	71 693 (1690)	45 912 (4076)	43 709 (8136)	21 411 (24 080)	1656 (46)	869 (31)	413 (31)
7	142 040 (1138)	70 262 (3192)	52 350 (7029)	20 070 (16 818)	6668 (37 072)	51 408 (92 167)	433 (41)	171 (28)
8	140 247 (1879)	58 164 (5185)	50 201 (14 411)	19 625 (29 218)	120 978 (75 008)	184 574 (184 212)	−303 700 (470 002)	92 (39)
i / j	2	3	4	5	6	7	8	

$10^6 X_{ij}$

The correction of order g^2 certainly goes in the right direction and indeed overshoots the numerical result. We do not consider the fact that the two-loop correction worsens the numerical agreement between theoretical and Monte Carlo results as significant, because the two-loop correction is in itself very large and thus cannot claim to give an accurate estimate of the finite g effects. To strengthen this point, we notice that rewriting Eq. (3.7) as

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\mathrm{TH}} \approx (1+g^2\delta)\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\mathrm{MC}}$$

[which neglects terms $O(g^4)$] we would get 0.752 for the predicted ratio between

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\mathrm{MC}}$$

and the asymptotic value

$$\left.\frac{\Lambda_M}{\Lambda_W}\right|_{\mathrm{TH}} .$$

The large value of the two-loop correction may in turn be attributed to the fact that, as the line in the fundamental-adjoint plane moves toward more and more negative slopes, it soon approaches the limit [$\beta_A=-\frac{1}{2}\beta_F$, cf. Eq. (3.9)] where perturbation theory ceases to make sense. A self-consistent treatment of the $|\mathrm{Tr}\,U_x^{\mu\nu}|^2$ term in the action may then be more appropriate.[3] Nevertheless, in so far as verification of universality is concerned, we find that this is well supported by the reasonable agreement between the numerical ratio of scales and its expected asymptotic limit, that the $O(g^2)$ corrections go in the right direction and are of a magnitude which can easily explain the residual discrepancy.

If we take $\beta=7.4$ (FA action) and $\beta=6.2$ (Wilson's action) to be the points where asymptotic scaling sets in, the corresponding values of the bare coupling constant are $g=1.10$ and $g=0.98$, respectively ($g^2=6/\beta$ with Wilson's action). Scaling appears to begin at a larger value of the unrenormalized coupling constant in the FA system, but this is *per se* of little relevance. More significant are the corresponding values of the lattice spacing. These are given by $a=0.20/\sqrt{\sigma}$ (FA action) and $a=0.19/\sqrt{\sigma}$ (W action), i.e., $a=0.096$ fm and $a=0.091$ fm assuming

TABLE V. Same as in Table I, but for $\beta=8.2$.

$E_F=0.351\,579$ (7), $E_A=0.551\,997$ (9)

$10^6 W_{ij}$

$i \backslash j$	1	2	3	4	5	6	7	8
1	648 370 (29)	457 267 (41)	328 401 (47)	236 873 (49)	171 030 (50)	123 539 (47)	89 246 (46)	64 484 (48)
2	188 149 (142)	267 180 (61)	166 473 (59)	105 579 (54)	67 301 (47)	43 013 (43)	27 511 (40)	17 612 (35)
3	142 058 (163)	84 082 (376)	95 360 (65)	56 568 (54)	33 974 (47)	20 486 (39)	12 385 (35)	7543 (32)
4	128 661 (231)	66 843 (455)	45 172 (990)	32 075 (53)	18 553 (42)	10 761 (39)	6302 (34)	3714 (29)
5	124 605 (305)	59 555 (830)	37 582 (1428)	31 682 (3445)	10 397 (46)	5827 (32)	3343 (28)	1911 (27)
6	122 382 (542)	58 180 (1138)	38 839 (2533)	34 359 (5094)	45 587 (14 048)	3120 (41)	1711 (27)	993 (26)
7	121 748 (812)	56 350 (1956)	31 758 (4441)	20 519 (8773)	45 014 (17 861)	−86 295 (46 062)	1023 (42)	487 (30)
8	121 021 (1215)	49 875 (3330)	32 914 (8141)	30 598 (13 314)	−14 851 (30 264)	198 024 (66 937)	−53 634 (192 135)	245 (41)
i / j	2	3	4	5	6	7	8	

$10^6 X_{ij}$

$\sqrt{\sigma}\simeq 420$ MeV. The difference between the two values is minimal and smaller than the variations which can be induced by using slightly different values of β to characterize the onset of scaling. Thus, we conclude that the asymptotic scaling behavior manifests itself in both systems at comparable values of the lattice spacing, namely, when a equals approximately 0.1 fm.

Another question which may be asked is whether the approach to scaling is different in the two systems. To get some quantitative insight into the matter, we assume that asymptotic scaling is characterized by the highest scales for both systems. In the FA system then an extrapolation to $\beta=6.6$ would give $\sigma a^2=0.132$ for the scaling value, whereas the MC calculation gives $\sigma a^2=0.197$. For a meaningful comparison, we must interpolate the results obtained with Wilson's action at $\beta=5.6$ ($\sigma a^2=0.279$) and $\beta=5.8$ ($\sigma a^2=0.111$). Given the approximatively linear behavior of σa^2 on a logarithmic scale, the interpolation is best done on the logarithms of MC results and gives a value $\sigma a^2=0.197$ for $\beta=5.675$. The corresponding scaling value is $\sigma a^2=0.118$. Thus, at the same physical value for a the deviation of the extrapolated scaling value of σa^2 from its actual value is larger in the system with Wilson's action than in the FA system and the latter appears to exhibit a somewhat smoother approach to scaling.

The agreement of the β function with its two-loop perturbative value is not a necessary requirement, however, for a good approximation to the physics of the continuum. The obvious criterion is that, once the relationship between lattice scaling a and coupling parameter β has been fixed, the physical values of all observables should be independent of β. The observables at our disposal, in the present analysis, are the values of the force as a function of separation. We can check, therefore, both scaling toward the continuum limit (but not necessarily asymptotic scaling) and universality by plotting all the results obtained for the force in physical units. This is done in Figs. 3 and 4 (we shall return later on to the meaning of the continuous line in the figures). In the graph of Fig. 3 (where the points for the mixed FA system duplicate those of Fig. 2, while those for the W system duplicate those of Fig. 5 in Ref. 1), the lower values for the scale parameters have been used. The rescaling of force and separation, we recall, is done according to the asymptotic formula for $\beta\geq 7$ (mixed FA system) and for $\beta\geq 6$ (W system), while the actual numerical results for σa^2 are used to set the scales at lower β. In Fig. 4 the rescaling is

TABLE VI. Same as in Table I, but for $\beta=8.6$.

$E_F=0.334\,251$ (8), $E_A=0.531\,759$ (10)

$10^6 W_{ij}$

$i \backslash j$	1	2	3	4	5	6	7	8
1	665 767 (33)	481 008 (51)	353 652 (63)	261 066 (68)	192 946 (68)	142 636 (66)	105 457 (61)	77 998 (58)
2	170 616 (132)	293 012 (74)	189 624 (77)	124 737 (75)	82 531 (68)	54 702 (59)	36 270 (51)	24 086 (43)
3	127 596 (150)	74 454 (311)	113 912 (82)	70 738 (71)	44 422 (63)	28 049 (50)	17 743 (40)	11 200 (33)
4	115 295 (198)	57 606 (384)	39 647 (945)	42 220 (66)	25 662 (55)	15 693 (40)	9661 (33)	5949 (33)
5	110 675 (274)	52 213 (562)	32 636 (978)	26 844 (2481)	15 185 (50)	9128 (36)	5480 (28)	3308 (30)
6	109 161 (376)	48 512 (893)	31 995 (1772)	17 108 (2930)	19 699 (7437)	5381 (38)	3176 (29)	1925 (29)
7	108 912 (628)	47 065 (1328)	27 204 (2260)	25 110 (4331)	16 818 (8414)	38 539 (21 620)	1804 (37)	1086 (28)
8	107 727 (870)	50 712 (1826)	24 721 (4355)	19 972 (8549)	−3890 (14 339)	6796 (29 196)	87 560 (67 156)	599 (36)
i / j	2	3	4	5	6	7	8	

$10^6 X_{ij}$

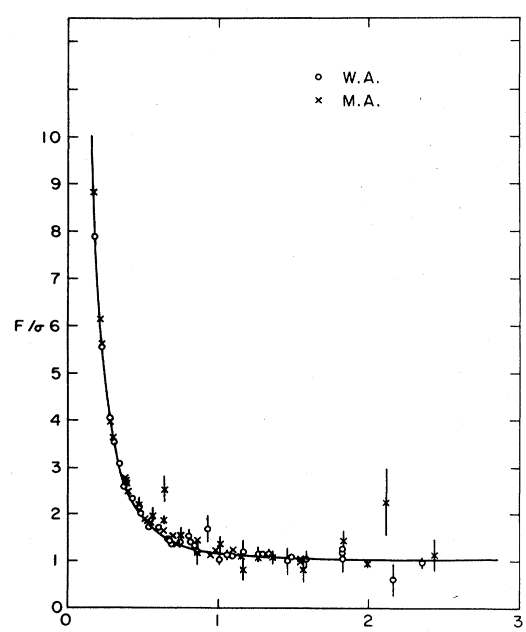

FIG. 3. Results for the force with Wilson's action and mixed FA action combined and interpolating curve.

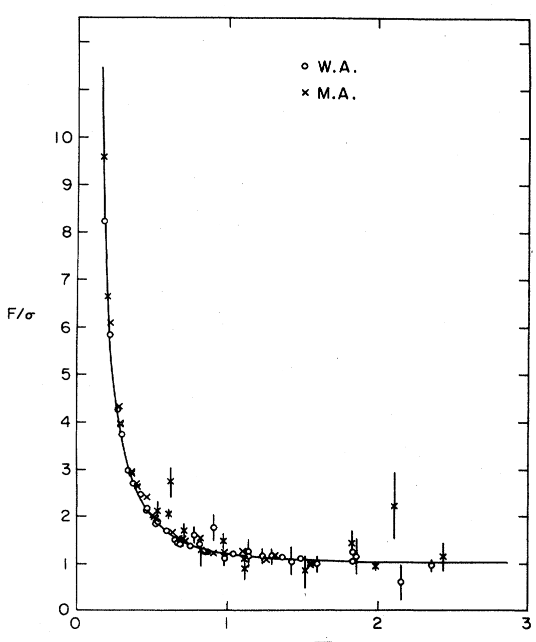

FIG. 4. Same as in Fig. 3, but with a slightly different choice of scale factors.

done according to the larger values for the scale parameters, using the asymptotic formula for $\beta \geq 7.4$ (mixed FA system) and $\beta \geq 6.2$ (W system).

The two graphs show that all Monte Carlo results are quite consistent with a definite functional form for $F(r)$, thus giving good evidence for scaling and for universality with respect to the choice of the action. The qualitative features of the two graphs are identical and it would seem very difficult to discriminate between the two choices for Λ on the basis of the rescaled data. We use this fact to establish a lower bound on the error in the determination of the scales, which we express by quoting a range of values:

$$\Lambda_M = (3.70\text{--}3.95)\times 10^{-2}\sqrt{\sigma} \tag{3.11}$$

for the system with mixed FA action (and $\beta_A = -\frac{1}{6}\beta_F$) and

$$\Lambda_W = (9.63\text{--}9.84)\times 10^{-3}\sqrt{\sigma} \tag{3.12}$$

for the system with Wilson's action. The values above should be considered as correlated: universality demands that if a lower scale is selected for one system, a lower scale should also be used for the other.

The degree of uncertainty in the scales becomes larger if one wishes to establish the true asymptotic scale, i.e., the one that would follow from going to the limit $\beta \to \infty$. We have no way to estimate the error in the absolute determination of Λ, but we have already shown that for the ratio of scales such error is $\simeq 13\%$. The deviation of the scale parameter from its theoretical value at $\beta \to \infty$ is immaterial so long as one compares observables obtained within the same scheme of renormalization, but is of course important if lattice results are to be compared with results derived by different regularizations, like those of perturbative QCD.

The Monte Carlo results provide a rather accurate determination of the force, $F(r)$, between static charges as a function of separation. For phenomenological applications it may be useful to represent $F(r)$ in terms of an approximate analytic expression, embodying the expected short-distance and long-distance behavior and containing a few parameters which may be fit to the MC data. To obtain such an expression we write first

$$F(r) = \frac{4\alpha(r)}{3r^2}\ , \tag{3.13}$$

and concentrate on the renormalization-group equation satisfied by $\alpha(r)$

$$\frac{d\alpha}{d\ln(r^2)} = f(\alpha)\ . \tag{3.14}$$

$f(\alpha)$, which can be straightforwardly related to the β function, has an expansion

$$f(\alpha) = f_0\alpha^2 + f_1\alpha^3 + O(\alpha^4) \tag{3.15}$$

for $\alpha \to 0$; on the other hand, $F(r) \to \sigma$ for $r \to \infty$ demands $\alpha(r) \approx 3r^2\sigma/4$ for $r \to \infty$ and therefore $f(\alpha) \approx \alpha$ for $\alpha \to \infty$. Thus, a rational approximation

$$f(\alpha) = \frac{P(\alpha)}{Q(\alpha)}\ , \tag{3.16}$$

where the degree of P is of one unit greater than the degree of Q, recommends itself. The information available on the small α and large α behavior of $f(\alpha)$ could be used to constrain the coefficients of P and Q, but this is in practice not necessary. It is more convenient to consider directly the integrated form of Eq. (3.14), namely,

$$\ln r^2 = \int \frac{Q(\alpha)d\alpha}{P(\alpha)} + \text{const}\ , \tag{3.17}$$

where a parametrization for the integral can be immediately determined on general grounds once the order of P is fixed. We found that a satisfactory fit to the MC data can be achieved assuming that $P(\alpha)$ is a fourth-order polynomial, with two roots at $-\alpha_0 \pm i\alpha_1$ and, of course, a double zero at the origin [cf. Eq. (3.15)]. This gives for the most general parametrization:

$$\ln r^2 = C_0 + \frac{C_1}{\alpha} + C_2\ln\alpha + C_3\ln[(\alpha+\alpha_0)^2 + \alpha_1{}^2] + C_4\tan^{-1}\left[\frac{\alpha+\alpha_0}{\alpha_1}\right]\ . \tag{3.18}$$

The constraints

$$\alpha(r) \approx \frac{4\pi}{11}[\ln(\Lambda_c r)^{-2} + \tfrac{102}{121}\ln\ln(\Lambda_c\sigma)^{-2}]^{-1}\ , \tag{3.19}$$

for $r \to 0$, where Λ_c is the scale parameter appearing in the short-distance behavior of the potential,[9] and

$$\alpha(r) \approx \frac{3\sigma r^2}{4} \tag{3.20}$$

for $r \to \infty$ may now easily be imposed. We thus arrive at

$$\ln(\Lambda_c r)^2 = -\frac{4\pi}{11\alpha} - \tfrac{102}{121}\ln\left[\frac{11\alpha}{4\pi}\right] + \tfrac{223}{242}\ln\left[\frac{(\alpha+\alpha_0)^2+\alpha_1{}^2}{\alpha_0{}^2+\alpha_1{}^2}\right] + C_4\left[\tan^{-1}\left[\frac{\alpha+\alpha_0}{\alpha_1}\right] - \tan^{-1}\left[\frac{\alpha_0}{\alpha_1}\right]\right]\ , \tag{3.21}$$

where α_0 and α_1 are free parameters and C_4 is given by

$$C_4 = \left[\ln\left[\frac{16\pi\Lambda_c{}^2}{33\sigma}\right] - \tfrac{223}{242}\ln\left[\frac{16\pi^2}{121(\alpha_0{}^2+\alpha_1{}^2)}\right]\right] \times \left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\alpha_0}{\alpha_1}\right]\right]^{-1}\ . \tag{3.22}$$

The scale parameter Λ_c can be determined theoretically:[9] it is related to the lattice scale parameters by

$$\Lambda_c \simeq 30.19\Lambda_W \tag{3.23}$$

and

$$\Lambda_c \simeq \frac{30.19}{4.46}\Lambda_M \simeq 6.77\Lambda_M \tag{3.24}$$

for the two actions, respectively. In our fitting procedure we have used for Λ_c the average of the two values which follow from Eqs. (3.23) and (3.24) and our MC results. We determined the parameters α_0 and α_1 by minimizing the χ^2 deviation. When computing the χ^2 we included only the points at separation $r > \sqrt{6}a$ (i.e., the first two points at all values of β have been left out). This we have done because at small lattice separations one expects the distortion due to the lattice to be the largest, while at the same time the statistical errors affecting the MC data are the least. Thus, the statistical errors can in no way account, at separations of very few lattice spacings, for the degree of uncertainty due to systematic effects and the inclusion of the corresponding points in the fitting procedure would bias the fit.

Our results for the interpolations of the force are illustrated by the continuous curves in Fig. 3 (lower choice of scales) and Fig. 4 (higher scales). The corresponding parameters are

$$\begin{aligned} &\Lambda_c = 0.270\sqrt{\sigma} , \\ &\alpha_0 = 0.068 , \\ &\alpha_1 = 0.231 \\ &(C_4 = -3.943) \end{aligned} \tag{3.25}$$

for Fig. 3, with

$$\chi^2 = 80.1 ; \tag{3.26}$$

and

$$\begin{aligned} &\Lambda_c = 0.279\sqrt{\sigma} , \\ &\alpha_0 = 0.084 , \\ &\alpha_1 = 0.202 \\ &(C_4 = -4.403) \end{aligned} \tag{3.27}$$

for Fig. 4, with

$$\chi^2 = 75.0 . \tag{3.28}$$

The quality of the fits is reasonably good (the number of degrees of freedom is 58), especially if one considers that the errors entering into the χ^2 are purely statistical and that, as discussed already, there are certainly systematic errors due to the lattice approximation and to our extrapolation procedure. Also, the curves corresponding to our interpolating formula appear to agree very well with the data at small lattice separation, which have been excluded from the fit. The short-distance behavior of the analytic formula is determined mainly by the value of Λ_c: it is thus almost completely determined by theoretical considerations and the agreement between the predicted short-distance asymptotically-free behavior and the MC data corroborates the validity of the numerical analysis. The statistical errors of the data at small lattice separation are, however, much smaller than the size of the corresponding points in the figures. They under-represent the true degree of uncertainty of the lattice results. If we included, for instance, the points at $r = \sqrt{6}a$ in the evaluation of the χ^2, with the parameters of Eq. (3.27), the χ^2 would be boosted to a value (in our opinion, unrealistic) of 286.

Summarizing, it would appear to us that our Monte Carlo results for the force between static quarks nicely confirm universality with respect to the choice of action and scaling toward the continuum limit, albeit not according to the perturbative two-loop formula at the lowest values of β. The Monte Carlo analysis provides a derivation of the force entirely from first principles, which is theoretically important and may have valuable phenomenological applications.

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