

Relativistic corrections to bound-state energies for two-fermion systems

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Holomorphy of the Breit-Fermi resolvent around the nonrelativistic limit is used to derive a new method for calculating first-order relativistic corrections to bound-state energies.

I. INTRODUCTION

The problem we are discussing is a rather old one and simple to state. Start with the Dirac operator with static external fields. Under mild assumptions on the potentials the spectrum will consist of continuous parts with a gap around energy zero and possibly bound states in the gap. The problem now is how to get information on the bound-state energies. In the absence of a variational principle, the most well-known procedure consists in formally expanding the Dirac operator for inverse powers of the velocity of light c . As a first step this Foldy-Wouthuysen procedure¹ gives highly singular operators. In a second step one applies first-order perturbation theory which, however, is completely unjustified. Although we have been able to give an interpretation to the second step in terms of spectral concentration,^{2,3} one should try to avoid the above-mentioned procedure completely. For the one-particle case we succeeded to prove holomorphy of the Dirac resolvent in c^{-1} . From the explicit expansion of the resolvent we then extracted corrections to bound-state energies of the Pauli operator.^{4,5}

In this paper we shall apply our method to the two-particle case. To do so, we have to have a Hamilton operator. In principle, one should start from a relativistic wave equation (Bethe-Salpeter equation or some other bound-state equations) and try to obtain a Hamiltonian by an instantaneous limit. Concerning this (certainly nonunique) step, we have nothing to say (except for some comments at the end of the paper). Here we assume that these steps can be done. For QED this yields the so-called Breit-Fermi Hamiltonian.⁶⁻⁸ At this point we should add a warning: In case the potential is determined up to a certain order in a coupling constant (or in $1/c^2$) it does not make sense to continue our procedure to higher orders in these parameters. We also remark that our procedure works for local as well as nonlocal interactions.

In order to clarify the above remarks, let us discuss the Hamiltonian for the positronium atom. Let $\tilde{H}(c) = H_0(c) + V$ with

$$H_0(c) = (c \vec{\alpha}_1 \cdot \vec{p} + \beta_1 m_1 c^2) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes (-c \vec{\alpha}_2 \cdot \vec{p} + \beta_2 m_2 c^2). \quad (1.1)$$

$H_0(c)$ represents the free Hamiltonian of two Dirac particles in the center-of-mass frame and acts in

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \otimes \mathbb{C}^4.$$

$\vec{\alpha}_1$ and β_1 ($\vec{\alpha}_2$ and β_2) are Dirac matrices for particle one (two). $\mathbb{1}_1$ and $\mathbb{1}_2$ are 4×4 unit matrices.

The main question concerns the potential V . Take, for instance,

$$V(\vec{x}) = v(r) \mathbb{1}_1 \otimes \mathbb{1}_2 + w_1(r) \vec{\alpha}_1 \otimes \vec{\alpha}_2 + w_2(r) (\vec{\alpha}_1 \cdot \hat{x}) \otimes (\vec{\alpha}_2 \cdot \hat{x}), \quad (1.2)$$

where \vec{x} denotes the relative coordinate, $r = |\vec{x}|$ and $\hat{x} = \vec{x}/r$; then for the special case

$$v(r) = -2w_1(r) = -2w_2(r) = -\frac{e^2}{r}, \quad (1.3)$$

Eqs. (1.1) to (1.3) represent the Breit-Fermi operator and the question of a nonrelativistic expansion of bound-state energies can be posed.

We shall also be concerned with more general interactions than (1.2) and (1.3), which are of interest for quarkonium spectroscopy. The general ansatz we take for V is of the form

$$V = \begin{pmatrix} v_+ & 0 & 0 & w_+ \\ 0 & v_- & w_- & 0 \\ 0 & w_- & v_- & 0 \\ w_+ & 0 & 0 & v_+ \end{pmatrix}, \quad (1.4)$$

where we have used a 16×16 matrix notation and each entry of the matrix in (1.4) represents a 4×4 matrix. It can be shown that special cases of (1.4) will account for general interactions coming from scalar, vector, pseudo-scalar, axial-vector, and tensor interactions, although we shall concentrate only on the two first mentioned.

There is a long list of publications concerned with relativistic corrections to bound-state energies and decay parameters (cf. e.g., Refs. 9–12 and the references therein). It is worth mentioning that for a scalar interaction there exist three different proposals for the correction terms.^{13–15} We shall see that our procedure gives a result in agreement with Ref. 15. For a determination of the structure of spin-dependent forces in QCD see Refs. 16–18.

We shall be dealing only with two-body systems; there are additional problems for the many-body case.¹⁹ In Sec. II we shall explain our method for interactions like (1.4). Section III is devoted to a discussion of special interactions and to a comparison with results in the literature.

II. POSITRONIUM

We use the representation

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.1)$$

then H_0 can be written in a 16×16 matrix form:

$$\begin{aligned} h_0(c) &= H_0(c) - m_1 c^2 - m_2 c^2 \\ &= \begin{pmatrix} 0 & cA_1 \\ cA_1 & -2m_1 c^2 \end{pmatrix} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \begin{pmatrix} 0 & cA_2 \\ cA_2 & -2m_2 c^2 \end{pmatrix} \end{aligned} \quad (2.2)$$

where we introduced the notation $A_1 = \vec{\sigma}_1 \cdot \vec{p}$, $A_2 = -\vec{\sigma}_2 \cdot \vec{p}$ and subtracted rest energies from $H_0(c)$. Next define the free resolvent $R_z^0(c)$:

$$R_z^0(c) = [h_0(c) - z]^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.3)$$

Since all entries in $h_0(c)$ commute with each other $h_0(c) - z$ can be inverted in a straightforward calculation. For notational simplicity we state the result in the special case $m_1 = m_2 = \frac{1}{2}$:

$$R_z^0(c) = \left[2p^2 - z - \frac{z^2}{2c^2} \right]^{-1} \left[P_+^1 \otimes P_+^2 + \frac{1}{c^2 + z} M(c) \right], \quad (2.4)$$

where we introduced the projection operators $P_+^{1,2} = (1 + \beta^{1,2})/2$, and the symmetric matrix $M(c)$ in a 4×4 notation is given by

$$\begin{aligned} M_{11} &= \frac{z}{2} + \frac{z^2}{2c^2} - p^2, & M_{22} &= z + \frac{z^2}{2c^2} - p^2 = M_{33}, \\ M_{12} &= A_1 \left[c + \frac{z}{2c} \right], & M_{24} &= A_2 \frac{z}{2c}, \\ M_{13} &= A_2 \left[c + \frac{z}{2c} \right], & M_{34} &= A_1 \frac{z}{2c}, \\ M_{14} &= A_1 A_2 = M_{23}, & M_{44} &= \frac{z}{2} + \frac{z^2}{2c^2} - p^2. \end{aligned} \quad (2.5)$$

In particular in the nonrelativistic limit $c \rightarrow \infty$, $R_z^0(c)$ converges to the free Pauli resolvent $(p^2/2\mu - z)^{-1}$ (where $\mu^{-1} = m_1^{-1} + m_2^{-1}$ is the reduced mass) times a projection operator:

$$\lim_{c \rightarrow \infty} R_z^0(c) = \left[\frac{p^2}{2\mu} - z \right]^{-1} P_+^1 \otimes P_+^2. \quad (2.6)$$

Equations (2.4) and (2.5) clearly demonstrate that $R_z^0(c)$ is holomorphic in c^{-1} around the nonrelativistic limit $c^{-1} = 0$.

But one notices a difference between the one-particle and the two-particle case. For the one-particle case even after multiplying the free resolvent with potentials which are relatively bounded with respect to $\vec{\alpha} \cdot \vec{p}$ norm convergence was obtained.^{4,5} Since this is not the case here we shall assume that V is a bounded operator in order to avoid problems of self-adjointness. Thus, we assume that v_\pm and w_\pm are essentially bounded from now on.

The main idea of our approach consists in discussing the resolvent of $H(c) = h_0(c) + V$ (which is a bounded operator),

$$R_z(c) = [H(c) - z]^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (2.7)$$

instead of dealing with the unbounded Hamiltonian $H(c)$. Next we shall extract in a two-step procedure parts of $H(c)$ which dominate in the limit $c \rightarrow \infty$ and write $H(c) - z = D(c) + E(c)$ with

$$\begin{aligned} D(c) &= \begin{pmatrix} -z & cA_1 & cA_2 & 0 \\ cA_1 & -2m_1 c^2 & 0 & 0 \\ cA_2 & 0 & -2m_2 c^2 & 0 \\ 0 & 0 & 0 & -2(m_1 + m_2)c^2 \end{pmatrix}, \\ E(c) &= \begin{pmatrix} v_+ & 0 & 0 & w_+ \\ 0 & v_- - z & w_- & cA_2 \\ 0 & w_- & v_- - z & cA_1 \\ w_+ & cA_2 & cA_1 & v_+ - z \end{pmatrix}. \end{aligned} \quad (2.8)$$

Since all elements of $D(c)$ commute it is trivial to invert $D(c)$. Making use of the identity

$$(D + E)^{-1} = (1 + D^{-1}E)^{-1} D^{-1} \quad (2.9)$$

allows, after multiplying out $D^{-1}(c)$ and $E(c)$, to conclude that in the limit $c \rightarrow \infty$, $R_z(c)$ tends to the (interacting) Pauli resolvent in norm:

$$\lim_{c \rightarrow \infty} R_z(c) = \left[\frac{p^2}{2\mu} + v_+ - z \right]^{-1} P_+^1 \otimes P_+^2, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.10)$$

Moreover, following the derivation of Eq. (2.15) in Ref. 5, the explicit form of $D^{-1}(c)$ finally yields our first main result.

Let $H(c) = h_0(c) + V$, V bounded, and $z \in \mathbb{C} \setminus \mathbb{R}$. Then $R(z) = [H(c) - z]^{-1}$ is holomorphic in c^{-1} around $c^{-1} = 0$. At first sight it might be surprising that the resolvent is analytic as a function of c^{-1} , while the folklore results for expansions of eigenvalues yield a series in c^{-2} . But as in the one-particle case one can go over to a modified resolvent $\tilde{R}_z(c)$,

$$\tilde{R}_z(c) = B(c)R_z(c)B^{-1}(c), \tag{2.11}$$

$$B(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix},$$

and with the help of the similarity transformation $B(c)$ one observes that $\tilde{R}_z(c)$ becomes holomorphic as a function of c^{-2} around $c^{-2} = 0$.

More explicitly we first note that $B(D^{-1}E)B^{-1}$ converges in norm to

$$d = \lim_{c \rightarrow \infty} B(c)D^{-1}(c)E(c)B^{-1}(c)$$

$$= \begin{pmatrix} \rho_z^0 v_+ & 0 & 0 & 0 \\ \rho_z^0 \frac{A_1}{2m_1} v_+ & 0 & 0 & 0 \\ \rho_z^0 \frac{A_2}{2m_2} v_+ & 0 & 0 & 0 \\ -\frac{w_+}{2(m_1+m_2)} & -\frac{A_2}{2(m_1+m_2)} & -\frac{A_1}{2(m_1+m_2)} & 0 \end{pmatrix},$$

$$\rho_z^0 = \left[\frac{p^2}{2\mu} - z \right]^{-1}. \tag{2.12}$$

In order to get a norm-convergent expansion of $\tilde{R}_z(c)$ in terms of c^{-2} , we apply formulas (2.9) once again and take out $1+d$ from $B(1+D^{-1}E)B^{-1}$:

$$\begin{aligned} \tilde{R}_z &= [1+d+e(c)]^{-1} \tilde{D}^{-1}(c) \\ &= [1+(1+d)^{-1}e(c)]^{-1} (1+d)^{-1} \tilde{D}^{-1}(c), \\ \tilde{D}(c) &= B(c)D(c)B^{-1}(c), \end{aligned} \tag{2.13}$$

where $e(c)$ denotes the corresponding remaining part. Explicit multiplication of the matrices in (2.13) then shows after some calculations that

$$\lim_{c \rightarrow \infty} \tilde{R}_z(c) = \begin{pmatrix} \rho_z & 0 & 0 & 0 \\ \frac{A_1}{2m_1} \rho_z & 0 & 0 & 0 \\ \frac{A_2}{2m_2} \rho_z & 0 & 0 & 0 \\ W \cdot \rho_z & 0 & 0 & 0 \end{pmatrix} = \tilde{R}_0(z),$$

$$W = \frac{w_+}{2(m_1+m_2)} + \frac{A_1 A_2}{4m_1 m_2}, \tag{2.14}$$

$$\rho_z = \left[\frac{p^2}{2\mu} + v_+ - z \right]^{-1},$$

and that $\tilde{R}_z(c)$ is indeed holomorphic with respect to c^{-2} around the nonrelativistic limit

$$\tilde{R}_z(c) = \tilde{R}_0(c) + c^{-2} \tilde{R}_1(z) + O(c^{-4}). \tag{2.15}$$

Since $\tilde{R}_1(z)$ has a quite complicated structure we only

quote those matrix elements that are needed for later purposes:

$$\begin{aligned} [\tilde{R}_1(z)]_{11} &= -\rho_z \left[\frac{A_1}{2m_1} (v_- - z) + \frac{A_2}{2m_2} w_- \right] \frac{A_1}{2m_1} \rho_z \\ &\quad - \rho_z \left[\frac{A_1}{2m_1} w_- + \frac{A_2}{2m_2} (v_- - z) \right] \frac{A_2}{2m_2} \rho_z \\ &\quad - \rho_z W^2 \rho_z, \end{aligned} \tag{2.16}$$

$$\begin{aligned} [\tilde{R}_1(z)]_{12} &= \rho_z A_1 / 2m_1, \quad [\tilde{R}_1(z)]_{13} = \rho_z A_2 / 2m_2, \\ [\tilde{R}_1(z)]_{14} &= 0. \end{aligned}$$

In order to derive our second main results, namely, an explicit formula for first-order relativistic corrections of bound-state energies, we follow the strategy developed in Refs. 4 and 5. From the fact that

$$[\tilde{R}_0(z) - \zeta]^{-1} = \begin{pmatrix} (\rho_z - \zeta)^{-1} & 0 & 0 & 0 \\ \zeta^{-1} \frac{A_1}{2m_1} \rho_z (\rho_z - \zeta)^{-1} & \zeta^{-1} & 0 & 0 \\ \zeta^{-1} \frac{A_2}{2m_2} \rho_z (\rho_z - \zeta)^{-1} & 0 & \zeta^{-1} & 0 \\ \zeta^{-1} \cdot W \cdot \rho_z (\rho_z - \zeta)^{-1} & 0 & 0 & \zeta^{-1} \end{pmatrix},$$

$$\zeta \neq 0, \tag{2.17}$$

we infer as in Lemma 2.2 of Ref. 5 that there is a one-to-one correspondence between nonzero eigenvalues of ρ_z and $\tilde{R}_0(z)$. In particular,

$$\left[\frac{p^2}{2\mu} + v_+ \right] f_0 = E_0 f_0, \text{ i.e., } \rho_z f_0 = (E_0 - z)^{-1} f_0,$$

$$E_0 \in \mathbb{R}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.18}$$

implies

$$[\tilde{R}_0(z) - (E_0 - z)^{-1}] \Phi_0 = 0, \quad \Phi_0 = \begin{pmatrix} f_0 \\ A_1 f_0 \\ \frac{A_2}{2m_2} f_0 \\ W f_0 \end{pmatrix}. \tag{2.19}$$

Since $\tilde{R}_z(c)$ is holomorphic with respect to c^{-2} around the nonrelativistic limit, analytic perturbation theory implies that eigenvalues and eigenfunctions of $\tilde{R}_z(c)$ and $H(c)$ [by the similarity transformation $B(c)$] are also analytic with respect to c^{-2} . Assuming for simplicity that E_0 is a nondegenerate discrete eigenvalue of the Pauli Hamiltonian $p^2/2\mu + v_+$ and that $E(c^{-2})$ is the corresponding relativistic eigenvalue of $H(c)$ [with $E(0) = E_0$] then $E(c^{-2})$ is also nondegenerate and we can expand

$$E(c^{-2}) = E_0 + c^{-2} E_1 + O(c^{-4}). \tag{2.20}$$

To calculate E_1 explicitly we apply first-order perturbation theory to Eq. (2.15). A standard procedure for

bounded (but nonself-adjoint) operators then leads to

$$\begin{aligned} [E(c^{-2})-z]^{-1} &= (E_0-z)^{-1} - c^{-2}(E_0-z)^{-2}E_1 + O(c^{-4}) \\ &= (E_0-z)^{-1} - c^{-2}(\psi_0, \tilde{R}_1(z)\Phi_0) + O(c^{-4}), \end{aligned} \quad (2.21)$$

where Φ_0 represents the eigenvector of $\tilde{R}_0(z)$ to the eigenvalue $(E_0-z)^{-1}$ [cf., e.g., (2.19)] and

$$\psi_0 = \begin{pmatrix} f_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.22)$$

is the eigenvector of $\tilde{R}_0(z)^+$ [to the eigenvalue $(E_0-\bar{z})^{-1}$]. Inserting Φ_0 and ψ_0 into Eq. (2.21) and using expressions (2.16) for $\tilde{R}_1(z)$ finally leads to

$$\begin{aligned} E_1 &= \left(f_0, \left[\frac{A_1}{2m_1}v - \frac{A_1}{2m_1} + \frac{A_2}{2m_2}v - \frac{A_2}{2m_2} + \frac{A_1}{2m_1}w - \frac{A_2}{2m_2} + \frac{A_2}{2m_2}w - \frac{A_1}{2m_1} \right. \right. \\ &\quad \left. \left. - E_0 \frac{p^2}{4} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) + 2(m_1+m_2)W^2 \right] f_0 \right), \\ W &= \frac{w_+}{2(m_1+m_2)} + \frac{A_1A_2}{4m_1m_2}. \end{aligned} \quad (2.23)$$

Remarks

(i) In the limit $m_2 \rightarrow \infty$ we obtain

$$E_1 = \left(f_0, \left[\frac{A_1}{2m_1}v - \frac{A_1}{2m_1} - E_0 \frac{p^2}{4m_1^2} \right] f_0 \right), \quad (2.24)$$

which is identical to our one-particle result.^{4,5}

(ii) In practice one has to discard the term

$$(f_0, w_+^2 f_0) / 2(m_1+m_2),$$

if the potential is determined up to first order in the coupling constant.

(iii) If E_0 is a degenerate eigenvalue of the Pauli Hamiltonian similar (but more involved) arguments apply.⁵

III. DISCUSSION

In order to compare with results obtained by completely different methods by various authors we first rewrite Eq. (2.23). Assuming v_{\pm} , w_{\pm} to be smooth enough and using the eigenvalue equation

$$\left[\frac{p^2}{2\mu} + v_+ \right] f_0 = E_0 f_0, \quad (3.1)$$

we can eliminate E_0 in Eq. (2.23) and obtain

$$\begin{aligned} E_1 &= - \left(f_0, \frac{p^4}{8} f_0 \right) \left[\frac{1}{m_1^3} + \frac{1}{m_2^3} \right] + \epsilon_1 \\ &\quad + (\text{second-order term in } w_+) \\ \epsilon_1 &= \left(f_0, \left[\frac{1}{8m_1^2} (2A_1v - A_1 - v_+p^2 - p^2v_+) \right. \right. \\ &\quad + \frac{1}{8m_2^2} (2A_2v - A_2 - v_+p^2 - p^2v_+) \\ &\quad + \frac{1}{4m_1m_2} (A_1w - A_2 + A_2w - A_1 + A_1A_2w_+ \\ &\quad \left. \left. + w_+ A_1A_2) \right] f_0 \right). \end{aligned} \quad (3.2)$$

For a comparison to results in the literature it remains to determine the potential; this is trivial for scalar (and pseudoscalar) interactions and needs more comments for the vector case.

A. Scalar interaction

There we have

$$V(r) = S(r)\beta_1 \otimes \beta_2 = S(r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.3)$$

So we identify $v_+ = -v_- = S$, $w_+ = w_- = 0$. After some straightforward algebra (assuming S is smooth enough) one gets

$$\begin{aligned} \epsilon_1 &= \frac{1}{8m_1^2} \left(f_0, [(\Delta S) + 4(\nabla S)\nabla + 4S\Delta \right. \\ &\quad \left. - \frac{2}{r} \frac{\partial S}{\partial r} (\vec{\sigma}_1 \cdot \vec{x} \times \vec{p})] f_0 \right) + (1 \leftrightarrow 2) \end{aligned} \quad (3.4)$$

which is the same result as that of Barnes and Ghau-dour;¹⁵ it disagrees with Gromes¹³ and with Olsson and Miller.¹⁴

B. Vector interaction

Here the identification is a bit more delicate, since there are different prescriptions for taking the instantaneous limit. It might therefore be of interest to quote the general result for the potential of Eq. (1.2), which means identifying

$$\begin{aligned} v_+ &= v_- = v, \\ w_+ &= w_- = w_1 \vec{\sigma}_1 \cdot \vec{\sigma}_2 + w_2 (\vec{\sigma}_1 \cdot \hat{x})(\vec{\sigma}_2 \cdot \hat{x}). \end{aligned} \quad (3.5)$$

We readily obtain from (3.2) the spin-independent ϵ_1^{SI} , the spin-orbit ϵ_1^{SO} , and the spin-spin ϵ_1^{SS} contributions to $\epsilon_1 = \epsilon_1^{\text{SI}} + \epsilon_1^{\text{SO}} + \epsilon_1^{\text{SS}}$.

$$\begin{aligned}\epsilon_1^{\text{SI}} &= \frac{1}{8m_1^2}(f_0, \Delta v f_0) - \frac{1}{8m_1 m_2}(f_0, [\{ \{ p_i w_1 p_i \} \} + \{ \{ p_i (\hat{x}_i w_2 \hat{x}_j) p_j \} \}] f_0) + (1 \leftrightarrow 2), \\ \epsilon_1^{\text{SO}} &= \left(f_0, \left[\frac{1}{4m_1^2} \frac{v'}{r} + \frac{1}{2m_1 m_2} \left[\frac{w_2}{r^2} - \frac{w_1'}{r} \right] \right] (\vec{\sigma}_1 \cdot \vec{x} \times \vec{p}) f_0 \right) + (1 \leftrightarrow 2), \\ \epsilon_1^{\text{SS}} &= \frac{1}{4m_1 m_2} \left(f_0, \left[(\vec{\sigma}_1 \cdot \vec{\sigma}_2) \left[-w_1'' - \frac{w_1'}{r} + \frac{w_2'}{r} \right] + (\vec{\sigma}_1 \cdot \hat{x})(\vec{\sigma}_2 \cdot \hat{x}) \left[-\frac{w_1'}{r} + w_1'' + 2\frac{w_2}{r^2} - \frac{w_2'}{r} \right] \right] f_0 \right),\end{aligned}\quad (3.6)$$

where we used the bracket notation of Ref. 13:

$$\{ \{ p_i F_{ij} p_j \} \} = p_i F_{ij} p_j + p_j F_{ij} p_i + p_i p_j F_{ij} + F_{ij} p_i p_j, \quad (3.7)$$

and summation over repeated indices is implied.

The most direct (and safest) approach concerning the connection to a Bethe-Salpeter kernel consists now in taking

$$V(r) = u(r)(\mathbb{1}_1 \otimes \mathbb{1}_2 - \vec{\alpha}_1 \otimes \vec{\alpha}_2) \quad (3.8)$$

which means one neglects the q_0 dependence of $\tilde{u}(-q_0^2 + \vec{q}^2)$ in the kernel. Then one identifies

$$v = u, \quad w_1 = -u, \quad w_2 = 0; \quad (3.9)$$

this yields agreement of ϵ_1^{SO} and ϵ_1^{SS} with Refs. 13 and 20; ϵ_1^{SI} differs.

If an expansion of $\tilde{u}(-q_0^2 + \vec{q}^2) \simeq \tilde{u}(\vec{q}^2) - q_0^2 \tilde{u}'(\vec{q}^2)$ is allowed one may use current conservation to transform the second term and obtain

$$v = u, \quad w_1 = -\frac{u}{2}, \quad w_2 = \frac{r}{2} u'. \quad (3.10)$$

Now all three contributions in (3.6) agree with Ref. 13.

It is interesting to note that both the spin-orbit and the spin-spin interaction terms are insensitive to the different limit descriptions. This is so because both expressions depend only on the sum $-w_1' + w_2/r$, which is equal to u' in both cases.

So let us finally conclude that we have given an unambiguous scheme to extract relativistic corrections to bound-state energies of two-particle systems if a time-independent potential is given; the step from a relativistic wave equation to the Hamiltonian description depends on the procedure one chooses.

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