Path integrals over the SU(2) manifold and related potentials

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A path-integral formalism is developed on the SU(2) manifold parametrized in terms of the Euler angles. The Green's function is studied for the Pöschl-Teller potential and the corresponding wave functions and energy spectrum are obtained.

I. INTRODUCTION

It is well known that only the linear and the harmonic-oscillator potentials are solvable exactly by path integrals in Cartesian coordinates. To do this one employs mid-point approximations in the time-graded formulations.[1] However, many other important potentials of quantum mechanics are also solved by relating their Green's functions to the Green's functions of harmonic oscillators. For example, the H atom and one-dimensional Morse-potential problems are solved by this method.[2,3] To convert these problems into the oscillator forms, point canonical transformations and accompanying time transformations are used. The action of the transformed Green's function usually includes additional quantum-mechanical terms resulting from the transformation of the path-integral measure. The radial parts of the Schrödinger equations of the potentials, which can be brought into some sort of harmonic-oscillator potentials, all have SU(1,1) dynamical symmetry[4] and their wave functions are of confluent hypergeometric type.

Peak and Inomata[5] studied the path integral for the three-dimensional rigid rotator and expanded the Green's function in terms of the SO(3) matrix elements, that is, in the spherical harmonics. Marinov and Terentyev[6] wrote the path integral on the n-dimensional sphere S^n, and expanded the Green's functions in terms of the Gegenbauer polynomials. The symmetric Rosen-Morse potential is solvable by bringing its Green's function into the Green's function of three-dimensional rigid rotator.[7]

In this paper we first establish the path integral over the SU(2) manifold parametrized in terms of the Euler angles and obtain the Green's function as an expansion in terms of the SU(2) matrix elements $e^{im\varphi}e^{in\varphi}P^l_{mn}(\cos\theta)$. For that, we first write and expand the short-time-interval Green's function for a particle moving in SU(2) space, that is, we follow the method of Ref. 5, which is used for path integration over SO(3) space.

In Sec. II the path integral of the Pöschl-Teller potential is expressed in terms of the Green's function of a free particle moving on the SU(2) manifold. The energy spectrum and the correctly normalized wave functions are obtained.

II. PATH INTEGRATION OVER THE SU(2) MANIFOLD

We parametrize the points of the SU(2) manifold, which is the three-dimensional unit sphere, in terms of the Euler angles:

$$\begin{aligned} u_1&=\cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2}\ ,\\ u_2&=\cos\frac{\theta}{2}\sin\frac{\varphi+\psi}{2}\ ,\\ u_3&=\sin\frac{\theta}{2}\cos\frac{\varphi-\psi}{2}\ ,\\ u_4&=\sin\frac{\theta}{2}\sin\frac{\varphi-\psi}{2}\ , \end{aligned} \tag{2.1}$$

where the ranges of the angles are given by

$$0\le\theta\le\pi,\quad 0\le\varphi\le 2\pi,\quad -2\pi\le\psi\le 2\pi\ .$$

The free Lagrangian for a particle moving on the unit SU(3) sphere with "rotational inertia" I is given as

$$L=\frac{I}{2}\dot{\bar{u}}^2=\frac{I}{2}(\dot u_1^2+\dot u_2^2+\dot u_3^2+\dot u_4^2)\ . \tag{2.2}$$

The kernel connecting the points $\bar u_{j-1}$ and $\bar u_j$ which are separated by a very small time interval $t_j-t_{j-1}=\epsilon$ can be approximated by

$$K_j=e^{iA_j}\ , \tag{2.3}$$

where the action for the short time interval is

$$A_j=\frac{I}{2\epsilon}(\bar u_j-\bar u_{j-1})^2=\frac{I}{\epsilon}(1-\cos\Theta_j) \tag{2.4}$$

with $\cos\Theta_j=\bar u_{j-1}\cdot\bar u_j$. Note that Θ_j is the angle that rotates the vector $\bar u_{j-1}$ onto $\bar u_j$ around an axis perpendicular to the "plane" defined by $\bar u_{j-1}$ and $\bar u_j$. To find the relation of Θ_j with the Euler angles of $\bar u_{j-1}$ and $\bar u_j$, we recall the correspondence between the vectors $\bar u$ and the SU(2) rotation matrices U:

$$U=\begin{bmatrix} u_1+iu_2 & iu_3-u_4 \\ iu_3+u_4 & u_1-iu_2 \end{bmatrix}$$

$$=\begin{bmatrix} \cos\frac{\theta}{2}e^{i(\varphi+\psi)/2} & i\sin\frac{\theta}{2}e^{i(\varphi-\psi)/2} \\ i\sin\frac{\theta}{2}e^{-i(\varphi-\psi)/2} & \cos\frac{\theta}{2}e^{-i(\varphi+\psi)/2} \end{bmatrix} . \quad (2.5)$$

It can be directly verified that the scalar product of two vectors $\bar{u}_a, \bar{u}_b$ is given by

$$\bar{u}_a\cdot\bar{u}_b=\tfrac{1}{2}\mathrm{tr}U_aU_b^{-1}$$
$$\equiv\tfrac{1}{2}\,\mathrm{tr}U=\cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2} , \quad (2.6)$$

where angles θ,φ,ψ corresponding to the rotation U depend on the Euler angles of the rotations U_a and U_b^{-1} by[8]

$$\cos\theta=\cos\theta_a\cos\theta_b+\sin\theta_a\sin\theta_b\cos(-\psi_b+\psi_a) , \quad (2.7a)$$

$$e^{i\varphi}=\frac{e^{i\varphi_a}}{\sin\theta}[\sin\theta_a\cos\theta_b-\cos\theta_a\sin\theta_b\cos(-\psi_b+\psi_a)-i\,\sin\theta_b\sin(-\psi_b+\psi_a)] , \quad (2.7b)$$

$$e^{i(\varphi+\psi)/2}=\frac{1}{\cos(\theta/2)}\left[\cos\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\exp\left[\frac{i}{2}[(-\psi_b+\varphi_a)+(-\varphi_b+\psi_a)]\right]\right.$$
$$\left.+\sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\exp\left[\frac{i}{2}[(\varphi_a+\psi_b)-(\psi_a+\varphi_b)]\right]\right] . \quad (2.7c)$$

On the other hand, since any matrix U of SU(2) can be brought into diagonal form by a similarity transformation,

$$\delta=\begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{bmatrix}=U'UU^{1-1};\quad U'\in\mathrm{SU}(2) . \quad (2.8)$$

Using the trace property

$$\mathrm{tr}U=\mathrm{tr}(U'^{-1}\delta U')=\mathrm{tr}\delta ,$$

we arrive at

$$\cos\frac{\alpha}{2}=\cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2} . \quad (2.9)$$

The geometrical meaning of α is that it is equal to the angle of rotation which corresponds to the matrix U.[8] Thus it is the same as the angle Θ_j defined by Eq. (2.4), that is,

$$\cos\frac{\Theta_j}{2}=\cos\frac{{}_j\theta_{j-1}}{2}\cos\frac{{}_j\varphi_{j-1}+{}_j\psi_{j-1}}{2} , \quad (2.10)$$

where ${}_j\theta_{j-1}$, ${}_j\varphi_{j-1}$, ${}_j\psi_{j-1}$ are related to the Euler angles of the vectors $\bar{u}_{j-1}$ and $\bar{u}_j$ according to Eqs. (2.7).

Now we can go back to Eq. (2.3) and rewrite it as

$$K_j=\exp\left[i\frac{I}{\epsilon}(1-\cos\Theta_j)+i\frac{\epsilon}{8I}\right] . \quad (2.11)$$

Here the extra term $\epsilon/8I$ in the action is the well-known ordering correction in the "polar" coordinate Θ representation of the path integral, which replaces the square of the conjugate momentum $P_\Theta{}^2$ by $P_\Theta{}^2-\frac{1}{4}$ in the Hamiltonian path integral.[5] If we expand K_j up to $O'(\epsilon^2)$ for $\epsilon\to 0$, by using the formula[5]

$$e^{(v/\epsilon)\cos\delta}\simeq\frac{\epsilon}{2v}\sum_{k=-\infty}^{\infty}\exp\left[ik\delta+\frac{v}{\epsilon}-\frac{\epsilon}{2v}(k^2-\tfrac{1}{4})\right] ,$$

we obtain

$$K_j=\frac{i\epsilon}{2I}e^{i\epsilon/8I}\sum_{k_j=-\infty}^{\infty}e^{-(i\epsilon/2I)(k_j{}^2-\frac{1}{4})}e^{ik_j\Theta_j} , \quad (2.12)$$

which can also be written, by changing the order of the summation, as

$$K_j\simeq\frac{i\epsilon}{2I}e^{i\epsilon/8I}\sum_{l_j=0}^{\infty}\left[\exp\left[-\frac{i\epsilon}{2I}(l_j{}^2-\tfrac{1}{4})\right]-\exp\left[-\frac{i\epsilon}{2I}[(l_j+1)^2-\tfrac{1}{4}]\right]\right]$$
$$\times\sum_{m_j=-l_j}^{l_j}e^{im_j\Theta_j} . \quad (2.13)$$

By using the formula[9]

$$e^{ax}-e^{bx}=(a-b)xe^{(a+b)x/2}\prod_{s=1}^{\infty}\left[1+\frac{(a-b)^2x^2}{4s^2\pi^2}\right] ,$$

we can write the difference of the two exponentials in Eq. (2.13) as

$$\exp\left[-\frac{i\epsilon}{2I}(l_j{}^2-\tfrac{1}{4})\right]-\exp\left[-\frac{i\epsilon}{2I}[(l_j+1)^2-\tfrac{1}{4}]\right]$$
$$=(2l_j+1)\frac{i\epsilon}{2I}e^{-(i\epsilon/2I)(l_j+\frac{1}{2})^2}\prod_{s=1}^{\infty}\left[1-\frac{(2l_j+1)\epsilon^2}{16I^2\pi^2s^2}\right]$$
$$=(2l_j+1)\frac{i\epsilon}{2I}e^{-(i\epsilon/2I)(l_j+\frac{1}{2})^2}\frac{\sin[\epsilon(2l_j+1)^{1/2}/4I]}{[\epsilon(2l_i+1)^{1/2}/4I]}$$
$$=(2l_j+1)\frac{i\epsilon}{2I}e^{-(i\epsilon/2I)(l_j+\frac{1}{2})^2}\quad\text{as }\epsilon\to 0 . \quad (2.14)$$

In the second step of the last equation we have used the product formula[10]

$$\prod_{s=1}^{\infty}\left[1-\frac{z^2}{s^2}\right]=\frac{\sin\pi z}{\pi z} .$$

The summation over m_j in Eq. (2.13) is, on the other hand, just the character of the SU(2) matrix representation with weight l, and can be expanded in terms of the SU(2) matrix elements,[8]

$$\sum_{m_j=-l_j}^{l_j} e^{im_j\Theta_j}=\sum_{m_j=-l_j}^{l_j} e^{-im_j({}_j\varphi_{j-1}+{}_j\psi_{j-1})}P^{l_j}_{m_jm_j}(\cos_j\theta_{j-1}) .$$

We can further expand it by using the addition theorem for the SU(2) representations:

$$\sum_{m_j=-l_j}^{l_j} e^{im_j\Theta_j}=\sum_{m_j=-l_j}^{l_j}\sum_{n_j=-l_j}^{l_j} e^{im_j(\varphi_{j-1}-\varphi_j)}e^{-in_j(\psi_{j-1}-\psi_j)}P^{l_j}_{m_jn_j}(\cos\theta_{j-1})P^{l_j}_{n_jm_j}(\cos\theta_j) . \tag{2.15}$$

Inserting Eqs. (2.14) and (2.15) into Eq. (2.13), we obtain for the short-time-interval kernel

$$K_j=\left[\frac{i\epsilon}{2I}\right]^2\sum_{l_j=0}^{\infty}\sum_{m_j=-l_j}^{l_j}\sum_{n_j=-l_j}^{l_j}(2l_j+1)e^{-(i\epsilon/2I)l_j(l_j+1)}e^{-im_j(\varphi_{j-1}-\varphi_j)}e^{-in_j(\psi_{j-1}-\psi_j)}P^{l_j}_{m_jn_j}(\cos\theta_{j-1})P^{l_j}_{n_j,m_j}(\cos\theta_j) . \tag{2.16}$$

We can now introduce the finite-time-interval kernel in the usual time-graded path-integral form:

$$K(\bar{u}_a,\bar{u}_b;T)=\lim_{\substack{n\to\infty\\ \epsilon\to 0}}\frac{1}{16\pi^2}\prod_{j=1}^{N+1}\left[\frac{2I}{i\epsilon}\right]^2\int\prod_{j=1}^{N}\left[\frac{1}{16\pi^2}\sin\theta_j d\theta_j d\varphi_j d\psi_j\right]\prod_{j=1}^{N+1}(K_j) , \tag{2.17}$$

where

$$d\Omega=\frac{1}{16\pi^2}\sin\theta\, d\theta\, d\varphi\, d\psi$$

is the invariant volume element of SU(2), $\bar{u}_a$ and $\bar{u}_b$ are the initial and final points corresponding to $j=0$ and $j=N+1$, respectively, and $T=(N+1)\epsilon$ is the total time interval.

Note that Eq. (2.17) could also be obtained by employing the usual path integral in Cartesian coordinates with the constraint $\sqrt{\bar{u}^{\,2}}=1$, using the method followed by Marinov and Terentyev[6] for the path integrations over the SO(n) manifold,

$$K(\bar{u}_b,\bar{u}_a;T)=\lim_{\substack{N\to\infty\\ \epsilon\to 0}}\int_0^{\infty}du_a\delta(u_a-1)\int_{-\infty}^{\infty}\prod_{j=1}^{N+1}\left[\frac{\epsilon}{2\pi}d\lambda_j\right]$$
$$\times\int_{-\infty}^{\infty}\prod_{j=1}^{N+1}\left[\left[\frac{I}{2\pi i\epsilon}\right]^{1/2}d^4u_j\right]$$
$$\times\exp\left[i\prod_{j=1}^{N+1}\left[\frac{2I}{\epsilon}(\bar{u}_j-\bar{u}_{j-1})^2-\lambda_j(\sqrt{\bar{u}_j^{\,2}}-1)\right]\right] . \tag{2.18}$$

From this last equation we can see that the measure of Eq. (2.17) is correct.

By virtue of the normalization relation

$$\frac{1}{16\pi^2}\int_0^{\pi}d\theta_j\sin\theta_j\int_0^{2\pi}d\varphi_j\int_{-2\pi}^{2\pi}d\psi_j e^{im_j\varphi_j+in_j\psi_j}e^{-i(m_{j+1}\varphi_j+n_{j+1}\psi_j)}$$
$$\times P^{l_j}_{n_jm_j}(\cos\theta_j)P^{l_{j+1}}_{m_{j+1}n_{j+1}}(\cos\theta_j)=\frac{1}{2l_j+1}\delta_{l_jl_{j+1}}\delta_{m_jm_{j+1}}\delta_{n_jn_{j+1}} ,$$

the integrals over

$$\prod_{j=1}^{N}(\sin\theta_j d\theta_j d\varphi_j d\psi_j)$$

can be calculated, and we end up with the result

$$K(\bar{u}_b,\bar{u}_a;T)=\frac{1}{16\pi^2}\sum_{l=0}^{\infty}(2l+1)e^{-(iT/2I)l(l+1)}\sum_{m=-l}^{l}\sum_{n=-l}^{l}e^{-i(m\varphi_a+n\psi_a)}e^{i(m\varphi_b+n\psi_b)}P^l_{mn}(\cos\theta_a)P^l_{nm}(\cos\theta_b) , \tag{2.19}$$

which displays the properly normalized wave functions

$$\psi^l_{mn}(\theta,\varphi,\psi)=\frac{(2l+1)^{1/2}}{4\pi}e^{-im\varphi}e^{-in\psi}P^l_{mn}(\cos\theta) \quad (2.20)$$

and the energy spectrum

$$E=\frac{1}{2I}l(l+1)\ . \quad (2.21)$$

To obtain a more compact expression for the kernel of Eq. (2.19), we sum over n by using Eq. (2.15), and arrive at the expression

$$K(\bar{u}_b,\bar{u}_a;T)=\frac{1}{16\pi^2}\sum_{l=0}^{\infty}(2l+1)e^{-(iT/2I)l(l+1)} \times \sum_{m=-l}^{l} e^{im\Theta_{ab}}\ , \quad (2.22)$$

where Θ_{ab} is defined by

$$\cos\frac{\Theta_{ab}}{2}=\cos\frac{{}_a\theta_b}{2}\cos\frac{{}_a\varphi_b+{}_a\psi_b}{2}$$

and ${}_a\theta_b$, ${}_a\varphi_b$, ${}_a\psi_b$ depend on the angles at the points $\bar{u}_a$ and $\bar{u}_b$ according to the relations of Eqs. (2.7). We can further simplify Eq. (2.22) by calculating the summation over m by using

$$\sum_{m=-l}^{l} e^{im\Theta_{ab}}=\frac{\sin(l+\frac{1}{2})\Theta_{ab}}{\sin(\Theta_{ab}/2)}$$

and obtain

$$K(\bar{u}_b,\bar{u}_a;T)=\frac{1}{16\pi^2\sin(\Theta_{ab}/2)} \times\sum_{l=0}^{\infty}(2l+1)e^{-(iT/2I)l(l+1)}\sin[(l+\tfrac{1}{2})\Theta_{ab}] \quad (2.23)$$

or

$$K(\bar{u}_b,\bar{u}_a;T)=-\frac{e^{iT/8I}}{16\pi^2\sin(\Theta_{ab}/2)}\frac{\partial}{\partial\Theta_{ab}} \times\theta_2\left[\frac{\Theta_{ab}}{2\pi},-\frac{T}{2\pi I}\right]\ , \quad (2.24)$$

where

$$\theta_2=2\sum_{l=0}^{\infty}e^{-i(T/2I)(l+\frac{1}{2})^2}\cos[(l+\tfrac{1}{2})\Theta_{ab}]$$

is the Jacobi theta function.[11] Expression of Eq. (2.24) is the same as the one obtained by Schulman.[12] If we insert the following form of the theta function

$$\theta_2=\sum_{l=-\infty}^{\infty}\left[\frac{2\pi I}{iT}\right]^{1/2}(-)^l\exp\left[2i\pi^2\left[\frac{\Theta_{ab}}{2\pi}+l\right]^2 I/T\right]$$

into Eq. (2.24), we get

$$K(\bar{u}_b,\bar{u}_a;T)=-\frac{e^{iT/8I}}{4\sin(\Theta_{ab}/2)}\left[\frac{I}{2\pi iT}\right]^{3/2}\sum_{l=-\infty}^{\infty}(-)^l(\Theta_{ab}+2\pi l)\exp\left[i\frac{I}{2T}(\Theta_{ab}+2\pi il)^2\right]\ , \quad (2.25)$$

which is equal to the sum over the classical paths. Similarly, the short-time propagator of Eq. (2.16) is expressible in the form of Eq. (2.25) with Θ_{ab} and T replaced by Θ_j and ϵ; and, as $\epsilon\rightarrow 0$ only the term with $l=0$ contributes:

$$K_j=-\frac{\Theta_i}{4\sin\Theta_j/2}e^{i\epsilon/8I}\left[\frac{I}{2\pi i\epsilon}\right]^{3/2}e^{i(\Theta_j/2\epsilon)I}\ ,$$

which is equal to the formula obtained by WKB approximations.[12] These equalities are in agreement with the work of Dowker who showed that the quasiclassical approximation is exact for path integrals on simple Lie groups.[13]

III. PATH INTEGRAL FOR POSCHL-TELLER POTENTIAL

In 1983 Pöschl and Teller introduced the potential[14]

$$V(x)=\frac{1}{2\mu}\left[\frac{K(K-1)}{\sin^2x}+\frac{\lambda(\lambda-1)}{\cos^2x}\right];\ K,\lambda>1\ . \quad (3.1)$$

It has been recently restudied by Nieto,[15] together with some other potentials and the normalization of its wave functions was found. Since the potential has infinite barriers at each $x=\pm n\pi/2$, $n=0,1,2,\ldots$, it is sufficient to study it in the range of $0\leq x\leq\pi/2$.

We start with the phase-space path integral

$$K(x_b,x_a;T) =\int_{[0]}^{[\pi/2]}\mathscr{D}x\frac{\mathscr{D}p_x}{[2\pi]}\exp\left[i\int_0^T dt\left[p_x\dot{x}-\frac{p_x{}^2}{2\mu}-V(x)\right]\right]\ , \quad (3.2)$$

which is as usual understood as the $\epsilon\rightarrow 0$, $N\rightarrow\infty$ limit of the time-graded definition of the measure

$$\int_{[0]}^{[\pi/2]}\mathscr{D}x\,\mathscr{D}p_x/[2\pi] =\int_0^{\pi/2}\prod_{j=1}^{N}dx_j\int_{-\infty}^{\infty}\prod_{j=1}^{N+1}\frac{dp_{xj}}{2\pi} \quad (3.3)$$

and of the action

$$A=\sum_{j=1}^{N+1}\epsilon[p_{xj}(x_j-x_{j-1})/\epsilon-p_{xj}/2\mu-V(x_j)] \quad (3.4)$$

with

$$x_a=x_0,\ \ x_b=x_{N+1},\ \ T=(N+1)\epsilon\ .$$

If we introduce the variable θ by

$$\theta=2x$$

the kernel of Eq. (3.2) becomes

$$K(x_b,x_a;T)=2\int_0^{[\pi]}\mathscr{D}\theta\frac{\mathscr{D}p_\theta}{[2\pi]}\exp\left[i\int_0^T dt\left[p_\theta\dot{\theta}-\frac{p_\theta{}^2}{2I}-\frac{\alpha^2+\beta^2-2\alpha\beta\cos\theta-\frac{1}{4}}{2I\sin^2\theta}\right]\right], \tag{3.5}$$

where

$$\alpha\equiv(\lambda+K-1)/2,\quad \beta\equiv(\lambda-k)/2,\quad I\equiv\mu/4\ , \tag{3.6}$$

and the factor 2 multiplying the expression comes from the $dp_{x_{N+1}}\rightarrow dp_{\theta_{N+1}}$ transformation. The Hamiltonian in the action of Eq. (3.5) immediately reminds one of the Hamiltonian for the "SU(2) rotator" written in terms of the Euler angles of Eq. (2.1):

$$H_{\mathrm{SU}(2)}=\frac{1}{2I}\left[p_\theta{}^2+\frac{p_\varphi{}^2+p_\psi{}^2-2p_\theta p_\psi\cos\theta}{\sin^2\theta}\right].$$

Motivated by this resemblance we introduce the following identity, which can be proven by direct calculation:[16]

$$\int_0^{2\pi}d\varphi_b e^{-i\alpha(\varphi_b-\varphi_a)}\int_{-2\pi}^{2\pi}d\psi_b e^{-i\beta(\psi_b-\psi_a)}\int_0^\infty du_a\delta(u_b-1)\int_0^\infty du_b$$
$$\times\int\prod_{j=1}^{N}(du_j d\varphi_j d\psi_j)\int_{-\infty}^{\infty}\prod_{j=1}^{N+1}\left[\frac{1}{(2\pi)^3}dp_{u_j}dp_{\varphi_j}dp_{\psi_j}\right]$$
$$\times\int_{-\infty}^{\infty}\left[\frac{\epsilon}{2\pi}d\lambda_j\right]\exp\left[i\int_0^T dt\left[p_u\dot{u}+p_\varphi\dot{\varphi}+p_\psi\dot{\psi}-\frac{p_u{}^2}{2Iu^2}\right.\right.$$
$$\left.\left.-\frac{p_\varphi{}^2+p_\psi{}^2-2p_\varphi p_\psi\cos\theta}{2I\sin^2\theta}-\lambda(u-1)\right]\right]$$
$$=\exp\left[i\int_0^T dt\left[-\frac{\alpha^2+\beta^2-2\alpha\beta\cos\theta}{2\sin^2\theta}\right]\right], \tag{3.7}$$

where the ranges of the variables are

$$0\le\varphi\le 2\pi,\quad -2\pi\le\psi\le 2\pi,\quad 0\le u<\infty\ .$$

Using this identity, we can write Eq. (3.5) as

$$K(x_b,x_a;T)=2\int_0^{2\pi}d\varphi_b\int_{-2\pi}^{2\pi}d\psi_b\exp[-i\alpha(\varphi_b-\varphi_a)-i\beta(\psi_b-\psi_a)]e^{-iT/8I}(\sin\theta_a\sin\theta_b)^{1/2}\mathsf{K} \tag{3.8}$$

with K defined as

$$\mathsf{K}(b,a;T)=(\sin\theta_a\sin\theta_b)^{-1/2}\int_0^\infty\int_0^\infty du_a du_b\delta(u_b-1)$$
$$\times\int\mathscr{D}(u\theta\varphi\psi)\frac{\mathscr{D}(p_\theta p_\varphi p_\psi p_u)}{[2\pi]^4}\frac{\mathscr{D}\lambda}{[2\pi]}$$
$$\times\exp\left[i\int_0^T dt\left[p_u\dot{u}+p_\theta\dot{\theta}+p_\varphi\dot{\varphi}+p_\psi\dot{\psi}\right.\right.$$
$$-\frac{p_u{}^2}{2I}-\frac{p_\theta{}^2-\frac{1}{4}}{2Iu^2}-\frac{p_\varphi{}^2+p_\psi{}^2-2p_\varphi p_\psi\cos\theta-\frac{1}{4}}{2Iu^2\sin\theta}$$
$$\left.\left.-\lambda(u-1)\right]\right], \tag{3.9}$$

which is the phase-space form of the path integral given by Eq. (2.18), and which describes the motion of a particle with "rotational inertia" I, on the surface of the SU(2) sphere. In writing Eq. (3.9) we have inserted

$$1=e^{-iT/8I}(\sin\theta_a\sin\theta_b)^{1/2}(\sin\theta_a\sin\theta_b)^{-1/2}\exp\left[i\int_0^T\frac{dt}{8I}\right].$$

Note that the four-dimensional volume element in terms of the Euler angles

$$d^4u=u^3du\,\sin\theta\,d\theta\,d\varphi\,d\psi\ ,$$

and the usual three-dimensional one in polar coordinates

$$d^3r=r^2dr\,\sin\theta\,d\theta\,d\varphi$$

have the same Jacobian factor $\sin\theta$, for $u=1$, $r=1$ constraints. Thus the "ordering" contributions to the action, i.e., the $(8Iu^2)^{-1}$ and $(8Iu^2\sin^2\theta)^{-1}$ terms, are the same in both cases.[5]

We can directly write the expression of K of Eq. (3.8) from Eq. (2.19), and obtain

$$K(x_b,x_a;T)=\frac{2}{16\pi^2}\int_0^{2\pi}d\varphi_b\int_{-2\pi}^{2\pi}d\psi_b e^{-i\alpha(\varphi_b-\varphi_a)}e^{-i\beta(\psi_b-\psi_a)} \times(\sin\theta_a\sin\theta_b)^{1/2}\sum_{l=0}^{\infty}(2l+1)\exp\left[-\frac{iT}{2I}[l(l+1)+\tfrac{1}{4}]\right] \times\sum_{m=-l}^{l}\sum_{n=-l}^{l}e^{im(\varphi_b-\varphi_a)}e^{in(\psi_b-\psi_a)}p^l_{mn}(\cos\theta_a)p^l_{mn}(\cos\theta_b)\ , \quad (3.10)$$

which gives after integrations over $d\varphi_b$ and $d\psi_b$

$$K(x_b,x_a;T)=(\sin\theta_a\sin\theta_b)^{1/2}\sum_{l=\max(|\alpha|,|\beta|)}^{\infty}(2l+1)e^{-i(T/2I)(l+\frac{1}{2})^2}P^l_{\alpha\beta}(\cos\theta_a)P^l_{\beta\alpha}(\cos\theta_b)\ . \quad (3.11)$$

Since λ and K are both positive, $|\alpha|=|(K+\lambda-1)/2|$ is larger than $|\beta|=|\frac{1}{2}(\lambda-K)|$, thus we can write l as

$$l=\tfrac{1}{2}(K+\lambda-1)+n;\quad n=0,1,2,\ldots \quad (3.12)$$

and arrive at the expression, with $I=\mu/4$ and $x=\theta/2$:

$$K(x_b,x_a;T)=\sum_{n=0}^{\infty}e^{-(iT/2\mu)(K+\lambda+2n)^2} \times\psi^n(x_a)\bar{\psi}^n(x_b)\ , \quad (3.13)$$

where

$$E_n=\frac{1}{2\mu}(K+\lambda+2n)^2 \quad (3.14)$$

and

$$\psi^n(x)=[2(\sin\theta\cos\theta)(K+\lambda+2n)]^{1/2} \times P^{(k+\lambda-1)/2+n}_{(k+\lambda-1)/2,(\lambda-k)/2}(1-2\sin^2x) \quad (3.15)$$

are the correct energy spectrum and wave functions for the Pöschl-Teller potential. By using the relation between the SU(2) reduced matrix elements $P^l_{mn}(\cos\theta)$ and the Jacobi polynomials $P^{(m-n,m+n)}_{l-m}(\cos\theta)$,[8] we obtain the more familiar and properly normalized wave functions given by Nieto:[15]

$$\psi^n(x)=[2(k+\lambda+2n)]^{1/2} \times\left[\frac{\Gamma(n+1)\Gamma(K+\lambda+n)}{\Gamma(K+n+\frac{1}{2})\Gamma(\lambda+n+\frac{1}{2})}\right]^{1/2} \times(\cos x)^{\lambda}(\sin x)^{K}P_n^{(K-1/2,\lambda-1/2)}(1-2\sin^2x)\ . \quad (3.16)$$

IV. CONCLUSIONS

In this paper we have studied the path integrals for the quantum-mechanical problems which have SU(2) symmetry. We did this by expanding the short-time-interval Green's function for the particle moving in the parameter space of SU(2) in terms of the matrix elements of the representations. The resulting Green's function then becomes available for direct use in solving the path integral for the Pöschl-Teller potential.

The SU(2)-group manifold S^3 considered in this work coincides with the quotient space $SO(4)/SO(3)\sim S^3$. However, the Green's functions for S^n may also be constructed using $SO(n+1)/SO(n)\sim S^n$.[6]

It is well known that all of the special functions which appear as solutions of problems in theoretical physics are the matrix elements of representations of some Lie groups.[17] Because of this fact, if one first parametrizes the problems suitably with their symmetries, it may be possible to relate their path integrals to the ones written for the motions on the appropriate group spaces. Working out the path integrals over the manifolds of Lie groups is thus important. It is also rather time saving and a technically easier task, because it allows one to employ several group properties.

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[1]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[2]I. H. Duru and H. Kleinert, Phys. Lett. **84B**, 185 (1979); Forschr. Phys. **30**, 401 (1982).

[3]I. H. Duru, Phys. Rev. D **28**, 2689 (1983).

[4]See, for example, A. O. Barut and H. Beker, Phys. Rev. Lett. **50**, 1560 (1983).

[5]D. Peak and A. Inomata, J. Math. Phys. **10**, 1422 (1969).

[6]M. S. Marinov and M. V. Terentyev, Fortschr. Phys. **27**, 511 (1979).

[7]I. H. Duru, ICTP, Trieste, Report No. IC/83/178 (unpublished).

[8]N. Ja. Vilenkin, *Special Functions and the Theory of Group Representation* (American Mathematical Society, Providence, Rhode Island, 1968).

[9]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

[10]E. R. Hansen, *A Table of Series and Products* (Prentice-Hall, Englewood Cliffs, N. J., 1975).

[11]W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. (Springer, Berlin, 1966).

[12]L. Schulman, Phys. Rev. **176**, 1558 (1968).

[13]J. S. Dowker, Ann. Phys. (N.Y.) **62**, 361 (1971).

[14]G. Pöschl and E. Teller, Z. Phys. **83**, 143 (1933).

[15]N. M. Nieto, Phys. Rev. A **17**, 1273 (1978).

[16]For path integrals over angular variables, see V. N. Popov, CERN Report No. TH2424, 1977 (unpublished).

[17]See, for example, in addition to Ref. 8, J. D. Talman, *Special Functions: A Group Theoretic Approach* (Benjamin, New York, 1968); W. Miller, *American Mathematical Society Memoir No. 50* (American Mathematical Society, Providence, Rhode Island, 1964).