# Charge of the vacuum in (1+1)-dimensional fermion theories with external fields

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In the framework (1+1)-dimensional fermion theories with scalar, pseudoscalar, and vector external fields we discuss the relation between the charge of the vacuum and the topological properties of the external fields. For theories with a gap above zero in the particle spectrum we prove the known relation between the charge of the vacuum and the behavior at infinity of the scalar and pseudoscalar external fields, and show that a vector potential alters the charge of the vacuum only if it is generated by a nonvanishing external charge. Some solvable models are also presented and discussed.

## I. INTRODUCTION

The classical work of Jackiw and Rebbi[1] has opened a wide spectrum of investigations on the problem of a quantized Dirac field in 1 + 1 dimensions interacting with a background field through scalar coupling, focusing in particular on the (fermion) charge of the vacuum sector. The original result of Ref. 1 on charge fractionalization was extended to realistic models in many-body theories[2] and critically reconsidered by evaluating the charge fluctuations and by studying the effect of boundary conditions.[3]

A further interesting development came by considering scalar as well as pseudoscalar background fields: it has been found[4] that (a) the charge of the vacuum is not restricted to $\pm\frac{1}{2}$, but can assume any real number; (b) the results can be extended to the interesting case in 3 + 1 dimensions, where the background is given by a monopole field.

These investigations open a certain number of interesting problems. We shall focus on some of them, and the general discussion will also be supported by the analysis of solvable models in 1 + 1 dimensions.

The paper is organized as follows. In Sec. II we review the C-invariant case of a scalar background field. We also examine the structure of the theory for a Majorana fermion, and show that the degeneracy of the vacuum present when charge fractionalization occurs is replaced in this case by the existence of two inequivalent vacuum states, related by the spontaneously broken symmetry $\psi(x)\rightarrow-\psi(x)$. In Sec. III we consider the general C-noninvariant case, i.e., with scalar, pseudoscalar, and vector external fields:

$$[i\partial\!\!\!/-A\!\!\!/-\lambda(x)e^{i\gamma_5\alpha(x)}]\psi(x)=0\ , \tag{1}$$

where $\lambda(x)>0$ and $\gamma_5=\gamma_0\gamma_2$ and discuss in the static case the validity of the relation

$$\langle 0\,|\,Q\,|\,0\rangle=\frac{1}{2\pi}[\alpha(+\infty)-\alpha(-\infty)] \tag{2}$$

which was proposed in Ref. 4 in a somewhat restricted situation. We prove the general validity of Eq. (2) under suitable conditions to be satisfied by the external vector potential. These conditions yield a simple interpretation of the result, i.e., that the charge of the vacuum is altered by the presence of an external vector field only if the total external charge is nonzero. In Sec. IV some technical aspects of the general discussion are illustrated for some specific models. The pathologies of the $\lambda=0$ case are also outlined.

## II. EXTERNAL SCALAR FIELD

Consider a Dirac particle in 1 + 1 dimensions interacting with a static scalar c-number field $\phi(x)$. The Hamiltonian for the first-quantized theory is given by

$$H=p\sigma_2+\phi(x)\sigma_1\ , \tag{3}$$

where $\sigma_1$, $\sigma_2$ are the usual Pauli matrices so that $\gamma^0=\sigma_1$, $\gamma^1=i\sigma_3$, and $p=-i\partial/\partial x$. In terms of components, the energy eigenvalue equation reads

$$u'+\phi u=Ev,\quad -v'+\phi v=Eu\ . \tag{4}$$

These yield the following, decoupled, second-order equations:

$$\begin{aligned}&u''+(E^2-\phi^2+\phi')u=0\ ,\\&v''+(E^2-\phi^2-\phi')v=0\ .\end{aligned} \tag{5}$$

For the second-quantized theory we assume equal-time anticommutation relations for the complex field $\psi=\binom{u}{v}$,

$$\begin{aligned}&\{\psi^\dagger(x^0,x),\psi(x^0,y)\}=\delta(x-y)\ ,\\&\{\psi(x_0,x),\psi(x_0,y)\}=0\ .\end{aligned} \tag{6}$$

The theory is charge-conjugation invariant, with

$$\psi^c(x)=C\psi(x)C^{-1}=\sigma_3\psi^\dagger(x)\ . \tag{7}$$

 

It is known that if $\phi(x)$ has an odd number of zeros and $|\phi(x)|>C$ for $|x|\to\infty$ the theory has an isolated zero mode with the wave function

$$f_0(x)=\text{const}\begin{bmatrix}1\\0\end{bmatrix}\exp\left[\int_0^x \phi(x')dx'\right], \quad \text{if } \lim_{x\to\infty}\phi(x)<0 \tag{8}$$

and

$$f_0(x)=\text{const}\begin{bmatrix}0\\1\end{bmatrix}\exp\left[-\int_0^x \phi(x')dx'\right], \quad \text{if } \lim_{x\to\infty}\phi(x)>0\,. \tag{9}$$

The existence of a zero mode requires the existence of two independent vacuum states (zero-energy states for the second-quantized theory), both belonging to the same irreducible representation of the field algebra. In fact, let us define

$$a=\int dx\,\psi(x,x)f_0(x)\,. \tag{10}$$

Then from (6) we get

$$\{a^\dagger,a\}=1,\quad \{a,a\}=0 \tag{11}$$

and since $f_0$ is a zero mode

$$[H,a]=[H,a^\dagger]=0\,. \tag{12}$$

If the vacuum were unique, from (11) and (12) we would have

$$2|\langle 0|a|0\rangle|^2=1 \quad\text{and}\quad 2\langle 0|a|0\rangle^2=0$$

which are incompatible. In fact the algebra (11) admits only a two-dimensional irreducible representation.[1] Let $|0_1\rangle$ and $|0_2\rangle$, defined by

$$a|0_1\rangle=0,\quad a^\dagger|0_1\rangle=|0_2\rangle \tag{13}$$

be a basis for a representation of the algebra (11). Let now $Q$ be the fermionic charge defined by

$$[Q,\psi(x)]=-\psi(x) \tag{14}$$

with the property

$$CQC^{-1}=-Q \tag{15}$$

and

$$[Q,H]=0\,. \tag{16}$$

From Eqs. (15) and (16) it follows that either $Q$ is zero on the $|0_1\rangle$, $|0_2\rangle$ manifold, or has nonzero opposite eigenvalues. From Eq. (14) and the fact that $\langle 0_1|\psi|0_2\rangle\neq 0$ [Eq. (13)] the first possibility is excluded, moreover the eigenvalues of $Q$ on the vacuum manifold must be $\pm\frac{1}{2}$.[5] Thus it is clear that charge fractionalization occurs as a consequence only of the existence of the zero mode $f_0$, charge-conjugation invariance of the theory, and the properties (14)–(16). In the sequel we examine what happens when the theory is not charge-conjugation invariant.

To examine further the present situation, we introduce the fermion current $j_\mu(x)$ defined by

$$j_\mu(x)=\lim_{\xi\to 0}\tfrac{1}{2}(\overline{\psi}(x+\xi)\gamma_\mu\psi(x)-\psi(x+\xi)\gamma_\mu^T\overline{\psi}(x)) \tag{17}$$

so that it anticommutes with the charge-conjugation operator. We also express $\psi$ in normal modes

$$\psi(x)=af_0(x)+\sum_n[c_nh_n(x)e^{-iE_nx^0}+d_n^\dagger g_n(x)e^{iE_nx^0}]\,, \tag{18}$$

where the summation is over the discrete and continuous spectrum.

From Eq. (6) we have the usual anticommutation relations for annihilation and creation operators; in addition, $a$ and $a^\dagger$ anticommute with $c_n$, $d_n$ and $c_n^\dagger,d_n^\dagger$. Positivity of the energy requires that the vacuum states should be annihilated by $c_n$ and $d_n$. Then the charge density on a general ground state

$$|\theta_1,\theta_2\rangle=\cos\theta_1|0_1\rangle+e^{i\theta_2}\sin\theta_1|0_2\rangle \tag{19}$$

is simply calculated and yields

$$\langle\theta_1,\theta_2|j_0(x)|\theta_1,\theta_2\rangle=-\tfrac{1}{2}\cos 2\theta_1|f_0(x)|^2\,. \tag{20}$$

The total charge

$$Q=\sum_n(c_n^\dagger c_n-d_n^\dagger d_n)+\tfrac{1}{2}(a^\dagger a-aa^\dagger) \tag{21}$$

has the vacuum expectation value

$$\langle\theta_1,\theta_2|Q|\theta_1,\theta_2\rangle=-\tfrac{1}{2}\cos 2\theta_1\,. \tag{22}$$

In order to confirm that the twofold degeneracy of the vacuum [Eq. (13)], with both vacuum states belonging to the *same* irreducible representation of the field algebra, is due to the existence of the fermion charge, it is important to analyze the theory where the field $\psi$ is a Majorana spinor. This is possible because $\phi(x)$ is real.

If we impose the condition

$$\psi^\dagger(x)=\sigma_3\psi(x) \tag{23}$$

or equivalently

$$\psi^c(x)=\psi(x) \tag{24}$$

then this implies that

$$c_n=d_n \quad\text{and}\quad a=a^\dagger \tag{25}$$

with $a$ anticommuting with $c_n$ and

$$a^2=\tfrac{1}{2}\,. \tag{26}$$

In this case the fermion charge [Eq. (21)] is zero. Furthermore, to obtain an irreducible representation of this algebra only one vacuum state is needed:

$$c_n|0\rangle=0 \quad\text{but}\quad a|0\rangle=\pm\frac{1}{\sqrt{2}}|0\rangle\,. \tag{27}$$

The $+$ sign and $-$ sign in these expressions correspond to two inequivalent representations of the field algebra, namely,

$$c_n\,|0_+\rangle=0,\quad a\,|0_+\rangle=\frac{1}{\sqrt{2}}\,|0_+\rangle \tag{28}$$

and

$$c_n\,|0_-\rangle=0,\quad a\,|0_-\rangle=-\frac{1}{\sqrt{2}}\,|0_-\rangle\ . \tag{29}$$

All elements of the field algebra generated by $\psi(x)$ have vanishing matrix elements between $|0_+\rangle$ and $|0_-\rangle$; any linear combination of these two states corresponds therefore to an impure vacuum state and if used as a cyclic vector would give rise to a *reducible* representation of the field algebra.

The existence of the two inequivalent representations [Eqs. (28) and (29)] is related to the spontaneous breaking of the symmetry [implemented in the previous situation by $\exp(i\pi Q)$]

$$\psi(x)\rightarrow-\psi(x)\ . \tag{30}$$

The vacuum expectation values of $\psi$ are, in fact, different from zero:

$$\begin{aligned}\langle 0_\pm\,|\,\psi(x)\,|\,0_\pm\rangle&=f_0(x)\langle 0_\pm\,|\,a\,|\,0_\pm\rangle\\&=\pm\frac{1}{\sqrt{2}}f_0(x)\end{aligned} \tag{31}$$

and are related to each other by the symmetry transformation (30). Thus we see that the twofold degeneracy of the vacuum for a charged field ($\psi\neq\psi^c$) is replaced, for a Majorana field ($\psi=\psi^c$), with the existence of two inequivalent vacua.

In order to illustrate some of the above conclusions and for further reference we discuss briefly the solvable model of a linearly varying background field:

$$\phi(x)=m+gx\quad(m,g>0)\ . \tag{32}$$

Then Eqs. (5) have the same structure as the equation for the quantum-mechanical harmonic oscillator. The zero mode is given by

$$\begin{aligned}f_0(x)&=\left[\frac{g}{\pi}\right]^{1/4}\begin{bmatrix}1\\0\end{bmatrix}e^{-(g/2)(x+m/g)^2}\\&=g^{1/4}\begin{bmatrix}1\\0\end{bmatrix}\psi_0(\xi)\ .\end{aligned} \tag{33}$$

If $h_n$ is a positive-energy solution, then

$$g_n(x)=\sigma_3 h_n(x) \tag{34}$$

is a negative-energy solution. The positive-energy solutions are given by

$$h_n(x)=\left[\frac{g}{4}\right]^{1/4}\begin{bmatrix}\psi_n(\xi)\\ \psi_{n-1}(\xi)\end{bmatrix}, \tag{35}$$

where $\psi_n$ are the harmonic-oscillator eigenfunctions and $\xi=\sqrt{g}\,(x+m/g)$. The corresponding energy is

$$E_n=\sqrt{2ng}\ . \tag{36}$$

In this case the charge density of the vacuum states is given by

$$\langle\theta_1,\theta_2\,|\,j_0(x)\,|\,\theta_1,\theta_2\rangle=-\tfrac{1}{2}\cos 2\theta_1\left[\frac{g}{\pi}\right]^{1/2}e^{-g(x+m/g)}\ . \tag{37}$$

A linear $\phi(x)$ can give a good approximation for the lowest-energy mode. Moreover we notice that by choosing a linear $\phi(x)$ all states are normalizable and no box quantization is required. In this way we automatically fulfill the limiting procedure proposed by Jackiw *et al.*[3] and Rajaraman and Bell.[3] Equation (37) shows that the charge density of the vacuum does not follow locally the profile of the background field also in the limit of small $g$ (see the slowly varying field approximation of Ref. 4).

## III. GENERAL EXTERNAL FIELDS

In this section we discuss the charge of the vacuum in the general case where charge conjugation is not a symmetry of the Lagrangian. The general situation for a Fermi field in $1+1$ dimensions can be described by the Lagrangian

$$\begin{aligned}\mathscr{L}[\lambda,\alpha,A_\mu]=\bar\psi[i\gamma^\mu\partial_\mu-\lambda(x)e^{i\gamma_5\alpha(x)}]\psi(x)\\-gj_\mu(x)A^\mu(x)\ ,\end{aligned} \tag{38}$$

where $A_\mu(x)$, $\lambda(x)$, and $\alpha(x)$ are external fields, and $\lambda(x)\geq 0$.

The gauge-invariant electric current $j_\mu(x)$ is defined by the split-point regularization procedure:

$$\begin{aligned}j_\mu(x)=\lim_{\xi\to 0}\tfrac{1}{2}[\bar\psi(x+\xi)e^{-i\xi_\rho A^\rho(x)}\gamma_\mu\psi(x)\\-\psi(x+\xi)\gamma_\mu^T e^{i\xi_\rho A^\rho(x)}\bar\psi(x)]\ ,\end{aligned} \tag{39}$$

where, to maintain formal gauge invariance, the limit on $\xi$ is to be understood as an average in the following sense:

$$\lim_{\xi\to 0}\equiv\tfrac{1}{2}\left[\lim_{\substack{\xi^0\to 0\\ \xi^1=0}}+\lim_{\substack{\xi^1\to 0\\ \xi^0=0}}\right]. \tag{40}$$

Consider now $x$-dependent gauge and chiral transformations on the field $\psi$:

$$\psi(x)\rightarrow\psi'(x)=e^{-i[\bar\Lambda(x)+\gamma_5\Lambda(x)]}\psi(x)\ . \tag{41}$$

Correspondingly, the Lagrangian (38) is transformed in the following way:

$$\mathscr{L}[\lambda,\alpha,A_\mu]\rightarrow\mathscr{L}[\lambda,\alpha+2\Lambda,A_\mu+\partial_\mu\bar\Lambda-\epsilon_{\mu\nu}\partial^\nu\Lambda] \tag{42}$$

[provided $\bar\Lambda(x)$ and $\Lambda(x)$ are continuous functions and have the additional regularity properties such that the equations of motion for both Lagrangians make sense]. Simultaneously, the current (39) transform as

$$j_\mu(x)\rightarrow j'_\mu(x)=j_\mu(x)+\frac{1}{\pi}\epsilon_{\mu\nu}\partial^\nu\Lambda(x)\ , \tag{43}$$

where $j'_\mu(x)$ is given by the expression (39) corresponding to the transformed Lagrangian, i.e., with $A_\rho$ in the exponent replaced by $A_\rho+\partial_\rho\bar\Lambda-\epsilon_{\rho\sigma}\partial^\sigma\Lambda$. The $c$-number term on the right-hand side of (43) comes from the regularization procedure adopted in the definition (39) of the current (anomaly), and the fact that the small-distance behavior of the two-point function is the same as in a free field theory:

$$\langle 0 \mid \psi(x+\xi)\bar{\psi}(x) \mid 0 \rangle = \frac{1}{2\pi i} \frac{\xi\cdot\gamma}{\xi^2} . \tag{44}$$

Moreover we have used the fact that in the limit $\xi \to 0$, $\xi_\rho \bar{\psi}_\alpha(x+\xi)\psi_\beta(x)$ reduces to a $c$ number, as can be seen by considering matrix elements between particle states.

From now on we restrict our analysis to static external fields (independent of the time variable) and $A_\mu = (A_0, 0)$ (static case). Therefore also the gauge and chiral functions $\bar{\Lambda}$ and $\Lambda$ in Eq. (41) will be restricted to be

$$\bar{\Lambda} = cx^0 \text{ and } \Lambda = \Lambda(x^1) .$$

The fundamental transformation equations (42) and (43) can then be rewritten in the form

$$\mathscr{L}[\lambda,\alpha,A_0] \to \mathscr{L}[\lambda,\alpha+2\Lambda,A_0+c-\partial_1\Lambda] , \tag{42'}$$

$$j_0(x) \to j_0(x) + \frac{1}{\pi}\partial_1\Lambda(x^1) . \tag{43'}$$

Clearly Eq. (43′) allows us to relate the charge of the vacuum for the different theories described by the Lagrangian $\mathscr{L}[\lambda,\alpha+2\Lambda,A_0+c-\Lambda']$:

$$Q[\alpha,A_0] = Q[\alpha+2\Lambda,A_0+c-\partial_1\Lambda] - \frac{1}{\pi}[\Lambda(+\infty)-\Lambda(-\infty)] . \tag{45}$$

In particular we see that as long as $\Lambda(+\infty)=\Lambda(-\infty)$ the charge operator (but not the current) remains invariant: an obvious result, since in this case the transformation from $\psi(x)$ to $e^{i\gamma_5\Lambda(x^1)}\psi(x)$ is an implementable transformation that commutes with the charge operator.

If $\alpha(x^1)$ is a continuous function, choosing $\Lambda(x) = -\frac{1}{2}\alpha(x)$, we get from Eq. (45)

$$Q[\alpha,A_0] = \frac{1}{2\pi}[\alpha(+\infty)-\alpha(-\infty)] + Q[0,A_0+\tfrac{1}{2}\alpha'] \tag{46}$$

and the problem of evaluating the charge of the vacuum in the general case is reduced to that of the charge of the vacuum induced only by an external vector field. We will show that under suitable conditions $\langle 0 \mid Q[0,A_0] \mid 0$ is indeed zero, thus confirming the validity (under the conditions we will specify) of Eq. (2).

To this end, consider the theory defined by the Lagrangian $\mathscr{L}[\lambda,0,A_0]$, with $\lambda(x) \ge 0$, $\lambda(x) > B > 0$ for $|x| > |\bar{x}|$, and $A_0(x)$ bounded: $|A_0(x)| < C$, and $A_0'(x) \to 0$ for $|x| \to \infty$. Under the above conditions we show that the charge of the vacuum $\langle 0 \mid Q[0,A_0] \mid 0 \rangle$ is zero.

To prove the above assertion consider first the theory with $A_0 = 0$. In this case, due to the conditions on $\lambda(x)$, the normal modes of the field $\psi(x)$ are separated from zero, therefore an energy gap separates the particle spectrum from the vacuum. Moreover, from charge-conjugation invariance we know that the charge of the vacuum is zero. We now prove that the charge remains zero to all orders in $A_0(x)$.[6] For the sake of clarity we begin by considering the first order.

Since we are considering the static case ($A_1 = 0$), it is convenient to take in the definition (39) of the regularized (gauge invariant) current the vector $\xi$ to be $(0,\xi^1)$. Thus the charge density takes the form

$$j_0(x) \lim_{\xi^1\to 0} \tfrac{1}{2}[\psi^\dagger(x+\xi)\psi(x) - \psi(x+\xi)\psi^\dagger(x)] . \tag{47}$$

Then the first-order contribution to the vacuum expectation value of $j_0$ is given by

$$\langle 0 | j_0^{(1)}(x) | 0 \rangle = -ig \lim_{\xi^1\to 0} \int d^2z \, \mathrm{Tr}\gamma_0 S(x,z)\gamma_0 S(z,x+\xi)A_0(z^1) , \tag{48}$$

where $S$ is the Feynman propagator satisfying the equation

$$[i\gamma^\mu\partial_\mu - \lambda(x)]S(x,y) = \delta^{(2)}(x-y) . \tag{49}$$

In order to prove that the RHS of (48) gives zero contribution to the charge (also for the $n$th-order case) we use the Ward identity

$$i\int d^2y \, S(x,y)\gamma^0 S(y,x') = S(x,x')(x^0 - x^{0'}) . \tag{50}$$

This identity can formally be derived as follows: from Eq. (49) we get

$$S(x,x')[\delta(x-y) - \delta(x'-y)] = -i\frac{\partial}{\partial y^\mu}[S(x,y)\gamma^\mu S(y,x')] .$$

By integrating the above identity over $d^2y$, multiplying by $y^0$, and by dropping the contributions at infinity, Eq. (50) is immediately obtained. However the recognition that, as a consequence of the energy gap, the terms at infinity do not contribute, is not obvious and in fact is not true in the massless or zero-gap case. Therefore it is safer to prove directly the identity (50) by starting from the representation of the propagator in terms of normal modes where the energy gap is explicitly exhibited:

$$iS(x,y) = \int_{E_0}^\infty dE \, [f_E(x)\bar{f}_E(y)e^{-iE(x^0-y^0)}\theta(x_0-y_0) - g_E(x)\bar{g}_E(y)e^{-iE(y^0-x^0)}\theta(y_0-x_0)] . \tag{51}$$

Then Eq. (50) is immediately derived. It is easily recognized that the integrations over $y^1$ and $y^0$ can be performed in arbitrary order and that only in the zero-gap case ($E_0 = 0$) do ambiguous infrared terms arise. Coming back to Eq. (48) we have

$$\int dx^1 \langle 0 | j_0^{(1)}(x) | 0 \rangle = -ig \lim_{\xi^1 \to 0} \int dx^1 \int d^2z \, \mathrm{Tr}(S(z,x+\xi)\gamma_0 S(x,z)\gamma_0 A_0(z^1)) \, . \tag{52}$$

Note that $S(x,y)$ depends on the time variables $x^0$, $y^0$ only through their difference (static case), and that for small $\xi$, $S(z,x+\xi)=S(z-\xi,x)$ [Eq. (44)]. Then by performing first the integration over $x^1$ and $z^0$ and by using the identity (50) we get zero, since $\xi^0=0$.

The delicate point is the justification for interchanging the integrations over the $x^1$ and $z^1$ variables: it is beyond the scope of this paper to give a rigorous and complete analysis of the conditions on $A_0(x)$ under which this possibility is guaranteed; however since the large-distance behavior of the integrand in (52) is determined by the energy gap, we assume (and feel justified in doing so) that the conditions are the same as in the $\lambda(x)=m>0$ case. In this case we find that the conditions $|A_0(x)|<C$ and $A_0'(x)\to 0$ for $|x|\to\infty$ are sufficient. This is shown at the end of this section where we also show that in the physically interesting case of $A_0(x)\approx |x^1|$, there is a finite contribution from the RHS of (52).

The proof for the $n$th order proceeds along similar lines:

$$\int dx^1 \langle 0 | j_0^{(n)}(x) | 0 \rangle = \frac{(-i)^n}{n!} g^n \lim_{\xi^1 \to 0} \sum_{\text{perm}} \int dx^1 d^1z_1 \cdots d^2z_n \times \mathrm{Tr}(\gamma_0 S(x,z_1+\xi_1)\gamma_0 S(z_1,z_2+\xi_2) \cdots \times S(z_n,x+\xi) A_0(z_1^1) A_0(z_2^1) \cdots A_0(z_n^1)) \, . \tag{53}$$

After integration over $x^1$ and $x^0-z_n^0$ (the differences $z_i^0-z_n^0$ held fixed), we get

$$(z_n^0-z_1^0)\mathrm{Tr}(S(z_n,z_1+\xi_1+\xi)\gamma_0 S(z_1,z_2+\xi_2) \cdots S(z_{n-1},z_n+\xi_n)\gamma_0 A_0(z_1^1)A_0(z_2^1) \cdots A_0(z_n^1)) \, .$$

Upon summation over the terms obtained from the above by cyclic permutations of the $z$'s, we get complete cancellation.

The result we have obtained can be expressed in simple physical terms; since the conditions on $A_0(x)$ imply that the total external charge generating $A_0(x)$ is zero, we have found that in this case also the induced charge on the vacuum vanishes. It is therefore interesting to analyze a situation in which the external charge is not zero: we do this in the case $\lambda(x)=m>0$, with $A_0(x)=kg\,|x^1|$. Equation (52) with $S(x,y)=S(x-y)$ gives[7]

$$\langle 0 | Q^{(1)} | 0 \rangle = \frac{-ig}{(2\pi)^2} \int dx^1 d^2q \, e^{-iqx} \widetilde{\Pi}_{00}(q) \widetilde{A}_0(q) = -ig \lim_{q\to 0} \widetilde{\Pi}_{00}(q) \widetilde{A}_0(q^1) \, , \tag{54}$$

where $\Pi_{\mu\nu}(x-y)$ is the vacuum polarization tensor. Near $q=0$ we have

$$\widetilde{\Pi}_{00}(q) = -\frac{i}{6\pi m^2}(q^1)^2[1+O(q^2)] \, .$$

Thus we see that the conditions we have imposed on $A_0(x)$ and its derivative are sufficient to ensure that $\langle Q^{(1)} \rangle = 0$.

In the present case we have

$$\widetilde{A}_0(q^1) = -\frac{2kg}{(q^1)^2}$$

and therefore

$$\langle 0 | Q^{(1)} | 0 \rangle = \frac{kg^2}{3\pi m^2} \tag{55}$$

while the external charge $Q^{\text{ext}}=-k$. Since higher orders do not contribute, we conclude that (55) gives correctly the charge of the vacuum.

The result that the induced charge on the vacuum is proportional to the external charge is of more general validity, as can be seen by the following argument. The charge of the vacuum, due to gauge invariance, is a functional of $A_0'$, the derivative of $A_0(x)$. We have shown that the addition of a bounded potential $\delta A_0$ with $\delta A_0' \to 0$ for $|x^1|\to\infty$ does not alter the charge of the vacuum, i.e.,

$$\langle 0 | Q[A_0'+\delta A_0'] | 0 \rangle = \langle 0 | Q[A_0'] | 0 \rangle \, .$$

Thus $\langle Q \rangle$ can depend only on the values of the electric field $E$ at infinity. Moreover, since there is a contribution to $\langle Q \rangle$ only from first order, $\langle Q \rangle$ is a linear functional of $E(\pm\infty)$. If $\lambda(x)=\lambda(-x)$ it is immediate to conclude that $\langle Q \rangle$ is proportional to $E(+\infty)-E(-\infty)$, that is to the external charge.

## IV. EXAMPLES

The conclusions of the previous sections are illustrated by a few examples. The pathologies of the no-mass-gap case are easily discussed in the extreme case of $\lambda(x)=0$. In the other examples we illustrate the subtleties of taking the limits and performing the integrations in the right order, and the role of the regularization (anomaly mechanism).

### A. External vector field

A massless fermion in the presence of a vector field obeys the equation

$$[i\gamma^\mu \partial_\mu - \gamma^\mu A_\mu(x)]\psi(x) = 0 \tag{56}$$

which is easily solved by the substitution

$$\psi(x)=e^{i[\overline{\Gamma}(x)+\gamma_5\Gamma(x)]}\chi(x)\ , \tag{57}$$

where

$$A_\mu(x)=-\partial_\mu\overline{\Gamma}(x)+\epsilon_{\mu\nu}\partial^\nu\Gamma(x) \tag{58}$$

and

$$i\partial\!\!\!/\chi(x)=0\ . \tag{59}$$

Let us fix the representation of $\chi$ by the two-point function

$$\langle 0\,|\,\chi(x)\overline{\chi}(0)\,|\,0\rangle=\frac{1}{2\pi i}\frac{\gamma\cdot x}{x^2-i\epsilon x^0}\ . \tag{60}$$

Then the gauge-invariant current (39) becomes [see Eq. (43)]

$$j_\mu(x)=:\overline{\chi}\gamma_\mu\chi:(x)-\frac{1}{\pi}\epsilon_{\mu\nu}\partial^\nu\Gamma(x)\ . \tag{61}$$

The charge of the vacuum is then

$$\langle 0\,|\,Q\,|\,0\rangle=-\frac{1}{\pi}[\Gamma(x_0,+\infty)-\Gamma(x_0,-\infty)]\ . \tag{62}$$

The charge of the vacuum is, however, ill defined: in fact the Lagrangian of the $m=0$ theory, even in the presence of a vector potential, is invariant under the transformation

$$\psi(x)\rightarrow e^{i[\overline{\Lambda}(x)+\Gamma_5\Lambda(x)]}\psi(x) \tag{63}$$

with

$$\partial_\mu\overline{\Lambda}(x)-\epsilon_{\mu\nu}\partial^\nu\Lambda(x)=0\ . \tag{64}$$

The current, however, does change under the above transformation, in fact

$$j_\mu(x)\rightarrow j_\mu(x)+\epsilon_{\mu\nu}\partial^\nu\Lambda(x)\ . \tag{65}$$

Therefore the charge of the vacuum depends not only on the external field, but also on the choice of the representation for $\psi$ (or equivalent for $\chi$). This example shows that the conclusions of Sec. III, in particular $\langle 0\,|\,Q\,|\,0\rangle=0$ for a vector potential with zero total external charge, rely crucially upon the assumption of a gap in the unperturbed spectrum.

### B. Vector and pseudoscalar external field

We now discuss a solvable model with the aim of emphasizing some points we did not touch on in the general discussion. Consider the first quantized Hamiltonian

$$H=p\sigma_2+\phi_1(x)\sigma_1+\phi_2(x)\sigma_3 \tag{66}$$

with $\phi_1(x)=gx$, $\phi_2(x)=m$ $(m,g>0)$ (this is the $C$-noninvariant version of the example discussed in Sec. III). Also

$$H=p\sigma_2+\lambda(x)\sigma_1 e^{i\gamma_5\alpha(x)} \tag{66$'$}$$

with

$$\lambda(x)=(m^2+g^2x^2)^{1/2},\quad \alpha(x)=\arctan\frac{gx}{m}\ .$$

Therefore, according to the general arguments of Sec. III we expect

$$\langle 0\,|\,Q\,|\,0\rangle=\frac{1}{2\pi}[\alpha(+\infty)-\alpha(-\infty)]=\tfrac{1}{2}\ . \tag{67}$$

The normal modes of the theory are easily found to be

$$f_0(x)=g^{1/4}\begin{bmatrix}1\\0\end{bmatrix}\psi_0(\xi),\quad \xi=\sqrt{g}\,x \tag{68}$$

with $E_0=-m$ and

$$h_n(x)=\left[\frac{g}{4}\right]^{1/4}\begin{bmatrix}\left[\dfrac{E_n+m}{E_n}\right]^{1/2}\psi_n(\xi)\\ \left[\dfrac{2ng}{E_n(E_n+m)}\right]^{1/2}\psi_{n-1}(\xi)\end{bmatrix} \tag{69}$$

with $E_n=(2ng+m^2)^{1/2}$, $n=1,2,\ldots$ [$\psi_n(\xi)$ are the eigenfunctions of the harmonic oscillator], as well as

$$g_n(x)=\sigma_3 h_n(x) \tag{70}$$

with $E_n=-(2ng+m^2)^{1/2}$, $n=1,2,\ldots$. Second quantization proceeds as usual. However, in a $C$-noninvariant theory there is an ambiguity in assigning creation or annihilation operators to the normal modes, a difficulty that in $C$-invariant theories is bypassed by the requirement of implementability of charge conjugation. To conform with the discussion of Sec. III we may give the following general prescription. We transform the Hamiltonian (66) or (66$'$) to the scalar-vector form

$$H'=p\sigma_2+\lambda(x)\sigma_1+\tfrac{1}{2}\alpha'(x) \tag{71}$$

and consider the term $\frac{1}{2}\alpha'(x)$ as an independent perturbation. Then the assignment of creation or annihilation operators is performed in such a way that after $\frac{1}{2}\alpha'(x)$ is switched off, charge conjugation is implementable. Any different prescription, resulting in a different assignment of $n$ isolated normal modes will alter the charge of the vacuum by an integer. This is not in contradiction with the results of Sec. III.

In the present case, since $\alpha'(x)>0$, it is immediate that our prescription leads to the following representation:

$$\psi(x)=b^\dagger f_0(x)e^{imt}+\sum_n[c_n h_n(x)e^{-iE_nt}+d_n^\dagger g_n(x)e^{iE_nt}]\ . \tag{72}$$

The vacuum expectation value of the current is

$$\langle 0\,|\,j_0(x)\,|\,0\rangle=\tfrac{1}{2}\,|\,f_0(x)\,|^2-\tfrac{1}{2}\sum_n[h_n^*(x)h_n(x)-g_n^*(x)g_n(x)]\ . \tag{73}$$

If we have to obtain $\langle Q\rangle=\frac{1}{2}$ [Eq. (67)], it should be true that

$$\int dx\sum_n({h_n}^2-{g_n}^2)=\sum_n\int dx({h_n}^2-{g_n}^2)=0\ . \tag{74}$$

Since in general, as we shall see in the following examples, there is a contribution to the charge from all the normal modes (i.e., the $x$ integral cannot be interchanged with the integral over the normal modes) we outline a proof of Eq. (74). We have

$$\sum_{n=1} (h_n{}^2-g_n{}^2)=\sqrt{g}\sum_{n=1} \frac{m}{|E_n|}(\psi_n{}^2-\psi_{n-1}{}^2)$$
$$=\lim_{N\to\infty} m\sqrt{g}\left[\frac{\psi_N{}^2}{|E_N|}-\frac{\psi_0{}^2}{|E_1|}+\sum_1^N \psi_n{}^2 c_n\right] \tag{75}$$

with

$$c_n=\frac{1}{|E_n|}-\frac{1}{|E_{n+1}|}>0\ . \tag{76}$$

The first term on the RHS of (75) goes to zero pointwise in the limit $N\to\infty$. The last term is a series of non-negative integrable functions such that the series of the integrals converges, therefore the integration can be performed term by term and we get

$$\int dx \sum_n (h_n{}^2-g_n{}^2)=m\sqrt{g}\left[\frac{1}{|E_1|}+\sum_{n=1} c_n\right]=0\ .$$

### C. Chiral external field

In the general discussion of Sec. III we have assumed regularity properties for the $\gamma_5$-phase $\alpha(x)$. In fact its derivative should give a vector potential such that the Dirac equation is meaningful. A discontinuity in $\alpha(x)$ would, for instance, given an ill-defined expression for the vector potential in Eq. (42) and then the equation needs a careful regularization. On the other hand we have seen that the charge does not depend on the details of $\alpha(x)$ but only on its limit values at infinity. Thus we expect Eq. (46) to be valid also in presence of steplike discontinuities in $\alpha(x)$.

Let us consider the following example:

$$(i\partial - m e^{i\gamma_5 \alpha\theta(x^1)})\psi(x)=0\ , \tag{77}$$

where the constant $\alpha$ is taken to be

$$0\le |\alpha| \le \pi\ .$$

The spectrum consists of a doubly degenerate continuum starting at $E=m$ and of a bound state located at $E_0$ given by the solution of the equation

$$\frac{(m^2-E_0{}^2)^{1/2}}{E_0}=\tan\frac{\alpha}{2}\ . \tag{78}$$

Then we have a bound state in the particle sector for $0<\alpha<\pi$ and in the antiparticle sector for $-\pi<\alpha<0$.

The charge density can be obtained in a straightforward manner by summing over the contribution of the normal modes

$$\langle 0|j_0(x)|0\rangle = -\frac{m^2}{2\pi}\sin\frac{\alpha}{2}\int_m^\infty \frac{dE}{k}\left[\frac{1}{\Delta}+\frac{1}{\Delta^*}\right]\sin 2k'|x^1| + \left[\frac{m^2-E_0{}^2}{2}\right]^{1/2} e^{-2(m^2-E_0{}^2)^{1/2}|x_1|}\epsilon(\alpha)\ , \tag{79}$$

where $\Delta=k\cos(\alpha/2)-iE\sin(\alpha/2)$ and $k=(E^2-m^2)^{1/2}$. The integration over $x^1$ can be performed by moving first the integration on $E$ to the imaginary axis and then by interchanging the order of integration. Then we get

$$\langle 0|Q|0\rangle=-\frac{m^2}{2\pi}\sin\frac{\alpha}{2}\int_{-i\infty}^{+i\infty}\frac{dE}{k^2}\frac{1}{\Delta}=\frac{\alpha}{2\pi}\ . \tag{80}$$

### D. Square-well vector potential

Our final example shows the importance of regularization and the necessity of performing the integration and of taking the limits in the right order. In fact in the previously discussed examples without vector potential we have calculated the charge of the vacuum ignoring the split-point regularization of the current, which actually was unnecessary to give a meaning to the current as an integral over the normal modes. In the presence of a vector potential a regularization is needed and can be performed either as in Eq. (40) or as in Eq. (47).

Consider a square-well vector potential:

$$A_0(x)=-g,\quad |x|<l$$
$$A_0(x)=0\ ,\ |x|>l \tag{81}$$

with $g>0$ and $g<m$ in the Dirac equation

$$[i\partial - m -\gamma_0 A_0(x)]\psi(x)=0\ . \tag{82}$$

The bound states are all in the particle sector and their energy is given by

$$kk'\cos 2k'l + i[m^2-E(E+g)]\sin 2k'l=0 \tag{83}$$

with $k=(E^2-m^2)^{1/2}$, $k'=[(E+g)^2-m^2]^{1/2}$. It can be shown that the zeros of the above equation are only on the real axis and lie on the interval

$$m-g<E<m\ .$$

The integral over the continuum for the charge density of the vacuum can be moved to the imaginary axis and the final result is

$$\langle 0|j_0(x)|0\rangle_{|x'|>l}=-i\frac{m^2 g}{2\pi}\int_{-i\infty}^{+i\infty} dE\frac{\sin 2k'l}{k\Delta}e^{-2ik(l-|x'|)}\ ,$$
$$\langle 0|j_0(x)|0\rangle_{|x'|<l}=\frac{m^2}{2\pi}P\int_{-i\infty}^{+i\infty} dE\, k\left[\frac{1}{E[m^2-E(E+g)]}+\frac{g}{kk'\Delta}\left[\frac{g(E+g)}{m^2-E(E+g)}\cos 2k'l-\cos 2k'x^1\right]\right]+\frac{g}{\pi}\ , \tag{84}$$

where $\Delta = kk'\cos 2k'l + i[m^2 - E(E+g)]\sin 2k'l$. The last term on the RHS of (84) comes from the regularization procedure and is the anomaly. Moreover we see that in the procedure of rotating the integration contour the continuum cancels the bound-state contribution and the final result allows the integration over $x^1$ first in order to evaluate the total charge:

$$\langle 0 | Q | 0 \rangle = \frac{m^2}{2\pi} \int_{-i\infty}^{+\infty} dE \left[ \frac{g \sin 2k'l}{k'^2 \Delta} + 2k'l \left[ \frac{1}{E[m^2 - E(E+g)]} + \frac{g^2 (E+g) \cos 2k'l}{kk'\Delta \cdot [m^2 - E(E+g)]} \right] - g \frac{\sin 2k'l}{k'^2 \Delta} \right] + \frac{2gl}{\pi} . \quad (85)$$

We have verified explicitly up to third order in $g$ that the total charge is zero.

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[5]As a consequence the representation of the gauge group is two-valued: $U(\alpha + 2\pi) = -U(\alpha)$.

[6]The proof essentially follows the lines of Bardeen *et al.* and Frishman *et al.* (Ref. 4) with some improvement due to the fact that it is not based on a slowly varying field approximation and is not limited to first order.

[7]To perform correctly the following steps, one should remember that the charge is defined as

$$Q = \lim_{R \to \infty} \int d^3x \, j_0(x) f \left[ \frac{x^1}{R} \right] \alpha(x^0)$$

with $f(x) = 1$ for $|x| < 1$, $f(x) = 0$, $|x| > 2$; $\int dx^0 \alpha(x^0) = 1$.