Illustrations of vacuum polarization by solitons

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The value and limitations of the adiabatic method for calculating induced charges are discussed in a general way and illustrated in some simple models in $1+1$ dimensions. The relevance of the size of solitons is emphasized.

I. INTRODUCTION

In a variety of contexts within condensed matter and particle physics, vacuum polarization effects involving fermion fields interacting with background solitons have been shown to induce unusual (including fractional) quantum numbers.[1,2]

A powerful method for studying such effects is the adiabatic method.[3] In this method, the background soliton field is imagined to be built up from the normal vacuum by slow changes of the fields in space and time. In this situation, the induced expectation value of any conserved current in the no-particle state is readily calculated by expanding in gradients of the background fields. By monitoring the flow of current at spatial infinity one calculates, invoking the current conservation law, the charge localized on the soliton. Several remarks of a general nature are important to the interpretation of results calculated by this method:

(i) Since the final soliton configuration is slowly varying at spatial infinity, the adiabatic approximation can be made arbitrarily good for the current flow at infinity.

(ii) The adiabatic result is valid even for solitons whose spatial gradients are not small. One could imagine forming such a soliton in two steps: first constructing a smooth soliton with the desired behavior at infinity, and then changing it locally to give it sharper features. The adiabatic method can be applied to the first step; the second involves only local changes and does not involve current flow at infinity although it may induce level crossings.

(iii) If level crossings do occur, they change the charge by an integer; thus the adiabatic method will accurately reproduce the *fractional part* of the charge of the ground state. It would, in fact, be rather silly to expect the simple adiabatic method to determine the fermionic charges induced by the soliton more fully than this; after all, one can construct many states with the same background scalar field and fermionic charges differing by integers by filling successive energy levels.

(iv) The spatial scale of the soliton at which level crossings may occur is set by the fermion effective mass; i.e., if the Compton wavelength of the fermion is much smaller than the characteristic spatial scale of background field variations, the adiabatic method is expected to be accurate.

In this paper, these remarks are illustrated in a theory of massive fermions interacting nonlinearly with a pseudoscalar field φ via the Lagrangian

$$\mathscr{L}=\bar{\psi}(i\partial-m_1-m_2e^{i\varphi(x)\gamma_5})\psi \ . \tag{1}$$

Although our emphasis here is on a mathematically complete analysis of simple models, some general remarks of a more physical nature are made at the end.

If $\varphi(x)$ is constant the fermion has a complex mass which can be made real by a redefinition of ψ. If $\varphi(x)$ has space dependence the mass becomes a complex function of space, $m(x)=m_1+m_2e^{i\varphi(x)}$. The resulting Dirac equation is solved and the energy spectrum is followed as φ changes slowly from $\varphi(x)=0$ to a nontrivial soliton configuration, for several (m_1,m_2) values. Care must be taken in choosing $\varphi(x)$ in order to have a manageable Dirac equation. Three cases are considered:

(a) infinitely thin soliton, where $\varphi(x)$ is constant everywhere except for one discontinuity;

(b) finite-width soliton, which is approximated by $\varphi(x)$ constant except for two discontinuities;

(c) infinitely wide soliton, where $\varphi(x)$ is slowly varying: $d\varphi/dx \ll m$.

In all cases, φ is taken to be odd so that a parity operation may be defined.[4]

The field operator can be expanded in terms of eigenstates of the solitonic Hamiltonian. The coefficients of this expansion are the creation and destruction operators of particles in solitonic energy eigenstates. In this way the creation and destruction operators in the presence and absence of a soliton can be expressed in terms of one another. The coefficients of this Bogoliubov transformation have a direct physical interpretation. They give the amplitude that a particle in a given eigenstate in the absence of the soliton will find itself in a given eigenstate in the presence of the soliton potential, if the potential is turned on suddenly. Expansion of the number operator in terms of the solitonic creation and destruction operators shows that the fractional part of the charge results from a

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change in the number of states in the negative-energy sea. The results agree with those obtained by the adiabatic method, which relates the charge of the soliton to the change in phase of the fermion mass, $\Delta\Theta$:

$$Q=-\frac{\Delta\Theta}{2\pi} . \tag{2}$$

In this way, the results from sudden and adiabatic switching are linked. $\Delta\Theta$ can easily be expressed in terms of m_1, m_2, and $\varphi(\pm\infty)$.

II. INFINITELY THIN SOLITON

$\varphi(x)$ is taken to be an odd function with a single discontinuity: $\varphi(x)=\alpha x/|x|$.

Consider first the case $m_1=0$. The change in phase of the mass is simply $\Delta\Theta=2\alpha$. For convenience set $m_2=m$. The Dirac equation is, in the representation $\gamma^0=\sigma_1$, $\gamma^1=i\sigma_3$, $\gamma^5=\gamma^0\gamma^1=\sigma_2$,

$$i\sigma_1\frac{\partial\psi}{\partial t}-\sigma_3\frac{\partial\psi}{\partial x}-m[\cos\varphi(x)-i\sigma_2\sin\varphi(x)]\psi=0 . \tag{3}$$

The equation is invariant under the parity operation $P\psi(x)=\sigma_1\psi(-x)$, since φ is odd. For a solution with energy E, for $x>0$,

$$\frac{d\psi}{dx}=\begin{bmatrix}-mc & E-ms\\ -E-ms & mc\end{bmatrix}\psi , \tag{4}$$

where $s=\sin\alpha$, $c=\cos\alpha$. This equation has plane-wave solutions for $|E|>m$ and exponential solutions for $|E|<m$. For $x<0$, one simply replaces s by $-s$.

Eigenstates are obtained by demanding continuity of ψ at the origin. This procedure yields a pair of solutions of opposite parity for each energy in the continuum, and a bound state of energy $E_b=m\cos\alpha$. The energy spectrum, as a function of α, is displayed in Fig. 1.

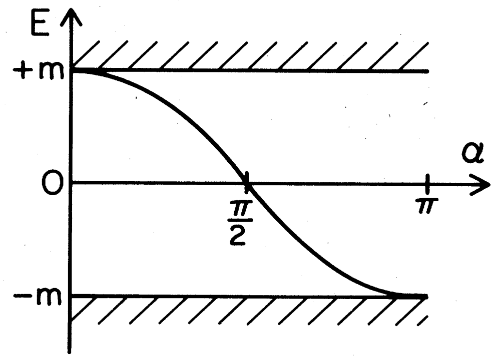

FIG. 1. Energy spectrum as a function of α for $m_1=0$, $m_2=m$.

Explicitly, the eigenfunctions are as follows. Bound state (energy $m\cos\alpha$):

$$\chi_b(x)=(ms/2)^{1/2}\begin{bmatrix}1\\1\end{bmatrix}[e^{msx}\Theta(-x)+e^{-msx}\Theta(x)] . \tag{5}$$

Energy $+E_p=(m^2+p^2)^{1/2}$, positive parity:

$$\mu_p^+(x)=N_p\left\{\left[(ipE_p+m^2sc)\begin{bmatrix}E_p+ms\\ mc+ip\end{bmatrix}e^{ipx}-(mc+ip)(ms-ip)\begin{bmatrix}E_p+ms\\ mc-ip\end{bmatrix}e^{-ipx}\right]\Theta(x)\right.$$
$$\left.+\left[-(mc+ip)(ms-ip)\begin{bmatrix}mc-ip\\ E_p+ms\end{bmatrix}e^{ipx}+(ipE_p+m^2sc)\begin{bmatrix}mc+ip\\ E_p+ms\end{bmatrix}e^{-ipx}\right]\Theta(x)\right\} . \tag{6}$$

Energy $+E_p$, negative parity:

$$\mu_p^-(x)=N_p\left\{\left[(ipE_p+m^2sc)\begin{bmatrix}E_p+ms\\ mc+ip\end{bmatrix}e^{ipx}-(ms+ip)(mc+ip)\begin{bmatrix}E_p+ms\\ mc-ip\end{bmatrix}e^{-ipx}\right]\Theta(-x)\right.$$
$$\left.+\left[(ms+ip)(mc+ip)\begin{bmatrix}mc-ip\\ E_p+ms\end{bmatrix}e^{ipx}-(ipE_p+m^2sc)\begin{bmatrix}mc+ip\\ E_p+ms\end{bmatrix}e^{-ipx}\right]\Theta(x)\right\} . \tag{7}$$

Here $\Theta(x)$ is the usual step function and the normalization factor is

$$N_p=[4E_p(E_p+ms)(m^2c^2+p^2)(m^2s^2+p^2)]^{-1/2} .$$

The negative-energy solutions, $\nu_p^{\pm}(x)$, are obtained from $\mu_p^{\pm}(x)$ by the substitution $E_p\rightarrow -E_p$. They are displayed explicitly in the Appendix.

These solutions are to be compared to the ordinary (i.e., $\alpha=0$) Dirac equation solutions:

$$u_p(x)=\frac{1}{\sqrt{2E_p}}\begin{bmatrix}m-ip\\ E_p\end{bmatrix}e^{ipx} ,$$
$$v_p(x)=\frac{1}{\sqrt{2E_p}}\begin{bmatrix}m+ip\\ -E_p\end{bmatrix}e^{-ipx} . \tag{8}$$

[Note the existence of an extra label (the parity) on the soliton eigenstates. Labeling them by momentum is actually a fraud: they are not momentum eigenstates. The states μ_p^+ and μ_{-p}^+, for example, are not independent. The "momentum" label on soliton eigenstates is understood not to take on negative values.]

The field operator can be expanded in terms of either $\{u_p, v_p, -\infty < p < \infty\}$ or $\{\chi_b, \mu_p^\pm, \nu_p^\pm, 0 \le p < \infty\}$, each of which is a complete, orthonormal set:

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} [b_p u_p(x) + d_p^\dagger v_p(x)]$$
$$= e\chi_b(x) + \int_0^\infty \frac{dp}{2\pi} \sum_{j=+,-} [a_p^j \mu_p^j(x) + c_p^{j\dagger} \nu_p^j(x)] \ . \tag{9}$$

Choosing the coefficient of χ_b to be e rather than $e^\dagger$ is purely conventional, analogous to the duality of viewing the Dirac sea as negative-energy electrons or positive-energy positrons.

From the fermion field anticommutation relations follow the soliton creation and destruction operator anticommutation relations, which will be needed below:

$$\{e^\dagger, e\} = 1 \ ,$$
$$\{a_p^{j\dagger}, a_q^l\} = 2\pi\delta(p-q)\delta_{jl} \ , \tag{10}$$
$$\{c_p^{j\dagger}, c_q^l\} = 2\pi\delta(p-q)\delta_{jl} \ .$$

All other anticommutators vanish.

One can relate the set $\{b_p, d_p^\dagger\}$ to the set $\{e, a_p^j, c_p^{j\dagger}\}$ using orthogonality of the eigenfunctions; for example,

$$b_k = \langle u_k \mid \chi_b \rangle e + \langle u_k \mid \mu_p^j \rangle a_p^j + \langle u_k \mid \nu_p^j \rangle c_p^{j\dagger} \ ,$$
$$d_k^\dagger = \langle v_k \mid \chi_b \rangle e + \langle v_k \mid \mu_p^j \rangle a_p^j + \langle v_k \mid \nu_p^j \rangle c_p^{j\dagger} \tag{11}$$

in bra-ket notation, with $\int_0^\infty (dp/2\pi) \sum_j$ implied. This is useful for writing combinations of one set of operators in terms of the other set. For example, the number operator is conventionally defined so that the state in which the ordinary Dirac sea is filled and the positive continuum is empty is an eigenstate with eigenvalue zero:

$$N = b_k^\dagger b_k - d_k^\dagger d_k \ . \tag{12}$$

(Again the momentum integral is implied.) In terms of the soliton operators, this is

$$N = (e^\dagger \langle \chi_b \mid u_k \rangle + a_p^{j\dagger} \langle \mu_p^j \mid u_k \rangle + c_p^j \langle \nu_p^j \mid u_k \rangle)$$
$$\times (e \langle u_k \mid \chi_b \rangle + a_q^l \langle u_k \mid \mu_q^l \rangle + c_q^{l\dagger} \langle u_k \mid \nu_q^l \rangle)$$
$$-(e \langle v_k \mid \chi_b \rangle + a_p^j \langle v_k \mid \mu_p^j \rangle + c_p^{j\dagger} \langle v_k \mid \nu_p^j \rangle)$$
$$\times (e^\dagger \langle \chi_b \mid v_k \rangle + a_q^{l\dagger} \langle \mu_q^l \mid v_k \rangle + c_q^l \langle \nu_q^l \mid v_k \rangle) \ . \tag{13}$$

This can be considerably simplified using the completeness and orthogonality relations and the anticommutation relations (8). The result is

$$N = e^\dagger e + a_p^{j\dagger} a_p^j - c_p^{j\dagger} c_p^j + \langle \nu_p^j \mid \nu_p^j \rangle - \langle v_k \mid v_k \rangle \ . \tag{14}$$

The last two terms represent the number of states in the Dirac sea in the presence of and absence of the soliton. While each of these integrals clearly diverges, the difference can be computed. This somewhat lengthy calculation may be found in the Appendix. One finds that the number of states in the Dirac sea decreases as α increases:

$$\langle \nu_p^j \mid \nu_p^j \rangle - \langle v_k \mid v_k \rangle = -\frac{\alpha}{\pi} = -\frac{\Delta\Theta}{2\pi} \ . \tag{15}$$

Thus,

$$N = e^\dagger e + a_p^{j\dagger} a_p^j - c_p^{j\dagger} c_p^j - \frac{\Delta\Theta}{2\pi} \ . \tag{16}$$

Imagine now that a soliton is formed infinitely slowly from the ground state. Referring to Fig. 1, a bound state emerges from the positive continuum and descends toward the negative continuum. Since the process is infinitely slow, the positive sea and the bound state remain empty; the negative sea remains fully occupied. Taking the expectation value of the number operator,

$$\langle N \rangle = -\frac{\Delta\Theta}{2\pi} \ . \tag{17}$$

When $0 < \alpha < \pi/2$, there is a bound state of positive energy. The state reached adiabatically is thus the ground state, and the adiabatic charge (17) is the ground state charge.

When $\Delta\Theta = \pi$, the bound state has zero energy. This is no accident; the theory has a charge conjugation symmetry since the mass is everywhere pure imaginary. [Specifically, $C\psi(x) = \sigma_1 \psi(x)^*$.] It is, in fact, a chirally rotated, infinitely thin version of the theory studied by Jackiw and Rebbi.[1] There are two degenerate ground states, with the bound state empty or full. The charge of the state with the bound state empty is $-\frac{1}{2}$ in agreement with Eq. (17).

Once $\Delta\Theta$ is beyond π, the bound state has negative energy. The state reached adiabatically is no longer the ground state, since a negative energy level is empty. The adiabatic state and the ground state therefore differ in charge by 1.

When $\Delta\Theta$ reaches 2π, the bound state joins the negative continuum, which then has one empty state. But at this point the fermion mass is everywhere $-m$; it can be made positive by a redefinition of ψ. The energy spectrum must then be identical to that when $\Delta\Theta = 0$. Since the negative-energy level which came from the bound state is empty, the fermion number is -1, in agreement with Eq. (17). Of course,the fermion does not even *see* the soliton, and the ground state has charge zero which differs from the adiabatic state's charge by 1, as stated above.

It is of interest to compute explicitly some of the Bogoliubov coefficients. For example, one can ask the following question: If the soliton were turned on suddenly rather than adiabatically, what is the probability with which the bound state χ_b would be filled? This quantity is given in terms of the Bogoliubov coefficients between the bound state and the states in the ordinary Dirac sea, v_k:

$$P(\alpha) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \mid \langle v_k \mid \chi_b \rangle \mid^2 \ . \tag{18}$$

With χ_b and v_k as given in Eqs. (5) and (8), the probability becomes, after evaluating an elementary integral,

$$P(\alpha) = \frac{1}{2} - \frac{\sin\alpha}{\pi \cos^2\alpha} \left[1 + 2\frac{\cos 2\alpha}{\sin 2\alpha} \left[\frac{\pi}{2} - \alpha \right] \right] \ . \tag{19}$$

When $\alpha = 0$, P is zero, indicating that the bound state, which has energy $+m$ and is in a sense the zero-momentum state in the positive continuum, has zero overlap with the Dirac sea. However, the probability must not be interpreted as an indication of how close the bound state energy is to the Dirac sea. For example, when $\alpha=\pi$, P is again zero, despite the fact that the bound state has reached the negative continuum. But the negative continuum when $\alpha=\pi$ is not the same as that when $\alpha=0$; indeed, the chiral transformation which changes the sign of the mass also exchanges positive- and negative-energy states.

Notice that $P(\alpha)$ has no very direct relation to N: the charge must be assigned to the whole sea, not to the bound state.

Consider next the more general case, where there is a bare mass $m_1\neq 0$. The Dirac equation is now

$$i\sigma_1\frac{\partial\psi}{\partial t}-\sigma_3\frac{\partial\psi}{\partial x}-[m_1+m_2\cos\varphi(x)+i\sigma_2 m_2\sin\varphi(x)]\psi=0\ . \tag{20}$$

The soliton is still infinitely thin: $\varphi(x)=\alpha x/|x|$. It is clear that this equation can be written in the previous form:

$$i\sigma_1\frac{\partial\psi}{\partial t}-\sigma_3\frac{\partial\psi}{\partial x}-m'[\cos\varphi'(x)+i\sigma_2\sin\varphi'(x)]\psi=0\ , \tag{21}$$

where $\varphi'(x)=\alpha' x/|x|$, if we assign to m' and α' the values

$$m'(\alpha)=(m_1{}^2+m_2{}^2+2m_1m_2\cos\alpha)^{1/2}\ , \tag{22}$$

$$\alpha'(\alpha)=\tan^{-1}\left[\frac{m_2\sin\alpha}{m_1+m_2\cos\alpha}\right]\ . \tag{23}$$

Because of this, much of the work done above can be used here by simply replacing α and m by α' and m'.

A qualitative difference between solitons with $m_1<m_2$ and those with $m_1>m_2$ is expected. If $m_1<m_2$, the soliton "winds around" the origin in the mass plane so that $\Delta\Theta=2\pi$ at $\alpha=\pi$, and hence from (2) the soliton charge is -1. If $m_1>m_2$, however, this is not the case. As α changes from zero to π, $\Delta\Theta$ increases from zero to a maximum (less than π) and then returns to zero; the charge of the soliton is zero.

The distinction can be illustrated neatly with the energy spectrum. Two features of the energy spectrum are of interest. First, the continua start at energies $\pm m'$ [see Eq. (22)], which changes with α. Second, there is a bound state at energy $E_b=m'\cos\alpha'=m_1+m_2\cos\alpha$. The energy spectrum as a function of α is displayed in Fig. 2, for $m_1<m_2$ and $m_1>m_2$.

For $m_1<m_2$ the buildup of the soliton proceeds in qualitatively the same manner as the $m_1=0$ case. As m_1 approaches m_2, the energy gap at $\alpha=\pi$ decreases until at $m_1=m_2$ they meet. The Dirac equation is that of a massless fermion. As m_1 becomes greater than m_2, the energy gap at $\alpha=\pi$ reappears.

The striking feature is that the bound state, instead of joining the negative continuum, rejoins the positive continuum. Keeping in mind the fact that α is changed infinitely slowly, the Dirac sea remains full throughout the entire process, rather than acquiring an empty state. The charge of the soliton at $\alpha=\pi$ is thus zero for $m_1>m_2$, in agreement with the adiabatic result.

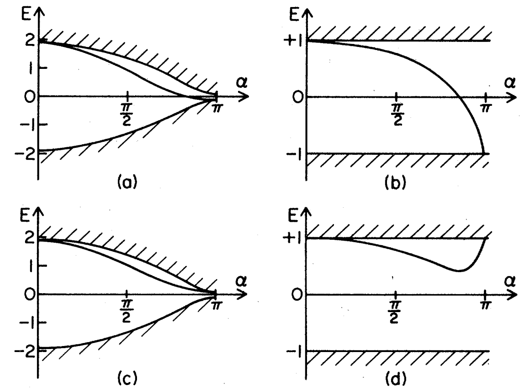

FIG. 2. Energy spectrum for $m_1=0.9$, $m_2=1.0$ [(a) and (b)], and $m_1=1.1$, $m_2=1.0$ [(c) and (d)]. (b) and (d) are normalized by dividing by $m'(\alpha)$ so that the energy gap appears to be independent of α.

Figure 2 thereby shows how a discontinuous change in the adiabatic charge as $m_1\gtrless m_2$ results from a continuous change in the motion of energy levels.

III. FINITE-WIDTH SOLITON

Consider next a scalar field which has two discontinuities rather than one:

$$\varphi(x)=\begin{cases}-\alpha & (x<-d),\\ 0 & (-d<x<d),\\ +\alpha & (x>d).\end{cases} \tag{24}$$

When $\alpha=0$ the fermion mass is everywhere m_1+m_2; as α increases the mass takes on space dependence, similar to above. The slight aesthetic advantage here is that the soliton with $\alpha=\pi$ has some structure, whereas the infinitely thin soliton with $\alpha=\pi$ is totally transparent to the fermions.

Similar to above, each of the three regions has plane-wave or exponential solutions depending on the energy; eigenstates of the Hamiltonian are found by imposing continuity of the wave function at $x=\pm d$. Here we will concentrate on looking for bound states. The form of a bound state is

$$\begin{aligned}\chi_b(x)&=A_1\begin{bmatrix}E+m_2s\\ m_1+m_2c+\kappa'\end{bmatrix}e^{\kappa'(x+d)}\quad(x<-d)\\ &=A_2\begin{bmatrix}E\\ m_1+m_2+\kappa\end{bmatrix}e^{\kappa x}\\ &\quad+A_3\begin{bmatrix}E\\ m_1+m_2-\kappa\end{bmatrix}e^{-\kappa x}\quad(-d<x<d)\\ &=A_4\begin{bmatrix}E-m_2s\\ m_1+m_2c-\kappa'\end{bmatrix}e^{-\kappa'(x-d)}\quad(x>d)\ ,\end{aligned} \tag{25}$$

where

$s=\sin\alpha,\quad c=\cos\alpha,\quad \kappa=[(m_1+m_2)^2-E^2]^{1/2}$,

and

$\kappa'=[(m_1+m_2c)^2+(m_2s)^2-E^2]^{1/2}$.

There are five unknowns: A_1, A_2, A_3, A_4, and the energy E. These are determined by the four continuity conditions and normalization. The result is a transcendental equation for the energy:

$$\tanh 2\kappa d=\frac{\kappa[\kappa' E-m_2 s(m_1+m_2c)]}{E^3-E[m_2{}^2(c+s^2)-m_1(m_1+m_2-m_2c)]\,\kappa' s(m_1+m_2)m_2}\ . \tag{26}$$

Clearly the infinitely thin result, $E_b=m_1+m_2\cos\alpha$, is recovered here if d is set to zero. Numerical solutions of the equation are presented in Fig. 3 for $m_1=0$, $m_1<m_2$, and $m_1>m_2$, for various thicknesses.

For $m_1<m_2$, as d increases the bound state no longer reaches the negative continuum when $\alpha=\pi$. A second bound state emerges from the positive continuum, resulting in a charge conjugation-symmetric energy spectrum at $\alpha=\pi$. As $d\to\infty$ the bound states get arbitrarily close together. In this case, as $\alpha=\pi$ the bound states are at infinitesimally positive and negative energies. The eigenfunctions are even and odd combinations of bound state eigenfunctions associated with each step. Except for the small energy differences, the ground state is fourfold degenerate.

For $m_1>m_2$, when d is nonzero the bound state returns to the positive continuum before $\alpha=\pi$. As d increases, the bound state's appearance gets more brief up until the point $d\to\infty$, where it does not appear at all.

IV. INFINITELY WIDE SOLITON

For the infinitely wide soliton we expect the adiabatic formula to apply directly as a good approximation for the charge density. We will now demonstrate this by calculating the charge density in the $m\bar{\psi}e^{i\varphi\gamma_5}\psi$ theory, using a special device which makes the calculation particularly transparent.

The interaction with the background field is simplified by redefining the fermion wave functions, according to

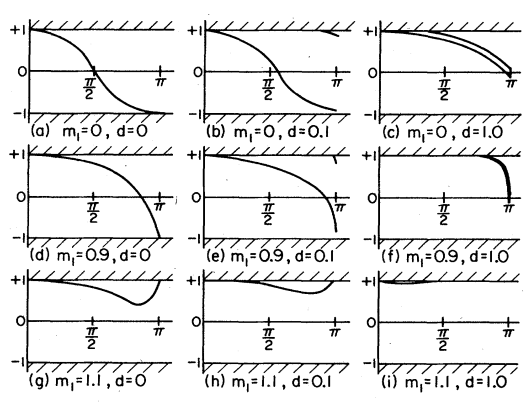

FIG. 3. Energy spectrum of finite-width solitons, normalized as in Fig. 2.

$$\tilde{\psi}(x)=e^{-i\varphi(x)\gamma_5/2}\psi(x)\ . \tag{27}$$

In terms of these functions the Dirac equation reads

$$(i\gamma^1\partial_1-m+\tfrac{1}{2}\gamma^1\gamma_5 m\partial_1\varphi)\tilde{u}=\gamma^0E\tilde{u}\ .$$

Now notice that the interaction term on the left-hand side can be brought over to the right-hand side—it represents a sort of position-dependent energy. Insofar as $\partial_1\varphi$ is slowly varying ($\partial_1\varphi\ll m$) we can simply reabsorb it as an effective energy. To calculate the Feynman graph giving the current density we simply replace the energy in the propagator by the effective energy, arriving at

$$\langle j_0\rangle=\int\frac{d^2k}{(2\pi)^2}\left[\frac{2(k_0-\Delta)}{(k_0-\Delta)^2+k_1{}^2+m^2}-\frac{2k_0}{k_0{}^2+k_1{}^2+m^2}\right], \tag{28}$$

where $\Delta=(m/2)\partial_1\varphi$ and the normal vacuum contribution has been subtracted off. Evaluating this to first order in Δ, we find

$$\langle j_0\rangle=-\partial_1\varphi/2\pi \tag{29}$$

as expected. By the same method, examining the energy-momentum tensor, we may evaluate the energy required to polarize the fermion sea, which is proportional to $(\partial_1\varphi)^2$.

We can check how the approximation involved in regarding $\partial_1\varphi$ as slowly varying is controlled, by imagining calculating the current density directly by summation of Feynman graphs. In such a calculation of $\langle j_0(x)\rangle$ interaction with the external field at y, involving $\partial_1\varphi(y)$, will be suppressed by position-space propagators behaving like $e^{-m|x-y|}$. Therefore, only $|x-y|\le 1/m$ need be considered.

Similar techniques have been used to gain an intuitive understanding of the response of the Fermi sea to a slowly varying background in condensed matter physics.[5]

V. CONCLUSIONS

Detailed analysis of simple examples has confirmed our main general points: the fractional part of the charge is given by the adiabatic method in general, as is the total charge for very wide solitons (on the scale of the Compton wavelength of the fermion). Narrow solitons require careful study of level crossings in the steepening step, and indeed may become transparent if they are narrow enough; thus, whatever charge was attached to the wide soliton must be liberated in the steepening process. Possible applications of our results to the strong and weak interactions are discussed in a companion paper.[6]

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APPENDIX

Here we compute the change in the number of states in the Dirac sea as an infinitely thin soliton is built up. The negative-energy solutions of the ordinary Dirac equation are

$$v_p(x)=\frac{1}{\sqrt{2}E_p}\begin{bmatrix} m+ip \\ -E_p \end{bmatrix} e^{-ipx}, \quad -\infty<p<\infty$$

$$\equiv V_p e^{-ipx} . \tag{A1}$$

The negative-energy solutions in the presence of the soliton given by $m_1=0$, $m_2\equiv m$, $\varphi(x)=\alpha x/|x|$ are

$$\begin{aligned} v_p^{\dagger}(x)=N_p'\Bigg\{ &\left[(-ipE_p+m^2sc)\begin{bmatrix} -E_p+ms \\ mc+ip \end{bmatrix} e^{ipx}-(mc+ip)(ms-ip)\begin{bmatrix} -E_p+ms \\ mc-ip \end{bmatrix} e^{-ipx}\right]\Theta(-x) \\ &+\left[-(mc+ip)(ms-ip)\begin{bmatrix} mc-ip \\ -E_p+ms \end{bmatrix} e^{ipx}+(-ipE_p+m^2sc)\begin{bmatrix} mc+ip \\ -E_p+ms \end{bmatrix} e^{-ipx}\right]\Theta(x)\Bigg\} \\ \equiv{}& (W_p^{(+)}e^{ipx}+Y_p^{(+)}e^{-ipx})\Theta(-x)+(\sigma_1 Y_p^{(+)}e^{ipx}+\sigma_1 W_p^{(+)}e^{-ipx})\Theta(x) \end{aligned} \tag{A2}$$

and

$$\begin{aligned} v_p^{-}(x)=N_p'\Bigg\{ &\left[(-ipE_p+m^2sc)\begin{bmatrix} -E_p+ms \\ mc+ip \end{bmatrix} e^{ipx}-(ms+ip)(mc+ip)\begin{bmatrix} -E_p+ms \\ mc-ip \end{bmatrix} e^{-ipx}\right]\Theta(-x) \\ &+\left[(ms+ip)(mc+ip)\begin{bmatrix} mc-ip \\ -E_p+ms \end{bmatrix} e^{ipx}-(-ipE_p+m^2sc)\begin{bmatrix} mc+ip \\ -E_p+ms \end{bmatrix} e^{-ipx}\right]\Theta(x)\Bigg\} \\ \equiv{}& (W_p^{(-)}e^{ipx}+Y_p^{(-)}e^{-ipx})\Theta(-x)-(\sigma_1 Y_p^{(-)}e^{ipx}+\sigma_1 W_p^{(-)}e^{-ipx})\Theta(x) . \end{aligned} \tag{A3}$$

Here

$$N_p'=[4E_p(E_p-ms)(m^2c^2+p^2)(m^2s^2+p^2)]^{-1/2}, \quad s=\sin\alpha, \quad c=\cos\alpha ,$$

and the momentum takes on only positive values. The two-component objects V_p, $W_p^{(\pm)}$, $Y_p^{(\pm)}$ are introduced to reduce the bulk of the calculation below. $W_p^{(\pm)}$ and $Y_p^{(\pm)}$ will be needed below; they are

$$\begin{aligned} W_p^{(\pm)}&=\frac{-ipE_p+m^2sc}{[4E_p(E_p-ms)(m^2c^2+p^2)(m^2s^2+p^2)]^{1/2}}\begin{bmatrix} -E_p+ms \\ mc+ip \end{bmatrix} , \\ Y_p^{(\pm)}&=-\frac{(mc+ip)(ms\mp ip)}{[4E_p(E_p-ms)(m^2c^2+p^2)(m^2s^2+p^2)]^{1/2}}\begin{bmatrix} -E_p+ms \\ mc-ip \end{bmatrix} . \end{aligned} \tag{A4}$$

The change in the number of states in the Dirac sea, i.e., the charge of the adiabatically-reached state, is

$$Q=\langle v_p^i \mid v_p^i \rangle-\langle v_p \mid v_p \rangle .$$

Showing the integrals and sums explicitly, this is

$$\begin{aligned} Q=\int_0^{\infty}\frac{dp}{2\pi}\Bigg[&\int_{-\infty}^{0} dx\,(W_p^{(+)\dagger}e^{-ipx}+Y_p^{(+)\dagger}e^{ipx})(W_p^{(+)}e^{ipx}+Y_p^{(+)}e^{-ipx}) \\ &+\int_0^{\infty} dx\,(Y_p^{(+)\dagger}\sigma_1 e^{-ipx}+W_p^{(+)\dagger}\sigma_1 e^{ipx})(\sigma_1 Y_p^{(+)}e^{ipx}+\sigma_1 W_p^{(+)}e^{-ipx}) \\ &+\int_{-\infty}^{0} dx\,(W_p^{(-)\dagger}e^{-ipx}+Y_p^{(-)\dagger}e^{ipx})(W_p^{(-)}e^{ipx}+Y_p^{(-)}e^{-ipx}) \\ &+\int_0^{\infty} dx\,(Y_p^{(-)\dagger}\sigma_1 e^{-ipx}+W_p^{(-)\dagger}\sigma_1 e^{ipx})(\sigma_1 Y_p^{(-)}e^{ipx}+\sigma_1 W_p^{(-)}e^{-ipx})\Bigg] \\ &-\int_{-\infty}^{\infty}\frac{dp}{2\pi}\int_{-\infty}^{\infty} dx\,(V_p^{\dagger}e^{+ipx})(V_pe^{-ipx}) . \end{aligned} \tag{A5}$$

Changing all the integrals to $\int_0^\infty dp \int_0^\infty dx$ and collecting like terms,

$$Q=2\int_0^\infty \frac{dp}{2\pi}\int_0^\infty dx[W_p^{(+)\dagger}W_p^{(+)}+Y_p^{(+)\dagger}Y_p^{(+)}+W_p^{(-)\dagger}W_p^{(-)}+Y_p^{(-)\dagger}Y_p^{(-)}-V_p^\dagger V_p-V_{-p}^\dagger V_{-p}]$$
$$+2\int_0^\infty \frac{dp}{2\pi}\int_0^\infty dx[(W_p^{(+)\dagger}Y_p^{(+)}+W_p^{(-)\dagger}Y_p^{(-)})e^{2ipx}+(Y_p^{(+)\dagger}W_p^{(+)}+Y_p^{(-)\dagger}W_p^{(-)})e^{-2ipx}]\ . \tag{A6}$$

The first term in brackets adds up to zero, so the charge is

$$Q=\int_0^\infty \frac{dp}{2\pi}\int_0^\infty dx[(W_p^{(+)\dagger}Y_p^{(+)}+W_p^{(-)\dagger}Y_p^{(-)})e^{ipx}+(Y_p^{(+)\dagger}W_p^{(+)}+Y_p^{(-)\dagger}W_p^{(-)})e^{-ipx}]\ . \tag{A7}$$

The x integrals are given by $\int_0^\infty dxe^{ipx}=i/p+\pi\delta(p)$. Q becomes

$$Q=\frac{\pi}{2}[W_0^{(+)\dagger}Y_0(+)+W_0^{(-)\dagger}Y_0^{(-)}+Y_0^{(+)\dagger}W_0^{(+)}+Y_0^{(-)\dagger}W_0^{(-)}]$$
$$+i\int_0^\infty \frac{dp}{2\pi p}(W_p^{(+)\dagger}Y_p^{(+)}+W_p^{(-)\dagger}Y_p^{(-)}-Y_p^{(+)\dagger}W_p^{(+)}-Y_p^{(-)\dagger}W_p^{(-)})\ . \tag{A8}$$

Using (A4) this becomes

$$Q=-\frac{1}{2}+\frac{m^2sc}{2\pi^2}\int_0^\infty \frac{dp}{(p^2+m^2s^2)(p^2+m^2)^{1/2}}\ . \tag{A9}$$

The integral can be evaluated by the substitution $(p^2+m^2)^{1/2}=t+p$; the result is

$$Q=-\frac{1}{2}+\frac{m^2sc}{\pi}\frac{1}{m^2sc}\left[\frac{\pi}{2}-\alpha\right]=\frac{-\alpha}{\pi} \tag{A10}$$

or

$$Q=-\frac{\Delta\Theta}{2\pi}\ . \tag{A11}$$

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