Ghost-free, nonlinear, spin-two, conformal gauge theory

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(Received 1 May 1984)

Conformal gravity is examined with the aim of constructing a renormalizable quantum field theory of gravitation. Acting on suggestions of ghost-free, linear conformal gravity, we examine conformal gauge theory. The correct choice of constraints is determined by comparison with the ghost-eliminating constraints of the linear theory. (These are quite different from those introduced previously in conformal gauge theories.) Einstein's field equation appears as the integrability condition for the constraint. Minimal coupling to massless matter preserves local, conformal symmetry.

I. INTRODUCTION

A. Purpose

The ultimate goal of this work is the reconciliation between quantum theory and gravitation.

The conflict between quantum theory and gravitation consists of the fact that general relativity, conventionally quantized, is not amenable to known procedures of renormalization. Although the theory is successful, within the rather limited range of experimental possibilities, an infinite multitude of renormalization constants robs it of predictive power in the strict sense. Nor is it advisable to take the success of low-order perturbation theory as an excuse for ignoring the problem. Recent progress in the electroweak domain shows dramatically how far one can go by paying attention to matters of principle.

B. Conformal invariance

It has often been claimed that conformally invariant field theories are "manifestly" renormalizable. It is a fact that the interesting renormalizable field theories—those that describe particles with spins exceeding one half, are smooth deformations of massless, conformally invariant theories. The Weinberg-Salam electroweak theory is of course the most conspicuous example. Note that there are probably no quantum field theories in Minkowski space that are truly conformally invariant. But there are classical field theories (in compactified Minkowski space) that are conformally invariant; and these, when carefully deformed and quantized, as by the Higgs-Kibble mechanism, sometimes turn out to be renormalizable.[1]

Einstein's classical theory of gravitation[2] in the absence of massive particles, describes only massless fields, yet it is not conformally invariant. The attempt made in this paper is inspired by the idea that prospects for renormalizability can be improved by taking a conformally invariant classical field theory as the point of departure, whence a realistic quantum field theory may be achieved by smooth deformation and quantization. Our approach is characterized by our insistence on maintaining conformal invariance as far as possible.

C. Unitarity

Up to now, conformal theories of gravitation have been plagued by ghosts. It is noteworthy, however, that this difficulty was understood only in the context of quantum field theories. It seems likely that classical field theories that violate unitarity upon quantization must be unhealthy to begin with, that is, even when they are considered as purely classical field theories. If so, it seems to have gone unrecognized. The early rejection of Weyl's conformal theory of gravitation[3] was not related to its spectral qualities, but was based instead on its failure to account for low-energy, observed gravitational phenomena. This theory would therefore have to undergo a radical deformation to make it phenomenologically relevant.

Utiyama and DeWitt[4] may have been the first to propose an Einstein-Weyl compromise. The action is a linear combination of Weyl's, conformally invariant and dominant at high energies to ensure renormalizability, and of Einstein's, to explain the observed gravitational effects, all of which are low-energy phenomena. (Utiyama and DeWitt started with Einstein's action and generated the Weyl term through renormalization. The reverse is also possible, as in some versions of induced gravity.[5]) It has been stressed that renormalizability of Weyl's theory is related less to its conformal invariance than to the simple fact that the propagator has dimension 4. The investigations by Stelle[6] confirm this, but unfortunately he found that this very feature of the theory is also responsible for the ghost. Ghosts can be avoided only at the very high price of admitting tachyons, thereby abandoning the principle that particles must have positive energy.

There is general agreement that unitarity (absence of ghosts) is a necessary feature of any viable physical theory. What distinguishes our approach is the use of this requirement as a constructive principle. What is natural and indeed standard procedure in quantum theory, both in nonrelativistic quantum mechanics and in relativistic field theories, seems not to have been applied systematically to the problem of conformal gravity.

D. The linear theory

In Ref. 7, unitarity and conformal invariance were used as constructive tools, to build a field theory with massless quanta with helicities ± 2. The result is a specific theory of free fields, together with precise requirements to be imposed on the sources (whether external or internal). We do not insist on the uniqueness of this linear theory, nor have we made an exhaustive analysis of alternatives, for destruction, on the basis of criteria that are not absolutely established, is even riskier than construction. If an acceptable alternative to the linear theory of Ref. 7 (henceforth " the linear theory") should be found, then it should be taken as the foundation for another constructive attempt along the lines of this paper.

The linear theory is here understood as a free, classical canonical field theory. It was quantized along the lines of the Gupta-Bleuler indefinite-metric quantization program.[7] The free quantum field operator was constructed from a space of free field modes that carries a nondecomposable representation of the conformal group, characterized by the following two essential requirements: (1) Inclusion as subquotients of the physical, massless, unitary irreducible representations with helicities ± 2 and (2) existence of a nondegenerate symplectic form, necessary for invariant quantization. The smallest representation of this type that can be realized on a space of field modes is the sum of two helicity-conjugate representations, one of which is the following Gupta-Bleuler triplet:

$$D(0,\tfrac{1}{2},\tfrac{3}{2})\rightarrow\begin{Bmatrix} D(3,2,0)\\ \oplus D(1,0,2)\\ \oplus D(-1,0,1)\end{Bmatrix}\rightarrow D(0,\tfrac{1}{2},\tfrac{3}{2}) .$$

Here $D(E_0,j_1,j_2)$ is a standard notation for an irreducible, positive-energy representation of so(4,2). The arrows have the following meaning. If A and B are representations, then $A\rightarrow B$ is a nondecomposable extension of B by A, in which B is an invariant submodule and A appears on the associated quotient. (The arrow represents the cochain map, or "leak," from A to B.) The representation on the left is analogous to the subquotient of scalar modes in electrodynamics; these modes are eventually eliminated by a gauge-fixing constraint that is analogous to the Lorentz condition. The representation on the right is carried by the invariant subspace of gauge fields; these decouple if the sources satisfy appropriate conservation laws. Next, $D(3,2,0)$ appears on a subquotient that is identified with the physical, propagating modes. The two remaining central subquotients are nonunitary; one is finite dimensional and perhaps not very important; the other, $D(1,0,2)$, is infinite dimensional and is the one that we shall refer to as the ghost. (We do not apply this epithet to the relatively innocuous scalar and gauge modes.) The ghost-suppressing constraints can be expressed as a differential equation for the classical field. It has nothing to do with gauge fixing, for it is completely gauge invariant in the sense of the local, Abelian gauge group. Among the modes used to define the quantum field operator, those of the submodule $D(3,2,0)\rightarrow D(0,\frac{1}{2},\frac{3}{2})$ satisfy the constraint, so the gauge modes are not restricted by it. In quantum field theory the constraint must be imposed as an initial condition on the physical states, in the manner of the Lorentz condition in QED.

The linear theory is thus characterized, not only by an action principle, but by a gauge-invariant constraint. In fact, the latter is stronger than the Euler-Lagrange equations and actually contains the essential dynamics. The program of this paper is to find a nonlinear theory of interacting fields that reduces to the linear theory just described in the first-order (i.e., linearized, free) approximation. The main concern is to generalize the gauge-invariant, ghost-suppressing constraint. There are very clear indications, in the structure of the linear theory, that the nonlinear generalization must be a local conformal gauge theory. Such theories usually need to be constrained; the linear theory will tell us how to choose constraints that eliminate the ghosts.

E. Summary

Section II introduces the local conformal algebra and the associated notions of connection, torsion, curvature, and metric. Section III deals with the problem of expanding around a fixed point in field space and defines the linear approximation. This enables us to establish an intertwining map between the linear approximation to the nonlinear gauge theory on the one hand, and the linear theory on the other. In Sec. IV we use this intertwining map to translate the ghost-eliminating, gauge-invariant constraint of the linear theory into constraints on the first-order torsion, curvature, and metric. These constraints are gauge invariant to lowest order and easily generalized to fully gauge-invariant constraints on the nonlinear conformal gauge theory. Up to this point, all calculations are done in Dirac's projective six-cone notation. In Sec. V we translate these partial results to Minkowski notation and examine the meaning of the constraints to first order. Einstein's linearized field equation appears as the integrability condition for the principal constraint. Weyl theory is seen to result from a wrong choice of constraints. Section VI deals with integrability in the full, nonlinear theory. The field structure is clarified and the nonlinear field equations determined. As in first order, the field equation is the integrability condition for the constraints. Finally, Sec. VII formulates matter coupling and an action principle.

II. LOCAL CONFORMAL GAUGE ALGEBRA

A. Local algebra

Given a Lie algebra $\mathfrak{G}$ acting in a differentiable manifold, we define an associated local algebra. Let $\{l_A\}$, $A=1,\ldots,n$ be a basis for $\mathfrak{G}$, and let $l_A\rightarrow M_A$ be the realization of $\mathfrak{G}$ by the vector fields that determine its action in the manifold. Let $l_A\rightarrow S_A$ be a faithful representation by matrices acting in a vector space V, and consider the realization by operators acting on V-valued functions given by

$$l_A\rightarrow L_A=M_A+S_A,\quad A=1,\ldots,n . \tag{2.1}$$

Finally, let V_L denote the vector space of $\mathfrak{G}$ and define the structure tensor:

$$[l_A,l_B]=C_{AB}^{C}l_C . \tag{2.2}$$

The associated local algebra is the space of V_L-valued differentiable functions $\Lambda=\{\Lambda^A\}$, $A=1,\ldots,n$, with the structure determined by that of the Lie algebra together with the action of the Lie algebra on the manifold:

$$[\Lambda'^A L_A, \Lambda^B L_B] = [\Lambda', \Lambda]^C L_C ,$$
$$[\Lambda', \Lambda]^C = \Lambda'^A \Lambda^B C^C_{AB} + \xi' \Lambda^C - \xi \Lambda'^C , \quad (2.3)$$
$$\xi \equiv \Lambda^A M_A , \quad \xi' \equiv \Lambda'^A M_A .$$

The map $\Lambda \to \xi$ is a homomorphism from the local algebra onto the algebra of differentiable vector fields, the diffeomorphism algebra of the manifold.

Let $l_A \to S_A$ be a faithful representation of the Lie algebra, by matrices acting in a vector space V, and consider V-valued differentiable tensor fields on the manifold. For any Λ in the local algebra, let $\mathscr{L}(\xi)$ denote the usual Lie derivative associated with the vector field $\xi = \Lambda \cdot M$. Then $\Lambda \to \mathscr{L}(\xi)$ is a realization of the local algebra, faithful only on the diffeomorphism algebra. Define the "extended Lie derivative" $\mathscr{L}_\Lambda$ by

$$\mathscr{L}_\Lambda \equiv \mathscr{L}(\xi) + \Lambda \cdot S , \quad \xi \equiv \Lambda \cdot M . \quad (2.4)$$

Then $\Lambda \to \mathscr{L}_\Lambda$ is a faithful realization of the local algebra. From now on the term "tensor" will be reserved for objects that transform this way under the action of the local algebra. If a tensor field is a vector-valued function on the manifold, then it will be called an "internal" tensor field.

B. Covariant derivative

Let the notation remain as above, except that $l_A \to S_A$ need not be faithful, and let there be given in addition a particular representation $l_A \to \widetilde{S}_A$, by matrices acting in a vector space $\widetilde{V}$, fixed from now on. We consider the space of differentiable functions valued in $\widetilde{V} \otimes V_L$:

$$\phi = \{\phi_\alpha{}^A\} , \quad A = 1, \ldots, n ,$$
$$\alpha = 0, 1, \ldots, d , \quad d+1 = \dim \widetilde{V} .$$

The fundamental object of local gauge theory is the covariant derivative Q. It is a map from the space of functions valued in V to the space of functions valued in $V \otimes \widetilde{V}$, and it is determined by the following operators:

$$Q_\alpha = \phi_\alpha{}^A L_A , \quad \alpha = 0, \ldots, d , \quad L_A = M_A + S_A . \quad (2.5)$$

An action of the local algebra on ϕ is defined by $\Lambda \to \delta_\Lambda$,

$$[\mathscr{L}_\Lambda, Q]_\alpha = [\Lambda, Q_\alpha] + \Lambda_\alpha{}^\beta Q_\beta \equiv (\delta_\Lambda \phi)_\alpha{}^A L_A , \quad (2.6)$$

$$\Lambda_\alpha{}^\beta \equiv (\Lambda^A \widetilde{S}_A)_\alpha{}^\beta . \quad (2.7)$$

Explicitly,

$$(\delta_\Lambda \phi)_\alpha{}^A = (\mathscr{L}_\Lambda \phi)_\alpha{}^A - e_\alpha \Lambda^A , \quad (2.8)$$

$$e_\alpha = \phi_\alpha{}^A M_A . \quad (2.9)$$

The last term in (2.8) shows that ϕ is not a tensor; we refer to this term as the "affine" term although δ_Λ is a linear operator. Now Eqs. (2.8) and (2.9) give

$$(\delta_\Lambda e)_\alpha = (\mathscr{L}_\Lambda e)_\alpha \equiv [\xi, e_\alpha] + \Lambda_\alpha{}^\beta e_\beta , \quad (2.10)$$

so that $e = (e_\alpha)$ is a tensor field according to our definition following Eq. (2.4), and more precisely a $\widetilde{V}$-valued vector field.

The term affine as used above may be justified as follows. Consider a coordinate neighborhood, let (x^a) denote coordinates, and write $\partial_a \equiv \partial/\partial x^a$. If Q_α factorizes,

$$Q_\alpha = e_\alpha + \phi_\alpha{}^A S_A = e_\alpha{}^a (\partial_a + \Gamma_a{}^A S_A) , \quad (2.11)$$

then (2.8) can be recovered from (2.10) and

$$(\delta_\Lambda \Gamma)_a{}^A = (\mathscr{L}_\Lambda \Gamma)_a{}^A - \partial_a \Lambda^A .$$

In the first term $\mathscr{L}_\Lambda$ acts on Γ as on Lie-algebra-valued one-forms. The last, affine term is closely associated with the last term in (2.8). Note that we shall not suppose that Q_α factorize as above.

C. Torsion and curvature

Recall that Q maps the space of functions valued in V to the space of functions valued in $V \otimes \widetilde{V}$, and that Q_α acts in the former space. Similarly, $Q \circ Q$ maps the space of functions valued in V to the space of functions valued in $V \otimes \widetilde{V} \otimes \widetilde{V}$, and is determined by operators $Q_{\alpha\beta}$ acting in the former space:

$$Q_{\alpha\beta} = Q_\alpha Q_\beta + (\phi_\alpha{}^A \widetilde{S}_A)_\beta{}^\gamma Q_\gamma . \quad (2.12)$$

Curvature and torsion are vector-valued functions (internal tensors) defined by

$$Q_{\alpha\beta} - Q_{\beta\alpha} = R_{\alpha\beta}{}^A S_A + t_{\alpha\beta}{}^\gamma Q_\gamma , \quad (2.13)$$

with R valued in $(\widetilde{V} \wedge \widetilde{V}) \otimes V_L$ and t valued in $(\widetilde{V} \wedge \widetilde{V}) \otimes \widetilde{V}$.

The formulas are

$$R_{\alpha\beta}{}^C = \phi_\alpha{}^A \phi_\beta{}^B C_{AB}{}^C + e_\alpha \phi_\beta{}^C - e_\beta \phi_\alpha{}^C - c_{\alpha\beta}{}^\gamma \phi_\gamma{}^C , \quad (2.14)$$

$$t_{\alpha\beta}{}^\gamma = c_{\alpha\beta}{}^\gamma + \phi_{\alpha\beta}{}^\gamma - \phi_{\beta\alpha}{}^\gamma , \quad (2.15)$$

with c defined (perhaps not uniquely) by

$$[e_\alpha, e_\beta] = c_{\alpha\beta}{}^\gamma e_\gamma \quad (2.16)$$

and

$$\phi_{\alpha\beta}{}^\gamma \equiv (\phi_\alpha{}^A \widetilde{S}_A)_\beta{}^\gamma . \quad (2.17)$$

[Note that we do not (yet) have a metric in $\widetilde{V}$.]

D. Generalizations

A more general affine realization of the local algebra is possible. Although it is rather trivial, this generalization will become useful later on, so we record the formulas here. We have agreed to reserve the term tensor to objects on which the action of the local algebra is given by $\mathscr{L}_\Lambda$. In view of the appearance of the affine term in its transformation law, let us call ϕ a "connection."

Keep all the above notations, and suppose in addition that $A = (A_\alpha{}^C)$ is an internal tensor with values in the same space as ϕ; that is, in $\widetilde{V} \otimes V_L$, so that $\delta_\Lambda A = \mathscr{L}_\Lambda A$. Then

$$\widetilde{\phi} \equiv \phi - A \quad (2.18)$$

is a new type of connection, transforming as follows:

$$(\delta_\Lambda \widetilde{\phi})_\alpha{}^A = (\mathscr{L}_\Lambda \widetilde{\phi})_\alpha{}^A - e_\alpha \Lambda^A ,$$
$$(\delta_\Lambda e)_\alpha = (\mathscr{L}_\Lambda e)_\alpha . \quad (2.19)$$

The local algebra acts on the pair $(\tilde{\phi},e)$ in the same way that it acts on (ϕ,e). What is new is that $\tilde{\phi}$ and e are not related to each other.

Now let

$$\tilde{Q}_\alpha = e_\alpha + \tilde{\phi}_\alpha{}^A S_A \ , \tag{2.20}$$

and proceed just as before:

$$\tilde{Q}_{\alpha\beta} - \tilde{Q}_{\beta\alpha} = \tilde{R}_{\alpha\beta}{}^A S_A + \tilde{t}_{\alpha\beta}{}^\gamma \tilde{Q}_\gamma \ , \tag{2.21}$$

$$\tilde{R}_{\alpha\beta}{}^C = \tilde{\phi}_\alpha{}^A \tilde{\phi}_\beta{}^B C_{AB}{}^C + e_\alpha \tilde{\phi}_\beta{}^C - e_\beta \tilde{\phi}_\alpha{}^C - c_{\alpha\beta}{}^\gamma \tilde{\phi}_\gamma{}^C \ , \tag{2.22}$$

$$\tilde{t}_{\alpha\beta}{}^\gamma = c_{\alpha\beta}{}^\gamma + \tilde{\phi}_{\alpha\beta}{}^\gamma - \tilde{\phi}_{\beta\alpha}{}^\gamma \ . \tag{2.23}$$

Finally, one may verify that

$$R_{\alpha\beta}{}^C - [(Q_\alpha A)_\beta{}^C - (\alpha,\beta)] = \tilde{R}_{\alpha\beta}{}^C - t_{\alpha\beta}{}^\gamma A_\gamma{}^C - A_\alpha{}^A A_\beta{}^B C_{AB}{}^C \ . \tag{2.24}$$

E. Conformal case

From now on the Lie algebra is the conformal Lie algebra so(4,2). In R^6 we choose coordinates $y=\{y^\alpha\}$, $\alpha=0,\ldots,5$. The pseudo-orthogonal metric is denoted $\delta=(\delta_{\alpha\beta})$ and is defined by

$$y^2 = \delta_{\alpha\beta} y^\alpha y^\beta = (y^0)^2 - (y^1)^2 - \cdots - (y^4)^2 + (y^5)^2 \ ,$$

it will be used to raise and lower greek indices. The Lie algebra so(4,2) acts by $y \to \Lambda y$, where $\Lambda = (\Lambda_\alpha{}^\beta)$ is a real matrix such that $\Lambda_{\alpha\beta} = -\Lambda_{\beta\alpha}$. A basis is obtained by writing $\Lambda = \frac{1}{2}\Lambda^{\alpha\beta} l_{\alpha\beta}$, with $l_{\alpha\beta} = -l_{\beta\alpha}, \alpha,\beta = 0,\ldots,5$. Then the structure is

$$[l_{\alpha\beta}, l_{\gamma\delta}] = \delta_{\beta\gamma} l_{\alpha\delta} - \delta_{\alpha\gamma} l_{\beta\delta} - \delta_{\beta\delta} l_{\alpha\gamma} + \delta_{\alpha\delta} l_{\beta\gamma} \ .$$

The representation denoted $l_A \to \tilde{S}_A$ is understood, from now on, to be just this defining representation of so(4,2). We have $V_L = \tilde{V} \wedge \tilde{V}$.

To adopt our notation to this case it is enough to replace each capital latin index by a pair of greek indices, and insert a factor $\frac{1}{2}$ in the case that summation on a capital index is replaced by summation on a pair of greek indices, since each pair is counted twice. We give explicitly the formulas that involve the structure tensor:

$$R_{\alpha\beta}{}^{\gamma\delta} = \phi_\alpha{}^{\gamma\sigma} \phi_{\beta\sigma}{}^\delta + e_\alpha \phi_\beta{}^{\gamma\delta} - (\alpha,\beta) - c_{\alpha\beta}{}^\sigma \phi_\sigma{}^{\gamma\delta} \ , \tag{2.25}$$

$$\tilde{R}_{\alpha\beta}{}^{\gamma\delta} = \tilde{\phi}_\alpha{}^{\gamma\sigma} \tilde{\phi}_{\beta\sigma}{}^\delta + e_\alpha \tilde{\phi}_\beta{}^{\gamma\delta} - (\alpha,\beta) - c_{\alpha\beta}{}^\sigma \tilde{\phi}_\sigma{}^{\gamma\delta} \ , \tag{2.26}$$

$$R_{\alpha\beta}{}^{\gamma\delta} - [(Q_\alpha A)_\beta{}^{\gamma\delta} - (\alpha,\beta)] = \tilde{R}_{\alpha\beta}{}^{\gamma\delta} - t_{\alpha\beta}{}^\sigma A_\sigma{}^{\gamma\delta} - [A_\alpha{}^{\gamma\sigma} A_{\beta\sigma}{}^\delta - (\alpha,\beta)] \ . \tag{2.27}$$

The notation $-(\alpha,\beta)$ means subtract the preceding terms, after interchanging α and β. The notation (2.17) is consistent with the present one.

Suppose in particular that $a = (a^\alpha)$ is an isotropic internal vector field:

$$a^\alpha a_\alpha = 0 \ , \tag{2.28}$$

and that A is the internal tensor field given by

$$A_\alpha{}^{\beta\gamma} = a^\beta \delta_\alpha{}^\gamma - a^\gamma \delta_\alpha{}^\beta \ . \tag{2.29}$$

In this case cancellations occur in (2.27) and one gets

$$R_{\alpha\beta}{}^{\gamma\delta} - [(Q_\alpha A)_\beta{}^{\gamma\delta} - (\alpha,\beta)] = \tilde{R}_{\alpha\beta}{}^{\gamma\delta} - \tilde{t}_{\alpha\beta}{}^\sigma A_\sigma{}^{\gamma\delta} \ . \tag{2.30}$$

It will turn out that the ghost-eliminating constraint is most directly expressed in terms of this quantity, rather than the curvature itself.

The manifold is Dirac's compactification $\overline{M}$ of Minkowski space M, realized as the projective cone $y^2=0$ in R^6.[8] The projection is given by $y \approx \lambda y$ for $\lambda \in \mathsf{R} - \{0\}$. The vector fields are

$$l_{\alpha\beta} \to M_{\alpha\beta} = y_\alpha \partial_\beta - y_\beta \partial_\alpha \ . \tag{2.31}$$

Here $\partial_\alpha = \partial/\partial y^\alpha$ is not well defined, since it is not tangent to the cone, but $M_{\alpha\beta}$ is nevertheless well defined. (For any differentiable ϕ, $\partial_\alpha \phi$ is defined modulo $y_\alpha \chi$, and this cancels in $M_{\alpha\beta}\phi$. In other words, $M_{\alpha\beta}$ is tangent to the cone.)

All fields are defined on the cone; the projection on $\overline{M}$ is accomplished by fixing the degrees of homogeneity, for example,

$$\hat{N} \phi_\alpha{}^{\beta\gamma} = N \phi_\alpha{}^{\beta\gamma}, \quad \hat{N} \equiv y \cdot \partial = y^\sigma \partial_\sigma \ , \tag{2.32}$$

where N is a real number. Minimal coupling, that is, the generation of interactions by substituting $\partial_\alpha \to Q_\alpha$ in the matter Lagrangian, suggests that $N=-1$. Additional justification for this choice will be found later. For the gauge parameter field Λ, the only fixed degree allowed by the commutation relations is zero:

$$\hat{N} \Lambda^{\alpha\beta} = 0 \ .$$

This is an absolute requirement, and it appeared to be a formidable obstruction against any significant comparison with linear conformal gravity.

The projection introduces some slight changes in the interpretation. Since the manifold is the projective cone, and not the cone itself, $\xi = \Lambda^{\alpha\beta} y_\alpha \partial_\beta$ and $e_\alpha = \phi_\alpha{}^{\beta\gamma} y_\beta \partial_\gamma$ are not strictly speaking vector fields; they have multiplier or fiber components as well. The map $\Lambda \to \xi$ is a homomorphism from the local algebra onto the local Weyl algebra (rather than the diffeomorphism algebra). Nevertheless, we continue to refer to ξ and to e_α as vector fields and other, similar abuses of terminology will also occur. Because of the close relationship between the manifold and the vector space $\tilde{V} = \mathsf{R}^6$ of the representation $l_A \to \tilde{S}_A$, one ends up using the same type of indices for the components of world tensors and internal tensors. Thus, $e_\alpha = e_\alpha{}^\beta \partial_\beta$, but the coefficients $e_\alpha{}^\beta$ are not the components of an internal tensor field. Instead, $(e_\alpha{}^\beta)$ is a $\tilde{V}$-valued (lower index) vector field (upper index). The distinction is irrelevant as far as global conformal transformations are concerned, but not for the action of the local algebra. The curvature and the torsion are pure internal tensors, that is, $(\tilde{V}\otimes)^4$-valued and $(\tilde{V}\otimes)^3$-valued functions, respectively. The components of e_α define a type of vierbein connection between internal and world tensors.

F. Minkowski notation

To translate from Dirac to Minkowski notation, introduce the coordinates $(x^a), a = +,0,\ldots,3,B$ in the usual way,

$$x^+=\ln\lambda\cdot y,\quad x^B=y^2/2(\lambda\cdot y)^2\ , \tag{2.33}$$

$$x^\mu=y^\mu/\lambda\cdot y,\quad \mu=0,\ldots,3\ , \tag{2.34}$$

where λ is a constant, isotropic ($\lambda^2=0$) six-vector, with $\lambda^\alpha=0$ for $\alpha=0,\ldots,3$. (It is also possible to regard $\overline{M}$ as a compactification of de Sitter space. To translate from Dirac to de Sitter notation proceed in the same way but with λ^2 fixed, positive). The equation $\lambda\cdot y=0$ determines a submanifold M^∞ of the projective cone $\overline{M}$. Minkowski space M can be identified with $\overline{M}$ after removal of M^∞: $M=\overline{M}-M^\infty$. The choice of λ is thus the choice of the location of Minkowski infinity in the homogeneous space $\overline{M}$.

If A is a $\widetilde{V}$-valued function, with components $A^\alpha=\delta^{\alpha\beta}A_\beta$, then we define $\check{A}^a$ and $\check{A}_a, a=+,0,\ldots,B$ by

$$A_\alpha dy^\alpha=(\lambda\cdot y)^{N+1}\check{A}_a dx^a\ ,$$

$$A^\alpha\partial/\partial y^\alpha=(\lambda\cdot y)^{N-1}\check{A}^a\partial/\partial x^a\ .$$

We find

$$\check{A}_a(x)=(\lambda\cdot y)^{-N}\mathscr{O}_a{}^\alpha A_\alpha(y)\ ,$$

$$\check{A}^a(x)=(\lambda\cdot y)^{-N}\mathscr{O}_\alpha{}^a A^\alpha(y)\ ,$$

$$\{\mathscr{O}_a{}^\alpha\}_{a=+,\mu,B}=\{\hat{y}^\alpha,\delta_\mu{}^\alpha-x_\mu\lambda^\alpha,\lambda^\alpha\}\ , \tag{2.35}$$

$$\{\mathscr{O}_\alpha{}^a\}_{a=+,\mu,B}=\{\lambda_\alpha,\delta_\alpha{}^\mu-\lambda_\alpha x^\mu,\hat{y}^\alpha\}\ ,$$

$$\hat{y}^\alpha=y^\alpha/\lambda\cdot y\ .$$

The extension to fields valued in tensor powers of $\widetilde{V}$ is obvious. For the metric $\delta=(\delta_{\alpha\beta})$ we find that the nonzero components are

$$\check{\delta}_{\mu\nu}=\delta_{\mu\nu},\quad \mu,\nu=0,\ldots,3,\quad \check{\delta}_{+B}=\check{\delta}_{B+}=1\ ,$$

($\delta_{\mu\nu}$) being the Minkowski metric. Note that

$$\check{A}^a\check{\delta}_{ab}=\check{A}_b\ .$$

Exactly the same formulas are used to relate a world vector field on R^6 (components A^α) to a Minkowski complex ($\check{A}^a$), and a one-form field (components B_α) to a Minkowski complex ($\check{B}_a$). Of course, the transformation properties are different. See Eqs. (2.36)—(2.42).

If the world vector field ξ is tangent to the cone, then $\xi^\alpha y_\alpha=0$ and $\check{\xi}^B=0$. All world vector fields will be tangential, so fields of one-forms will be defined up to $y_\alpha\chi$. This means that if ξ is a one-form then $\check{\xi}_B$ is irrelevant.

The same rules are applied to define $\check{\Lambda}^{ab}$ from $\Lambda^{\alpha\beta}$ when Λ is a gauge parameter. If $\xi=\Lambda^A M_A=\Lambda^{\alpha\beta}y_\alpha\partial_\beta$, then the nonzero components of $\check{\Lambda}$ are ($\mu=0,\ldots,3$ and $a=+,0,\ldots,3$)

$$\check{\Lambda}^{Ba}=\check{\Lambda}_+{}^a=\check{\xi}^a,\quad \check{\Lambda}^{+\mu}=\check{\Lambda}_B{}^\mu,\quad \check{\Lambda}^{\mu\nu}\ .$$

If (A_α) is an internal vector field (a function valued in $\widetilde{V}$), then

$$(\mathscr{L}_\Lambda\check{A})_+=[\check{\xi}+(N+1)\check{\xi}^+]\check{A}_+,\quad \check{\xi}=\check{\xi}^\mu\partial_\mu\ , \tag{2.36}$$

$$(\mathscr{L}_\Lambda\check{A}_\mu)=(\check{\xi}+N\check{\xi}^+)\check{A}_\mu+\check{\Lambda}_\mu{}^\nu\check{A}_\nu+\check{\Lambda}_\mu{}^+\check{A}_+\ , \tag{2.37}$$

$$(\mathscr{L}_\Lambda\check{A})_B=[\check{\xi}+(N-1)\check{\xi}^+]\check{A}_B+\check{\Lambda}_B{}^\nu\check{A}_\nu\ . \tag{2.38}$$

This is a nondecomposable, faithful representation of the local algebra. The invariant subspaces are $\check{A}_+=0$ ($y^\alpha A_\alpha=0$) and $\check{A}_+=\check{A}_\mu=0$ ($A_\alpha=y_\alpha\chi$).

If (A^α,B_β) are a world vector field and a one-form field, then

$$(\mathscr{L}_\Lambda\check{A})^+=[\check{\xi}+(N-1)\check{\xi}^+]\check{A}^+-\check{\xi}^+_{,\mu}\check{A}^\mu\ , \tag{2.39}$$

$$(\mathscr{L}_\Lambda\check{A})^\mu=(\check{\xi}+N\check{\xi}^+)\check{A}^\mu-\check{\xi}^\mu_{,\nu}\check{A}^\mu\ , \tag{2.40}$$

$$(\mathscr{L}_\Lambda\check{B})_+=[\check{\xi}+(N+1)\check{\xi}^+]\check{B}_+\ , \tag{2.41}$$

$$(\mathscr{L}_\Lambda\check{B})_\mu=(\check{\xi}+N\check{\xi}^+)\check{B}_\mu+\check{\xi}^\nu_{,\mu}\check{B}_\nu+\check{\xi}^+_{,\mu}\check{B}_+\ , \tag{2.42}$$

where $\check{\xi}^a{}_{,b}=\partial\check{\xi}^a/\partial x^b$. These are two nondecomposable representations of the local algebra, faithful only on the Weyl group. The invariant subspaces are $\check{A}^\mu=0$ ($A^\alpha=y^\alpha\chi$) and $B_+=0$ ($y^\alpha B_\alpha=0$).

G. Metric and action functional

It is easy to construct scalar fields in terms of $R_{\alpha\beta}{}^{\gamma\delta}$, but a little more difficult to find a gauge-invariant density. Consider the world tensor with components

$$g^{ab}=(\lambda\cdot y)^{-2N}\delta^{\alpha\beta}e_\alpha{}^a e_\beta{}^b\ , \tag{2.43}$$

where N is the degree of ϕ. The action $\delta_\Lambda=\mathscr{L}_\Lambda$ of the local algebra is indicated by (2.39) and (2.40) and in particular

$$(\delta_\Lambda g)^{\mu\nu}=(\check{\xi}+2N\check{\xi}^+)g^{\mu\nu}-\check{\xi}^\mu_{,\rho}g^{\rho\nu}-\check{\xi}^\nu_{,\rho}g^{\mu\rho}\ . \tag{2.44}$$

This is a representation of the local algebra, faithful on the Weyl algebra. (We ignore the + components, which amounts to passing to a quotient by an invariant subspace).

If $|g|\equiv\det(g^{\mu\nu})$, then in the physically interesting case $d=5$ ($\mu=0,\ldots,3$),

$$\delta_\Lambda|g|=(\check{\xi}+8N\check{\xi}^+-2\check{\xi}^\rho{}_{,\rho})|g|\ .$$

If H is any scalar field, then

$$I=\int d^4x(-|g|)^{-1/2}\check{H} \tag{2.45}$$

is invariant under diffeomorphisms. It is invariant under the Weyl algebra, and thus fully invariant under the local algebra, if the degree of H is $4N$. This suggests that the action of the theory is obtained by taking for H an algebraic expression of the second order in $R_{\alpha\beta}{}^{\gamma\delta}$.

The covariant metric tensor is also of interest. We shall give a definition that will turn out to be relevant later, but that cannot be motivated at this point. Suppose that $\phi=A+\widetilde{\phi}$, as in Eq. (2.18), and that A has the particular form (2.29), with the vector field (a^α) subject to the following restrictions:

$$a^\alpha a_\alpha=0,\quad a^\alpha y_\alpha=1\ . \tag{2.46}$$

In this case

$$e_\alpha{}^\beta\equiv\phi_\alpha{}^{\sigma\beta}y_\sigma=\delta_\alpha{}^\beta-y_\alpha a^\beta+\widetilde{e}_\alpha{}^\beta,\quad \widetilde{e}_\alpha{}^\beta\equiv\widetilde{\phi}_\alpha{}^{\gamma\beta}y_\gamma\ . \tag{2.47}$$

Recall that ($e_\alpha{}^\beta$) is a $\widetilde{V}$-valued (lower index) vector field (upper index). Now let $(\theta_\beta{}^\gamma)$ be a $\tilde{V}$-valued (upper index) one-form field (lower index). The quantities

$$(e\theta)_\alpha{}^\gamma = e_\alpha{}^\beta \theta_\beta{}^\gamma$$

are then the components of an internal tensor.

As explained in Sec. II F, all vector fields are tangential, in particular $e_\alpha{}^\beta y_\beta = 0$, so $(e_\alpha{}^\beta)$ cannot be invertible. Similarly, one-forms are always defined modulo their components along (y_α). Now define $(\theta_\beta{}^\gamma)$, modulo such components, by

$$e_\alpha{}^\beta \theta_\beta{}^\gamma = \delta_\alpha{}^\gamma - y_\alpha a^\gamma \ . \tag{2.48}$$

The solution of this equation is

$$\theta_\beta{}^\gamma = \delta_\beta{}^\gamma - y_\beta a^\gamma - \tilde{\theta}_\beta{}^\gamma \ , \tag{2.49}$$

$$\tilde{\theta} \approx (1+\tilde{e})^{-1}\tilde{e} \ . \tag{2.50}$$

Here $\tilde{\theta}$ and $\tilde{e}$ stand for the matrices $(\tilde{\theta}_\beta{}^\gamma)$ and $(\tilde{e}_\alpha{}^\beta)$, and $\approx$ means equality up to the inherent ambiguity of one-forms that we have just explained.

Finally, define $(\theta_b{}^\gamma)$ and $(\check{g}_{ab})$ by

$$\begin{aligned} &\theta_\beta{}^\gamma dy^\beta = \theta_b{}^\gamma dx^b \ , \\ &\check{g}_{ab} = (\lambda\cdot y)^{-2} \theta_a{}^\alpha \theta_b{}^\beta \delta_{\alpha\beta} \ . \end{aligned} \tag{2.51}$$

Compare with Eq. (2.43); in the case at hand $N=-1$. If we expand (2.50), then we obtain (with $y_\alpha dy^\alpha = 0$, $dx^B = 0$)

$$\begin{aligned} \check{g}_{ab} dx^a dx^b &= (\lambda\cdot y)^{-2}(\delta_{\alpha\beta} - \tilde{e}_{\alpha\beta} - \tilde{e}_{\beta\alpha} + \cdots) dy^\alpha dy^\beta \\ &\equiv g_{\alpha\beta} dy^\alpha dy^\beta \ . \end{aligned} \tag{2.52}$$

Notice that $g^{aB} = 0$ and $\check{g}_{a+} = 0$, so that $(\check{g}_{\mu\nu})$ is the inverse of $(g^{\mu\nu})$. Also $\theta_a{}^\alpha y_\alpha = 0$ and $\theta_+{}^\alpha = y^\alpha$.

III. CONNECTION TO THE LINEAR THEORY

A. Formulation of the problem

The general framework of conformal gauge theory does not contain a prescription for choosing the correct action functional. Gauge invariance is a strong condition, but it is by no means strong enough. An additional input is needed, and this leaves room for hoping that unitarity can be incorporated. Actually, the theory is characterized by an action and a constraint, and it is the latter that is most directly responsible for unitarity. The determination of the constraint will be our main concern, the answer will be obtained at the end of Sec. IV.

By unitarity of a quantum field theory is meant, roughly speaking, the absence of interacting, propagating ghosts. When unitarity is provable, then it is proved in perturbation theory. Strong requirements need to be satisfied already at the lowest level of perturbation theory; there are requirements on the free propagators, the constraints, and the currents. Under conditions that tend to prevail in gauge theories, these may turn out to be sufficient to ensure that unitarity can be extended to all orders of perturbation theory. Here we do not study unitarity beyond the lowest level of perturbation theory; at this level, quantization is determined by the propagator and the constraints. Our program is to choose the action and the constraints so as to be consistent with the unitarity of the quantized field theory at the lowest level.

Since unitarity is not a property of classical field theories, it can be discussed only in connection with quantization, and therefore only in perturbation theory. It follows that the classical field theory must be developed as a perturbation theory, before any sensible discussion of unitarity can be made. We are thus forced to introduce perturbation expansions, and to study the properties of our field theory in the linear approximation. Suppose, then, that

$$\phi_\alpha{}^{\beta\gamma} = \eta_\alpha{}^{\beta\gamma} + h_\alpha{}^{\beta\gamma} \ , \tag{3.1}$$

where η is a fixed "background" field. Our first problem is to make a sensible choice of this fixed field.

Equation (3.1) has an analog in the expansion of the metric field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, in Weyl's conformal theory of gravitation.[3] Here η is a fixed, background metric. If it is flat, then it is conformally invariant; that is, it is a fixed point, with respect to conformal transformations, in the space of metric fields. This has the very satisfactory consequence of making the classical theory conformally invariant to each order of the perturbation expansion. Our case is different. The only fixed point in the space of tensor fields of the type of ϕ is given by $\eta_\alpha{}^{\beta\gamma}(y) = y^\beta \delta_\alpha{}^\gamma - y^\gamma \delta_\alpha{}^\beta$, and this is unsuitable. First, the degree of homogeneity is $+1$, instead of the preferred value -1; see Sec. II E and below. Second, the vector fields e_α would reduce to $y_\alpha \hat{N}$, and this would trivialize the first-order curvature, as will be seen below. We must conclude, therefore, that η cannot be a fixed point, and that our theory is probably not conformally invariant to each order of perturbation theory.

At this point one may be tempted to give up, but in fact there are several reasons to think that this would be premature. First, the fact that η cannot be conformally invariant in our theory simply means that it is closer to gauge theories cum Higgs-Kibble mechanism than to Weyl gravity. Second, manifest covariance in the sense of Dirac's six-cone formalism seems to be a stronger condition than conformal covariance *per se*, as is seen when one tries to write Maxwell's theory in Dirac's notation.

Our choice for η is given by

$$\eta_\alpha{}^{\beta\gamma}(y) = (\lambda\cdot y)^N (\lambda^\beta \delta_\alpha{}^\gamma - \lambda^\gamma \delta_\alpha{}^\beta) \ , \tag{3.2}$$

where λ is the constant, isotropic vector field introduced in Sec. II F, and N will be taken equal to -1. The appearance of a fixed direction in six-space cannot be avoided; the fact that this direction is determined by λ is expected to minimize the damage. After all, any imbedding of Minkowski space into the projective cone involves a fixed direction; in fact, $\lambda\cdot y = 0$ is the equation that determines the points of $\bar{M} - M$, that is, the points at infinity (Sec. II F).

Once is has been decided to make use of λ to manufacture η, then (3.2) is not the only possibility. Lengthy calculations have been carried out with the most general expression that relates η covariantly to λ, that is, expressions that are invariant under the stabilizer of λ in so(4.2) (the global Weyl algebra). The result is that every modification of (3.2) is detrimental; in particular, only (3.2) leads to vanishing zeroth-order curvature and torsion, the importance of which will be discussed in detail below.

We now expand, using (3.1):

$$e_\alpha = {}^0e_\alpha + {}^1e_\alpha, \quad c_{\alpha\beta}{}^\gamma = \sum {}^n c_{\alpha\beta}{}^\gamma ,$$
$$R_{\alpha\beta}{}^{\gamma\delta} = \sum {}^n R_{\alpha\beta}{}^{\gamma\delta} , \tag{3.3}$$

where the summation is over $n = 0, 1, \ldots,$ and the nth term is homogeneous of order n in the field h. To zeroth order in h we find, using (3.2) and (2.9), (2.16), and (2.31),

$${}^0e_\alpha = D_\alpha \equiv \partial_\alpha - \hat{y}_\alpha \lambda\cdot\partial, \quad \hat{y}_\alpha \equiv y_\alpha/\lambda\cdot y , \tag{3.4}$$

$${}^0c_{\alpha\beta}{}^\gamma = 0, \quad {}^0R_{\alpha\beta}{}^{\gamma\delta} = 0 . \tag{3.5}$$

Here we have set $N = -1$, otherwise the torsion would not vanish; this choice will be adhered to from now on. As was pointed out in Sec. II E, consideration of minimal coupling also points to $N = -1$. [Strictly, ${}^0c_{\alpha\beta}{}^\gamma$ is defined by (2.16) only up to expressions of the form $q_{\alpha\beta}\lambda^\gamma$.]

If, as is expected, the action will turn out to be a bilinear functional of the curvature and the torsion, then the disappearance of ${}^0R$ and of ${}^0c$ are welcome. In this case the action will contain no term linear in h, so that $\phi = \eta$ will be an extremum of the action and a solution of the field equation, as required for the internal consistency of the perturbation theory. In addition, it means that ${}^2R$ and ${}^2c$ will be irrelevant for the linear approximation. The quantities to be studied in the linear approximation are therefore ${}^1R$ and ${}^1c$. Inspection of these quantities will suggest our approach to the problem of unitarity. Note also that the vanishing of ${}^0R$ and ${}^0c$ are necessary and sufficient conditions for ${}^1R$ and ${}^1c$ to be gauge invariant.

To begin with,

$${}^1e_\alpha = h_\alpha{}^{\cdot\gamma}\partial_\gamma, \quad h_\alpha{}^{\cdot\gamma} \equiv h_\alpha{}^{\beta\gamma} y_\beta , \tag{3.6}$$

$${}^1c_{\alpha\beta}{}^\gamma = D_\alpha h_\beta{}^{\cdot\gamma} - (\alpha,\beta) . \tag{3.7}$$

Notation: A dot replacing an index will always have the same meaning as in (3.6). The second term in (3.7) is equal to the first term, except that α and β are to be interchanged, and is to be subtracted. Below will appear covariant components $h_{\alpha\beta\gamma}$ of h; indices are raised and lowered by means of the pseudo-orthogonal six-dimensional metric, without changing their order.

The expression obtained for ${}^1R$ can be arranged as follows:

$$\lambda\cdot y\, {}^1R_{\alpha\beta\gamma\delta} = D_\alpha X_{\beta\gamma\delta} + \eta_{\alpha\beta}{}^\sigma X_{\sigma\gamma\delta} + \eta_{\alpha\gamma}{}^\sigma X_{\beta\sigma\delta} + \eta_{\alpha\delta}{}^\sigma X_{\beta\gamma\sigma} - (\alpha,\beta) , \tag{3.8}$$

$$X_{\alpha\beta\gamma} = \lambda\cdot y h_{\alpha\beta\gamma} - \lambda_\beta h_{\alpha\cdot\gamma} - \lambda_\gamma h_{\alpha\beta\cdot} , \tag{3.9}$$

where D_α is the differential operator defined by (3.4). [If we had expanded around the fixed point $y^\beta\delta_\alpha{}^\gamma - y^\gamma\delta_\alpha{}^\beta$, instead of (3.2), then D_α would have been replaced by $y_\alpha\hat{N}$, and the first-order curvature would have been a purely algebraic expression containing no derivatives.]

The linear, conformal theory bears a certain resemblance to the theory developed here. This linear theory contains a ghost that can be controlled by means of a constraint. The basic field is a third-rank Dirac six-tensor. The important constraint, that effectively eliminates the ghost from the physical sector, looks very much like ${}^1R_{\alpha\beta\gamma\delta} + (\alpha\beta,\gamma\delta) = 0$. It is reasonable to suppose that this linear theory is closely related to the first-order approximation of our nonlinear gauge theory. Our strategy will be to try to find an intertwining between the two theories, and to profit from our understanding of the former, to identify and eliminate the ghost from the latter. To discover the intertwining map we shall compare the gauge groups. This will be done in this section; then, in the next section, we shall use this mapping to discover the constraints that must be imposed on the full, nonlinear gauge theory.

Since η is fixed, the local algebra acts on h according to

$$(\delta_\Lambda h)_\alpha{}^{\beta\gamma} = (\delta_\Lambda \phi)_\alpha{}^{\beta\gamma} .$$

Only the affine term is relevant in the first approximation. (Compare linearized Einstein theory.) It is found by replacing ϕ by η in the expression (2.8) for $\delta_\Lambda\phi$. The result is

$$(\delta_\Lambda h)_\alpha{}^{\beta\gamma} = (\lambda\cdot y)^{-2}[(\lambda\cdot y\Lambda^{\beta\sigma} - \lambda^\beta\Lambda^{\cdot\sigma})\lambda_\sigma\delta_\alpha{}^\gamma - (\beta,\gamma)] - D_\alpha\Lambda^{\beta\gamma} . \tag{3.10}$$

This formula will be used to find the intertwining map.

B. Review of the linear theory

The field of the linear theory is a third rank six-tensor of mixed symmetry,

$$\Psi'_{\alpha\beta\gamma} + \Psi'_{\alpha\gamma\beta} = 0, \quad \Psi'_{\alpha\beta\gamma} + \Psi'_{\beta\gamma\alpha} + \Psi'_{\gamma\alpha\beta} = 0 . \tag{3.11}$$

(The field component $\Psi'_{\alpha\beta\gamma}$ was denoted $\Psi_{\beta\gamma\alpha}$ in Ref. 7.) The free field satisfies a wave equation and several constraints. Most important is the constraint that eliminates the ghost. This constraint is gauge invariant, and its nonlinear analog is therefore expected to have an intrinsic, geometric meaning, unlike the remaining constraints that are of the usual gauge-fixing type. Only this gauge-invariant, ghost-eliminating constraint will be discussed here.

Let

$$\text{grad}_\alpha = y_\alpha\partial^2 - (2\hat{N} + 4)\partial_\alpha, \quad \hat{N} \equiv y\cdot\partial , \tag{3.12}$$

and let d' be the operator (denoted d in Ref. 7) that acts on antisymmetric tensor fields of rank 2 and degree $+1$, and on tensor fields of rank 3 and mixed symmetry of degree 0, according to

$$(d'\overline{\Lambda})_{\alpha\beta\gamma} = 2\text{grad}_\alpha\overline{\Lambda}_{\beta\gamma} - \text{grad}_\beta\overline{\Lambda}_{\gamma\alpha} - \text{grad}_\gamma\overline{\Lambda}_{\alpha\beta} , \tag{3.13}$$

$$(d'\Psi')_{\alpha\beta\gamma\delta} = \text{grad}_\alpha\Psi'_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) . \tag{3.14}$$

The constraint that exorcises the ghost is

$$d'\Psi' = 0 . \tag{3.15}$$

The operator d' is a coboundary operator, $d'\circ d' = 0$. The physical, massless modes, with helicities ± 2, are in the cohomology space $\text{Ker}\,d'/\text{Im}\,d'$. That $d'\circ d' = 0$, means that $d'\Psi'$ is invariant under the "gauge transformations" defined by

$$\delta\Psi' = d'\bar{\Lambda} \ . \tag{3.16}$$

The tensor $d'\Psi'$ looks much like a symmetrized curvature tensor, and our intertwining map will relate $d'\Psi'$ of the linear theory to the first-order curvature of the nonlinear theory.

A number of obstructions prevent a direct confrontation between the two theories. (i) The degree and the symmetry type of Ψ' are different from those of h. (ii) The gauge transformations are quite different, involving first-order differential operators in one case and the second-order operator grad in the other. (iii) Finally, the degrees of the gauge parameters are different: this is perhaps the most fundamental difficulty. We shall solve these problems, one at a time.

C. Reformulation of the linear theory

We shall deal with the problem of symmetry first. Let Ψ be a tensor field of degree 0, antisymmetric in the last two indices:

$$\Psi_{\alpha\beta\gamma} + \Psi_{\alpha\gamma\beta} = 0 \ .$$

Then Ψ is the sum of two fields with definite symmetry:

$$3\Psi_{\alpha\beta\gamma} = \Psi'_{\alpha\beta\gamma} + \Psi''_{\alpha\beta\gamma} \ , \tag{3.17}$$

$$\begin{aligned} \Psi'_{\alpha\beta\gamma} &= 2\Psi_{\alpha\beta\gamma} - \Psi_{\beta\gamma\alpha} - \Psi_{\gamma\alpha\beta} \ , \\ \Psi''_{\alpha\beta\gamma} &= \Psi_{\alpha\beta\gamma} + \Psi_{\beta\gamma\alpha} + \Psi_{\gamma\alpha\beta} \ , \end{aligned} \tag{3.18}$$

Ψ' being of mixed symmetry, satisfying (3.11), and Ψ'' being totally antisymmetric. The new field Ψ has the same symmetry as h. We must now reformulate the linear theory in terms of Ψ.

Define a new coboundary operator d by

$$(d\bar{\Lambda})_{\alpha\beta\gamma} = \text{grad}_\alpha \bar{\Lambda}_{\beta\gamma} \ , \tag{3.19}$$

$$(d\Psi)_{\alpha\beta\gamma\delta} = \text{grad}_\alpha \Psi_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) \ , \tag{3.20}$$

and impose the constraint

$$d\Psi = 0 \ . \tag{3.21}$$

Since $d \circ d = 0$, $d\Psi$ is gauge invariant under the generalized gauge transformation

$$\delta\Psi = d\bar{\Lambda} \ . \tag{3.22}$$

Like Ψ, $d\Psi$ is a mixture of two symmetry types, one part has box symmetry and the other is completely antisymmetric. The first part depends only on Ψ', therefore (3.21) implies (3.15) and eliminates the ghost from Ψ'. As for the skew part Ψ'', it is trivialized by (3.21). This will be discussed in more detail later on. For now, let us suppose, for simplicity, that Ψ satisfies the gauge-fixing conditions $y^\alpha \Psi_{\alpha\beta\gamma} = 0$ and $\text{grad}^\alpha \Psi_{\alpha\beta\gamma} = 0$. Then contraction of (3.21) with y^α tells us that $\Psi'' = 0$. Equations (3.19)—(3.22) are therefore no more than a reformulation of the linear theory.

D. The intertwining map

We next confront the gauge transformation (3.22) of the reformulated linear theory with the gauge transformation (3.10) of the first approximation to the nonlinear theory. The most dramatic difference is that the operator grad_α in (3.19) is a second-order differential operator, while D_α in (3.10) is first order. In view of this situation we cannot hope to find a simple intertwining between the two gauge groups without accepting some restrictions. It will be sufficient to intertwine (large) subgroups. Henceforth, we restrict the gauge group of the linear theory by the condition

$$\lambda^\alpha \text{grad}_\alpha \bar{\Lambda}_{\beta\gamma} = 0 \tag{3.23}$$

on the gauge parameter $\bar{\Lambda}$.

In this case we have

$$\text{grad}_\alpha \bar{\Lambda}_{\beta\gamma} = -4 D_\alpha \bar{\Lambda}_{\beta\gamma} \ .$$

[D_α was defined in Eq. (3.4) and grad_α in (3.12)]. The gauge transformation (3.22) can therefore be replaced by

$$(\delta\Psi)_{\alpha\beta\gamma} = D_\alpha \bar{\Lambda}_{\beta\gamma} \ . \tag{3.24}$$

Since $\bar{\Lambda}$ has degree $+1$ and Λ has degree 0, we now map the restricted gauge group determined by (3.23) into the group (3.10) by

$$\bar{\Lambda}_{\beta\gamma} = \lambda \cdot y \Lambda_{\beta\gamma} \ . \tag{3.25}$$

The image is the subgroup determined by $(\partial^2)^2 \Lambda_{\beta\gamma} = 0$.

Now it is easy to intertwine (3.24) with (3.10). Define E and f by [see notation defined in (3.6)]

$$\begin{aligned} E^{\beta\gamma} &\equiv (\lambda \cdot y)^{-1} \Psi^{\cdot\beta\gamma} \ , \\ f^\beta &\equiv (\lambda \cdot y)^{-1} (\lambda \cdot y E^{\beta\gamma} - \lambda^\beta E^{\cdot\gamma}) \lambda_\gamma \ , \end{aligned} \tag{3.26}$$

then (3.24) gives $\delta E = \Lambda$, and it is easy to see that the map $\Psi \rightarrow h$ defined by

$$h_\alpha{}^{\beta\gamma} = (\lambda \cdot y)^{-1} (f^\beta \delta_\alpha{}^\gamma - f^\gamma \delta_\alpha{}^\beta + \lambda_\alpha E^{\beta\gamma} - \Psi_\alpha{}^{\beta\gamma}) \tag{3.27}$$

transforms the action (3.24) to the action (3.10). This is not the most general possibility, for any gauge-invariant quantity (for example, such as can be constructed from $\lambda^\alpha \Psi_\alpha{}^{\beta\gamma}$, which is gauge invariant) can be added on the right. But (3.27) is the simplest intertwining map, and attempts at adding complications have not been encouraging.

Equation (3.27) intertwines the action of the gauge groups, but it does not intertwine the global conformal transformations. Indeed, the conformal Lie algebra is just the subalgebra of the local algebra that is obtained by restricting the functions $\Lambda^{\alpha\beta}$ to be constant over the manifold. The action on h includes the affine part given by the first term on the right-hand side of (3.10); this term arises from the fact that the background field η is not conformally invariant. In the linear theory, on the other hand, the conformal algebra acts linearly on Ψ, and has nothing directly to do with gauge transformations. Suppose we ignore the affine part of $\delta_\Lambda h$, and retain only the linear part, so that h transforms like a Dirac tensor of degree -1. Even then, Eq. (3.27) fails to intertwine this action with that of the linear theory. This is inevitable, since the degrees of h and Ψ are not the same.

This situation is much the same as in Higgs-Kibble theory. There, the starting point is a theory of massless fields, conformally invariant except perhaps for the Higgs-Kibble potential.[1] The shift of the origin in field space, $u(x) \to \text{const} + u'(x)$, does not destroy conformal invariance but the action of the conformal algebra on u' includes an affine term, and invariance is therefore not satisfied to each order of perturbation theory. Nevertheless, in spite of the noninvariance of the background η, there is a sense in which the constraints of the first-order gauge theory are invariant. The conformal action on Ψ defines, through Eq. (3.27), a new conformal action on h. The first-order constraints are a covariant set of equations with respect to this action of the conformal algebra on h. We shall return to this point at the end of Sec. IV.

IV. GHOST-ELIMINATING CONSTRAINT

A. Formulation of the problem

The map (3.27), that relates the field h of the nonlinear theory to the field Ψ of the linear theory, is gauge invariant in the sense that the gauge transformation (3.24) induces the affine gauge transformation (3.10). It follows that any gauge-invariant constraint on Ψ induces a gauge-invariant constraint on h. Our task will be to find the gauge-invariant constraint on h that is induced by the map (3.27) from the gauge-invariant, ghost-eliminating constraint $d\Psi=0$. We are here concerned only with the first-order approximation to the nonlinear theory, but later we must generalize this constraint on h to a constraint on ϕ that is gauge invariant in the sense of the full, nonlinear gauge theory, and there lurks an obstruction that we had better take into account at this stage.

There is a strong presumption that the constraint $d\Psi=0$ corresponds to a constraint on the curvature. The curvature, as well as the torsion, are manufactured from ϕ and its first derivatives, while $d\Psi$ contains second-order derivatives of Ψ. It is clear that the intertwining map (3.27) cannot relate a second-order differential operator to a first-order differential operator. We shall therefore replace the constraint $d\Psi=0$ by an equivalent condition that contains only first-order derivatives. (The new condition will actually be weaker; it is equivalent to $d\Psi=0$ only in the sense that it is equally effective in separating the ghost from the physical modes.)

B. Elimination of higher derivatives

Consider Eq. (3.21), namely,

$$(d\Psi)_{\alpha\beta\gamma\delta} \equiv (y_\alpha \partial^2 - 2\partial_\alpha)\Psi_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) = 0 \ . \tag{4.1}$$

It is evidently equivalent to the pair

$$\partial_\alpha \Psi_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) = y_\alpha \chi_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) \ , \tag{4.2}$$

$$2\chi_{\beta\gamma\delta} = \partial^2 \Psi_{\beta\gamma\delta} \ . \tag{4.3}$$

Equation (4.2), taken by itself, and understood in the sense that "there exists a field χ such that (4.2) holds," is already a very strong condition on Ψ.

Since Eq. (4.1) incorporates the wave equation, one might think that (4.2) contains only the constraints and that the second-order equation (4.3) is the wave equation. Instead, as will be shown in Sec. IV E, Einstein's linearized field equation actually follows from (4.2). The additional information that may be contained in Eq. (4.3) is therefore practically negligible, and we shall henceforth consider (4.2) instead of (4.1). In this way we avoid constraints that contain second-order derivatives.

Finally, let us note that Eq. (4.2) can be written

$$(D\Psi)_{\alpha\beta\gamma\delta} \equiv D_\alpha \Psi_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) \approx 0 \ , \tag{4.4}$$

by which is meant that the quantity on the left is equal to an expression of the form

$$(y\chi)_{\alpha\beta\gamma\delta} \equiv y_\alpha \chi_{\beta\gamma\delta} - (\alpha,\beta) + (\alpha\beta,\gamma\delta) \ . \tag{4.5}$$

The differential operator D_α was defined by (3.4). It turns out to be convenient for our subsequent analysis to break up (4.4) into several parts.

If B is any tensor field (in the linear or linearized theory), let ${}^tB$ denote its transverse part, defined by

$${}^tB_{\alpha\cdots} \equiv (\delta_\alpha{}^\sigma - \lambda_\alpha \hat{y}^\sigma) \cdots (\cdots) B_{\sigma\cdots} \ , \qquad \hat{y}^\sigma \equiv y^\sigma / \lambda \cdot y \ . \tag{4.6}$$

A tensor field is determined by its transverse part and by one or more contractions of the form $y^\alpha B_{\alpha}\ldots$. In the case of (4.4) it is equivalent to the pair

$${}^t(D\Psi)_{\alpha\beta\gamma\delta} \approx 0, \quad y^\alpha (D\Psi)_{\alpha\beta\gamma\delta} \approx 0 \ . \tag{4.7}$$

We shall refer to the first (second) equation as the principal (secondary) part of the constraint (4.4).

Applying y^γ to (4.7) one gets

$$\Psi_{\beta\delta\cdot} + \Psi_{\delta\beta\cdot} \approx D_\beta B_\delta + D_\delta B_\beta, \quad B_\beta \equiv \Psi_{\cdot\beta\cdot} \ , \tag{4.8}$$

where a dot in the place of an index stands for contraction with y^α, as in Eq. (3.6). We substitute this back into (4.7), and separate the two symmetry types, to obtain the two equations

$$3\Psi''_{\beta\gamma\delta} \approx \sum_{\text{cycl}} D_\beta \Psi''_{\cdot\gamma\delta} \ , \tag{4.9}$$

$$2D_\beta P_{\gamma\delta} - D_\gamma P_{\delta\beta} - D_\delta P_{\beta\gamma} \approx 0 \ , \qquad P_{\beta\gamma} \equiv \Psi'_{\cdot\beta\gamma} + D_\beta B_\gamma - D_\gamma B_\beta \ . \tag{4.10}$$

The projections Ψ' and Ψ'' were defined in Eq. (3.18). The set (4.8)—(4.10) is equivalent to the second equation in (4.7).

Close examination of (4.10) shows that the space of harmonic solutions of this equation for P is finite dimensional. It carries the finite-dimensional representation that appears in the nondecomposable field representation of the linear theory. To simplify our task of translating the constraints into equations for h, we shall eliminate this finite space of unphysical modes. We do this by replacing (4.10) by the slightly stronger condition

$$P_{\beta\gamma} = 0 \ . \tag{4.11}$$

Equations (4.9)—(4.11) are gauge-invariant restrictions on

$\Psi''_{\alpha\beta\gamma}$ and on $\Psi'_{\cdot\beta\gamma}$. They cannot be strengthened further without blocking the gauge.

C. The first-order constraints

To express the constraints in terms of h we must use (3.27). Let us write that equation as follows:

$$h_{\alpha\beta\gamma}=(\lambda\cdot y)^{-1}(f_\beta\delta_{\alpha\gamma}-f_\gamma\delta_{\alpha\beta})+\tilde{h}_{\alpha\beta\gamma}\ , \tag{4.12}$$

$$\tilde{h}_{\alpha\beta\gamma}=-(\lambda\cdot y)^{-1}(\Psi_{\alpha\beta\gamma}-\lambda_\alpha\hat{y}^\sigma\Psi_{\sigma\beta\gamma})\ . \tag{4.13}$$

All the constraints can be expressed simply in terms of $\tilde{h}$.

The simplest constraint on h follows directly from (4.12):

$$\tilde{h}_{\cdot\beta\gamma}\equiv y^\alpha\tilde{h}_{\alpha\beta\gamma}=0\ . \tag{4.14}$$

Equation (4.13) is equivalent to the statement that there exists a field E, such that Ψ has the representation

$$\Psi_{\alpha\beta\gamma}=-(\lambda\cdot y)\tilde{h}_{\alpha\beta\gamma}+\lambda_\alpha E_{\beta\gamma}\ , \tag{4.15}$$

with $\tilde{h}$ satisfying (4.14). We insert this representation into the constraint equations and eliminate E.

We begin with (4.11); this gives the means to eliminate $E_{\beta\gamma}$:

$$2E_{\beta\gamma}=\tilde{h}_{\gamma\beta\cdot}-\tilde{h}_{\beta\gamma\cdot}+D_\gamma E_{\beta\cdot}-D_\beta E_{\gamma\cdot}\ .$$

Equation (4.9) takes the form

$$\sum_{\text{alt}}(\tilde{h}_{\beta\gamma\delta}-D_\beta\tilde{h}_{\gamma\delta\cdot})\approx 0\ , \tag{4.16}$$

and (4.8) becomes

$$\tilde{h}_{\beta\delta\cdot}+\tilde{h}_{\delta\beta\cdot}+D_\beta E_{\delta\cdot}+D_\delta E_{\beta\cdot}\approx 0\ . \tag{4.17}$$

The secondary part of the constraint is given by (4.14), (4.16), and (4.17). In (4.17) $E_{\beta\cdot}$ cannot be eliminated and stands for an arbitrary vector field of degree zero, satisfying $y^\beta E_{\beta\cdot}=0$.

Turning to the principal part of the constraint, we have

$$^t(D_\alpha\Psi_{\beta\gamma\delta})=-(\lambda\cdot y)^t(D_\alpha\tilde{h}_{\beta\gamma\delta})\ ,$$

$$^t(D_\alpha\tilde{h}_{\beta\gamma\delta})=D_\alpha{}^t\tilde{h}_{\beta\gamma\delta}+(\lambda\cdot y)^{-1}\lambda_\alpha{}^t\tilde{h}_{\beta\gamma\delta}+(\lambda\cdot y)^{-1}(\delta_\alpha{}^\rho-\hat{y}_\alpha\lambda^\rho)(\lambda_\beta{}^t\tilde{h}_{\rho\gamma\delta}+\lambda_\gamma{}^t\tilde{h}_{\beta\rho\delta}+\lambda_\delta{}^t\tilde{h}_{\beta\gamma\rho})\ ,$$

$$^t(D_\alpha\tilde{h}_{\beta\gamma\delta})-(\alpha,\beta)\approx{}^1R'_{\alpha\beta\gamma\delta}\ ,$$

with the definitions (correct to lowest order)

$$^1R'_{\alpha\beta\gamma\delta}\equiv D_\alpha\tilde{h}_{\beta\gamma\delta}-(\alpha,\beta)-\tilde{t}_{\alpha\beta}{}^\sigma\eta_{\sigma\gamma\delta}\ , \tag{4.18}$$

$$\tilde{t}_{\alpha\beta}{}^\gamma\equiv c_{\alpha\beta}{}^\gamma+\tilde{h}_{\alpha\beta}{}^\gamma-\tilde{h}_{\beta\alpha}{}^\gamma\ . \tag{4.19}$$

The principal constraint is the expression (4.18), symmetrized with respect to exchange of the pairs $(\alpha\beta,\gamma\delta)$:

$$^1R'_{\alpha\beta\gamma\delta}+(\alpha\beta,\gamma\delta)\approx 0\ . \tag{4.20}$$

It should be emphasized that the linear approximation (3.8) of R cannot be used in place of $^1R'$ in Eq. (4.21). We have made a thorough investigation of this possibility, and we have found that the physical modes would in this case be eliminated along with the ghost.

D. Nonlinear gauge-invariant constraints

Returning to the nonlinear gauge theory, let us write the expansion (3.1) as follows:

$$\phi_\alpha{}^{\beta\gamma}=A_\alpha{}^{\beta\gamma}+\tilde{\phi}_\alpha{}^{\beta\gamma},\ \ \tilde{\phi}_\alpha{}^{\beta\gamma}=\tilde{h}_\alpha{}^{\beta\gamma}\ , \tag{4.21}$$

$$A_\alpha^{\beta\gamma}=a^\beta\delta_\alpha{}^\gamma-a^\gamma\delta_\alpha{}^\beta,\ \ a^\beta=(\lambda\cdot y)^{-1}(\lambda^\beta+f^\beta)\ . \tag{4.22}$$

Here (a^β) is an internal vector field and $(A_\alpha{}^{\beta\gamma})$ is an internal tensor field. The expansion point is given by $\tilde{\phi}=f=0$. Note that the formulas are exact, though the constraints that the fields must satisfy are so far known only to lowest order in $\tilde{\phi}$ and f.

The expression (3.26) for f^β shows that $f^\beta y_\beta=0$, so that

$$a^\beta y_\beta=1\ . \tag{4.23}$$

Also $f^\beta\lambda_\beta=0$, so that to first order,

$$a^\beta a_\beta=0\ . \tag{4.24}$$

We shall take both of these equations to be exact. This does not amount to a loss of generality; it only restricts the decomposition in (4.21). For e_α we now have

$$e_\alpha=\partial_\alpha-y_\alpha a\cdot\partial+\tilde{e}_\alpha\ , \tag{4.25}$$

$$\tilde{e}_\alpha\equiv\tilde{\phi}_\alpha{}^{\beta\gamma}y_\beta\partial_\gamma\ . \tag{4.26}$$

The simplest part of the constraint, Eq. (4.14), has a very simple generalization, namely,

$$y^\alpha\tilde{\phi}_\alpha{}^{\beta\gamma}=0\ . \tag{4.27}$$

This is gauge invariant, for the affine term in $\delta_\Lambda\phi$ is $-e_\alpha\Lambda^{\beta\gamma}$, and

$$y^\alpha e_\alpha\Lambda^{\beta\gamma}=\hat{N}\Lambda^{\beta\gamma}=0$$

when (4.27) holds.

Equation (4.16) also generalizes easily, for it is the first approximation to

$$\sum_{\text{cycl}}t_{\alpha\beta\gamma}\approx 0\ . \tag{4.28}$$

This is evidently gauge invariant, since $(t_{\alpha\beta\gamma})$ is an internal tensor. In (4.28) we may indifferently write $\tilde{t}_{\alpha\beta\gamma}$, defined by (2.23), instead of $t_{\alpha\beta\gamma}$, defined by (2.15).

Equation (4.17) also has a straightforward generalization. If we define the covariant metric $(g_{\mu\nu})$ as in Sec. II G, then (4.17) is the first-order expression for the following condition:

$$(g_{\mu\nu})\text{ is conformally flat}\ . \tag{4.29}$$

This is seen by comparing (4.17) with (2.52).

Finally, (4.20) evidently is the linear approximation to

$$R'_{\alpha\beta\gamma\delta}+(\alpha\beta,\gamma\delta)\approx 0\ , \tag{4.30}$$

where $R'_{\alpha\beta\gamma\delta}$ is the quantity that was expressed in two different ways in (2.30), the simplest being

$$R'_{\alpha\beta\gamma\delta}=\widetilde{R}_{\alpha\beta\gamma\delta}-\widetilde{t}_{\alpha\beta}{}^{\sigma}A_{\sigma\gamma\delta} \ . \qquad (4.31)$$

To summarize, Eqs. (4.27)—(4.30) is a set of nonlinear constraints, gauge invariant in the sense of the full, conformal gauge theory, that reduces to the linear constraints (4.14), (4.16), (4.17), and (4.20) in the first-order approximation around the fixed point $f^{\alpha}=0=\widetilde{\phi}_{\alpha}{}^{\beta\gamma}$.

It should be stressed, however, that (4.30) is not the only possible generalization of (4.20). One may add any internal tensor, with the right symmetry and degree, that vanishes in the zeroth and in the first orders. Such a tensor is, for example,

$$\widetilde{t}_{\alpha\beta}{}^{\sigma}\widetilde{t}_{\gamma\delta}{}^{\sigma} \ . \qquad (4.32)$$

Later, we shall make good use of this freedom to improve (4.30).

E. Global conformal invariance

It was pointed out, in Sec. III D, that (global) conformal invariance is probably lost in perturbation theory. Yet, something close to full conformal invariance is recovered in the first order. The nonlinear constraints (4.27)—(4.30) of the full gauge theory reduce to Eqs. (4.14)—(4.20) in the first order. Now, if new transformation properties are assigned to h, determined by the map (3.27) and the transformation properties of Ψ in the linear theory, then this set of constraints is conformally invariant. This is satisfying, but on reflection it seems to be not quite sure whether it is essential. We would like to stress that the central point of our derivation has been to make use of the fact that the linear theory is ghost free; its conformal covariance is almost coincidental. The resolution of this paradox must be sought in the circumstance that the first-order constraints that were derived from the linear theory all turned out to have natural generalizations to fully gauge-invariant nonlinear constraints.

V. FIRST-ORDER CONSTRAINTS AND WAVE EQUATION

A. Constraints in Minkowski notation

One of the important conclusions that must be drawn from the linear theory is that ghost-free conformal gauge theory cannot be based merely on an action principle. Rather, its formulation includes an action principle together with a gauge-invariant constraint. Furthermore, the constraints are likely to be stronger than the Euler-Lagrange equations, which leads us to believe that the choice of the action is essentially fixed by the constraints. This is not to discount the role that is to be played by the action principle, nor to claim that the determination of the action is entirely straightforward, but only to suggest that the constraints may reveal some of the essential features of the theory. In this section we shall study the constraints, their consequences, and their completeness. We show that they contain Einstein's linearized field equation and discover the relationship of our theory to Weyl's conformal metric theory.

The nonlinear, gauge-invariant constraints were given by Eqs. (4.27)—(4.30). The geometric interpretation of all these equations is not yet understood, but they do at least have the virtue of being invariant under the infinitesimal transformations of the local conformal algebra (gauge invariance). The main criterion for choosing the constraints does not, however, come from geometry, but from the requirement of unitarity. This is a criterion that we know how to apply only in perturbation theory; let us therefore return to the linearized constraints given in Sec. IV C, Eqs. (4.14), (4.16), (4.17), and (4.20).

The meaning of (4.14)—(4.20) will become clearer when these equations are expressed in more familiar notation. The translation from Dirac to Minkowski notation was explained in Sec. II F. We shall omit the caret and use the same letter $\widetilde{h}$ for the Dirac complex $(\widetilde{h}_{\alpha\beta\gamma}),\alpha,\beta,\gamma=0,\ldots,5$ and for the Minkowski complex $(\widetilde{h}_{abc}),a,b,c=+,0,\ldots,3,B$. Since only the latter appears below there will be no ambiguity. Indices μ,ν,λ,ρ take the values $0,\ldots,3$.

Equation (4.14) tells us that $\widetilde{h}_{+bc}=0$ for all values of b and c, and (4.16) means that all the components $\widetilde{h}''_{\mu\nu\lambda}$ of the antisymmetric projection $\widetilde{h}''$ of $\widetilde{h}$ can be expressed in terms of $\widetilde{h}''_{\mu\nu+}=\widetilde{h}_{\mu\nu+}-\widetilde{h}_{\nu\mu+}$. Next, Eq. (4.17) is an expression for the symmetric part of $\widetilde{h}_{\mu\nu+}$ in terms of $E_{\mu+}$ and E_{B+}. Here is all this information collected:

$$\widetilde{h}_{+ab}=0,\ a,b=+,0,\ldots,3,B \ , \qquad (5.1)$$

$$\widetilde{h}''_{\mu\nu\lambda}=\tfrac{1}{2}\sum_{\text{cycl}}\partial_{\mu}(\widetilde{h}_{\nu\lambda+}-\widetilde{h}_{\lambda\nu+}) \ , \qquad (5.2)$$

$$\widetilde{h}_{\mu\nu+}+\widetilde{h}_{\nu\mu+}=\partial_{\mu}E_{\nu}+\partial_{\nu}E_{\mu}+\delta_{\mu\nu}E_{B} \ , \qquad (5.3)$$

with $E_a\equiv -E_{a+}$, $a=0,\ldots,3,B$.

Finally,

$$R'_{\mu\nu\lambda\rho}=\partial_{\mu}\widetilde{h}_{\nu\lambda\rho}+\delta_{\mu\lambda}\widetilde{h}_{\nu B\rho}+\delta_{\mu\rho}\widetilde{h}_{\nu\lambda B}-(\mu,\nu) \ , \qquad (5.4)$$

and the principal constraint (4.20) reduces to

$$G_{\mu\nu\lambda\rho}\equiv\partial_{\mu}\widetilde{h}_{\nu\lambda\rho}+\delta_{\mu\lambda}h_{\nu\rho}-(\mu,\nu)+(\mu\nu,\lambda\rho)$$
$$=0 \ , \qquad (5.5)$$

$$h_{\mu\nu}\equiv\widetilde{h}_{\mu B\nu}+\widetilde{h}_{\nu B\mu} \ . \qquad (5.6)$$

This tells us that $(\widetilde{h}_{\mu\nu\lambda})$ is determined up to a gauge transformation by $(h_{\mu\nu})$; but it also places a restriction on $(h_{\mu\nu})$. Integrability of (5.5) implies that $(h_{\mu\nu})$ satisfies Einstein's linearized field equation.

B. Einstein's linearized field equation

Equation (5.5), seen as a differential equation for $(\widetilde{h}_{\mu\nu\rho})$, is integrable if and only if

$$\epsilon^{\mu\nu\lambda\rho}\epsilon^{\pi\kappa\sigma\tau}\delta_{\mu\pi}\partial_{\nu}\partial_{\kappa}h_{\lambda\sigma}=0 \ .$$

This is precisely Einstein's linearized field equation:

$$R(h)_{\rho\tau}-\tfrac{1}{2}\delta_{\rho\tau}R(h)_{\sigma\sigma}=0 \ , \qquad (5.7)$$

$$R(h)_{\rho\tau}\equiv\partial^2h_{\rho\tau}-\partial_{\rho}\partial_{\sigma}h_{\sigma\tau}-\partial_{\tau}\partial_{\sigma}h_{\sigma\rho}+\partial_{\rho}\partial_{\tau}h_{\sigma\sigma} \ . \qquad (5.8)$$

It was obtained from the principal constraint (4.20), without any help from the other constraints. It is very satisfying to find that this field equation is the condition of integrability of the constraint.

C. The wrong constraint, Weyl theory

In the linearized theory, all constraints can be interpreted as a means of cutting down the space of modes of the field. The smallest space that can be defined this way will include the physical modes and some longitudinal or "gauge" modes that can be gotten rid of only by passing to a quotient space. By definition, it does not contain any ghost. True ghosts, as opposed to "scalar" or gauge modes, can be eliminated by gauge-invariant constraints, while gauge fixing will take care of the scalar modes.

It is evident, however, that the constraints imposed so far are incomplete, in the sense that the remaining redundant field components cannot be removed by gauge fixing, for the theory contains gauge-invariant quantities such as $\tilde{t}$ that have not yet been constrained. We shall now examine the consequences of setting $\tilde{t}\approx 0$.

According to (4.12) and (3.7), to first order, in Dirac notation,

$$c_{\alpha\beta\gamma}\approx\partial_\alpha\tilde{e}_{\beta\gamma}-\partial_\beta\tilde{e}_{\alpha\gamma},\quad \tilde{e}_{\alpha\beta}\equiv\tilde{h}_{\alpha\sigma\beta}y^\sigma ,$$

and the definition (4.19) gives

$$\tilde{t}_{\alpha\beta\gamma}\approx\tilde{h}_{\alpha\beta\gamma}+\partial_\alpha\tilde{e}_{\beta\gamma}-(\alpha,\beta) .$$

The sign $\approx$ means, here as always, equality up to y_α terms. The equation $\tilde{t}_{\alpha\beta\gamma}\approx 0$ can be solved for $\tilde{h}_{\alpha\beta\gamma}$, and one obtains

$$2\tilde{h}_{\alpha\beta\gamma}\approx-\partial_\beta g_{\alpha\gamma}-\partial_\alpha\tilde{e}_{\beta\gamma}-(\beta,\gamma) . \tag{5.9}$$

The quantity $g_{\alpha\beta}$ was defined by (2.52); here the contributing terms are $-\tilde{e}_{\alpha\beta}-\tilde{e}_{\beta\alpha}$.

In Minkowski notation it reads

$$2\tilde{h}_{\mu\nu\lambda}=-\partial_\nu g_{\mu\lambda}-\partial_\mu\tilde{e}_{\nu\lambda}+2\delta_{\mu\nu}\tilde{e}_{\lambda B}-(\nu,\lambda) .$$

When this is used to eliminate $\tilde{h}_{\mu\nu\lambda}$ from (5.5) one gets

$$\begin{aligned}G_{\mu\nu\lambda\rho}&=-\partial_\mu\partial_\lambda g_{\nu\rho}+\delta_{\mu\lambda}(h_{\nu\rho}-\partial_\nu\tilde{e}_{\rho B}-\partial_\rho\tilde{e}_{\nu B})\\&\quad-(\mu,\nu)+(\mu\nu,\lambda\rho)\\&=0 .\end{aligned}$$

The traceless part of this is the first-order conformal curvature tensor of $(g_{\mu\nu})$, so this metric is conformally flat, in agreement with one of the secondary constraints. The trace is

$$\begin{aligned}G_{\mu\nu\mu\rho}&=\partial^2 g_{\nu\rho}-\partial_\nu\partial_\sigma g_{\sigma\rho}-\partial_\rho\partial_\sigma g_{\sigma\nu}+\partial_\nu\partial_\rho g_{\sigma\sigma}\\&\quad+2(h_{\nu\rho}-\partial_\nu\tilde{e}_{\rho B}-\partial_\rho\tilde{e}_{\nu B})+\delta_{\nu\rho}(h_{\mu\mu}-2\partial_\sigma\tilde{e}_{\sigma B})\\&=0 .\end{aligned} \tag{5.10}$$

When this is solved for $h_{\mu\nu}$, and the solution is inserted into Einstein's equation (5.7), then one finds that the field $\tilde{e}$ cancels out and the equation obtained for g is the linearized wave equation of Weyl's theory.

In this way some contact with Weyl's theory is made, but it is highly spurious. First of all, the constraint $\tilde{t}_{\alpha\beta\gamma}\approx 0$ was not suggested by our study of ghost-free linear theory. Second, a secondary constraint requires that $(g_{\mu\nu})$ be conformally flat. Not only that; if $\tilde{t}_{\alpha\beta\gamma}\approx 0$, then the principal constraint also requires that $(g_{\mu\nu})$ be conformally flat. Therefore, to preserve the dynamical content of Weyl's theory, we must discard most of our ghost-eliminating constraints. No wonder, therefore, that Weyl's theory has ghosts.

If $\tilde{t}_{\alpha\beta\gamma}=0$, then our principal constraint tells us, in Eq. (5.10), that $(h_{\mu\nu})$ is flat, and the theory loses its main dynamical content.

D. The theory of Kaku *et al.*

The theory Kaku *et al.*[9] can be obtained by assuming factorization, as in Eq. (2.11). In the conformal case,

$$Q_\alpha=e_\alpha{}^a(\partial_a+\tfrac{1}{2}\Gamma_a^{\beta\gamma}S_{\beta\gamma}),\quad \phi_\alpha{}^{\beta\gamma}=e_\alpha^a\Gamma_a^{\beta\gamma} .$$

Now set

$$\mathscr{D}_a\equiv\partial_a+\tfrac{1}{2}\Gamma_a^{\beta\gamma}S_{\beta\gamma} ,$$

$$[\mathscr{D}_a,\mathscr{D}_b]=\tfrac{1}{2}\Omega_{ab}{}^{\gamma\delta}S_{\gamma\delta} ,$$

then

$$\tilde{R}_{\alpha\beta}{}^{\gamma\delta}=e_\alpha^a e_\beta^b\Omega_{ab}{}^{\gamma\delta} .$$

The restriction of Ω to the group of translations and dilatations is $\Omega_{ab\gamma\delta}y^\delta$. The constraint introduced by Kaku *et al.* is $\Omega_{ab\gamma\delta}y^\delta\approx 0$, which says that the restriction of Ω to the translations is zero. This is the same as $\tilde{t}=0$. Using this constraint, they reduced their theory to Weyl's theory.

We can give a tentative geometrical interpretation of our principal constraint. The restriction of Ω to the Lorentz group is $\Omega_{\mu\nu\lambda\rho}$. In the first approximation, the principal constraint means that $\Omega_{\mu\nu\lambda\rho}+\Omega_{\lambda\rho\mu\nu}=0$.

E. Additional constraints and gauge fixing

Taking into account all constraints imposed so far, we notice that the "B-components" $\tilde{h}_{Bab}$ are not involved. The role that is played by the field (a_α) also remains mysterious. These quantities are irrelevant to lowest order and do not describe the physical modes. They can be constrained in a fully gauge-invariant manner, for example, by postulating that

$$a^\alpha\phi_{\alpha\beta\gamma}=0 , \tag{5.11}$$

$$Q_\alpha a_\beta-Q_\beta a_\alpha=0 . \tag{5.12}$$

The remaining free components of $\tilde{h}$ are $\tilde{h}_{\mu\nu+}+\tilde{h}_{\nu\mu+}$ (zero up to a gauge transformation), $\tilde{h}_{\mu\nu+}-\tilde{h}_{\nu\mu+}$, $\tilde{h}_{\mu\nu B}+\tilde{h}_{\nu\mu B}$ (meaningful up to a gauge transformation), $\tilde{h}_{\mu\nu B}-\tilde{h}_{\nu\mu B}$, $\tilde{h}_{\mu+B}$, and $\tilde{h}_{\mu\nu\lambda}$ (fixed up to a gauge transformation). First-order gauge transformations affects these quantities in the following way:

$$\delta_\Lambda(\tilde{h}_{\mu\nu+}+\tilde{h}_{\nu\mu+})=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu+2\delta_{\mu\nu}\xi_B ,$$

$$\delta_\Lambda(\tilde{h}_{\mu\nu+}-\tilde{h}_{\nu\mu+})=\partial_\mu\xi_\nu-\partial_\nu\xi_\mu-2\Lambda_{\mu\nu} ,$$

$$\delta_\Lambda(\tilde{h}_{\mu\nu B}\pm\tilde{h}_{\nu\mu B})=\partial_\mu\Lambda_{B\nu}\pm\partial_\nu\Lambda_{B\mu} ,$$

$$\delta_\Lambda \tilde{h}_{\mu+B}=\Lambda_{\mu B}-\partial_\mu \xi_B \ ,$$
$$\delta_\Lambda h'_{\mu\nu\lambda}=\partial_\nu\Lambda_{\lambda\mu}+\partial_\lambda\Lambda_{\mu\nu}-2\partial_\mu\Lambda_{\nu\lambda}+3\delta_{\mu\nu}\Lambda_{\lambda B}-3\delta_{\mu\lambda}\Lambda_{\nu B} \ .$$

We also have

$$\delta_\Lambda f_\mu=\Lambda_{\mu B} \ ,$$

so that, if we fix the gauge by setting $f_\mu=0$, then we loose the option of choosing $\partial_\mu h_{\mu\nu}=0$. We may transform $\tilde{h}_{\mu\nu+}$ to zero, but then we are no longer free to impose conditions on $h'_{\mu\nu\lambda}$. The analysis cannot be carried further than this until the couplings between field components have been clarified by a study of the higher orders.

VI. INTEGRABILITY

A. Integrability in perturbation theory

The first-order constraints were determined by unitarity, and the nonlinear extrapolation given in Sec. IV D is fully gauge invariant with respect to the local, conformal gauge algebra. Nevertheless, there is some ambiguity left, for there exist tensor fields of the right type that vanish in the two lowest orders of the perturbation expansion (zeroth and first order). In particular, we may modify the principal constraint (4.30) as follows:

$$R'_{\alpha\beta\gamma\delta}+c[\tilde{t}_{\gamma\sigma\alpha}\tilde{t}_{\sigma\delta\beta}-(\alpha,\beta)]+(\alpha\beta,\gamma\delta)\approx 0 \ , \qquad (6.1)$$

c being any constant, this having no effect in zeroth or first order.

To resolve this ambiguity we examined the integrability of (6.1) in the sense of the perturbation expansion. (Exact solutions that cannot be expanded in a perturbation series have no interest in quantum field theory.) In the simplest case, when the metric g is flat and the vector field $\check{a}$ vanishes, terms appear in the second order that make it seem very unlikely that any solution of (6.1) exist to this order, unless $c=-1$. The final choice of principal constraint is thus

$${}^tG_{\alpha\beta\gamma\delta}\approx 0 \ , \qquad (6.2)$$

$$G_{\alpha\beta\gamma\delta}\equiv e_\alpha\tilde{\phi}_{\beta\gamma\delta}+(\tilde{\phi}_{\alpha\gamma\sigma}\tilde{\phi}_{\beta\sigma\delta}-\tilde{t}_{\gamma\sigma\alpha}\tilde{t}_{\sigma\delta\beta})-(\alpha,\beta)-c_{\alpha\beta}{}^\sigma\tilde{\phi}_{\sigma\gamma\delta}+(\alpha\beta,\gamma\delta) \ , \qquad (6.3)$$

where ${}^tG$ means transverse part of G and $\approx$ is equality up to y terms.

B. Simplifications

Now it turns out that this modification of the principal constraint solves the integrability problem to all orders. In fact, when the secondary constraint (4.28) (vanishing of the antisymmetric part of the torsion) is taken into account, it is readily verified that

$$G_{\alpha\beta\gamma\delta}=e_\alpha\tilde{\phi}_{\beta\gamma\delta}+\gamma_{\alpha\beta}{}^\sigma\tilde{\phi}_{\sigma\gamma\delta}+\gamma_{\alpha\gamma}{}^\sigma\tilde{\phi}_{\beta\sigma\delta}+\gamma_{\alpha\delta}{}^\sigma\tilde{\phi}_{\beta\gamma\sigma}-(\alpha,\beta)+(\alpha\beta,\gamma\delta)+[\gamma_{\alpha\gamma}{}^\sigma\gamma_{\beta\delta}{}^\sigma-(\alpha,\beta)+(\alpha\beta,\gamma\delta)] \ . \qquad (6.4)$$

Here γ is determined by e_α:

$$\gamma_{\alpha\beta\gamma}\equiv-\tfrac{1}{2}(c_{\alpha\beta\gamma}+c_{\gamma\alpha\beta}-c_{\beta\gamma\alpha}) \ . \qquad (6.5)$$

There are no longer any terms of higher order in $\tilde{\phi}$.

Next, by means of a gauge transformation, we convert $\tilde{\phi}$ to a tensor. The action of the local algebra on γ is

$$(\delta_\Lambda\gamma)_{\alpha\beta\gamma}=(\mathscr{L}_\Lambda\gamma)_{\alpha\beta\gamma}-e_\alpha\Lambda_{\beta\gamma} \ , \qquad (6.6)$$

so that the quantity π defined by

$$\pi_{\alpha\beta\gamma}\equiv\tilde{\phi}_{\alpha\beta\gamma}-\gamma_{\alpha\beta\gamma} \qquad (6.7)$$

is a tensor field. It is antisymmetric in (β,γ) and has mixed symmetry in $(\alpha\beta\gamma)$.

Using (6.7) to eliminate $\tilde{\phi}$ from (6.4) we get

$$G_{\alpha\beta\gamma\delta}=q_\alpha\pi_{\beta\gamma\delta}-(\alpha,\beta)+(\alpha\beta,\gamma\delta)+[r_{\alpha\beta\gamma\delta}+(\alpha\beta,\gamma\delta)] \ , \qquad (6.8)$$

where the small covariant derivative q is

$$q_\alpha=e_\alpha+\tfrac{1}{2}\gamma_\alpha{}^{\beta\gamma}\tilde{S}_{\beta\gamma} \ , \qquad (6.9)$$

the small torsion vanishes identically and r is the small curvature,

$$r_{\alpha\beta\gamma\delta}=e_\alpha\gamma_{\beta\gamma\delta}+\gamma_{\alpha\gamma}{}^\sigma\gamma_{\beta\sigma\delta}-(\alpha,\beta)-c_{\alpha\beta}{}^\sigma\gamma_{\sigma\gamma\delta} \ . \qquad (6.10)$$

C. The small curvature

We suspect that the small curvature (6.10) vanishes, in view of the secondary constraint (4.29) ($g_{\mu\nu}$ conformally flat). Only the transverse part contributes to the constraint, but it would be very convenient to have the whole tensor vanish, since this would make q_α commutative and greatly facilitate the investigation of integrability. This turns out to be essentially a matter of choosing among the solutions of

$$[e_\alpha,e_\beta]=c_{\alpha\beta}{}^\gamma e_\gamma \ ,$$

the defining equation for the quantity $(c_{\alpha\beta}{}^\gamma)$. Since

$$e_\alpha=\partial_\alpha-a_\alpha\hat{N}-\tilde{e}_\alpha, \ \ y^\alpha\tilde{e}_\alpha=0, \ \ \hat{N}=y^\alpha\partial_\alpha \ ,$$
$$(e_\beta{}^\alpha-\delta_\beta{}^\alpha-c_{\cdot\beta}{}^\alpha)e_\alpha{}^\gamma=0 \ . \qquad (6.11)$$

Recall that a dot in the place of an index indicates a contraction against y^α.

Now consider

$$y^\alpha r_{\alpha\beta\gamma\delta}=(e_\beta{}^\alpha-\delta_\beta{}^\alpha-c_{\cdot\beta}{}^\alpha)\gamma_{\alpha\gamma\delta}-(e_\beta\gamma_{\cdot\gamma\delta}+\gamma_{\beta\gamma}{}^\sigma\gamma_{\cdot\sigma\delta}+\gamma_{\beta\delta}{}^\sigma\gamma_{\cdot\gamma\sigma}) \ . \qquad (6.12)$$

In view of (6.11) it is very natural to ensure the vanishing of the first term by assuming factorization of γ:

$$\gamma_\sigma{}^{\gamma\delta}=e_\sigma{}^a\overline{\gamma}_a{}^{\gamma\delta} \ , \qquad (6.13)$$

especially since this also gets rid of the second term, for (6.13) gives

$$y^\alpha\gamma_\alpha{}^{\beta\gamma}=0 \ . \qquad (6.14)$$

It follows that π is transverse in the first index and that

$$(e_\beta{}^\alpha-\delta_\beta{}^\alpha-\gamma_\beta{}^{\cdot\alpha})e_\alpha{}^\gamma=0\ . \tag{6.15}$$

Next

$$y^\gamma r_{\alpha\beta\gamma\delta}=e_\alpha\gamma_{\beta\cdot\delta}-e_\alpha{}^\gamma\gamma_{\beta\gamma\delta}+\gamma_\beta{}^{\cdot\gamma}\gamma_{\alpha\delta\gamma}-(\alpha,\beta)-c_{\alpha\beta}{}^\sigma\gamma_{\sigma\cdot\delta}\ . \tag{6.16}$$

To make this vanish we need some specialization of $\gamma_{\beta\cdot\delta}$. Equation (6.15) strongly suggests that

$$\gamma_{\beta\cdot\alpha}=e_{\beta\alpha}-\delta_{\beta\alpha}+y_\beta a_\alpha=\tilde{e}_{\beta\alpha}\ , \tag{6.17}$$

$$a^\alpha e_\alpha=0\ . \tag{6.18}$$

Equation (6.17) is anyhow very desirable, since it is equivalent to $\pi_{\alpha\cdot\gamma}=0$, so that

$$(\pi_{\alpha\beta\gamma})\text{ is completely transverse}\ . \tag{6.19}$$

In view of (6.18), $(c_{\alpha\beta}{}^\gamma)$ is determined up to $q_{\alpha\beta}a^\gamma$ ($q_{\alpha\beta}$ arbitrary), and this freedom can be exploited to enforce (6.13) and (6.17). The essential assumption and new constraint is Eq. (6.18). It is consistent with the perturbation expansion.

We need one more constraint to make (6.16) vanish, namely,

$$q_\alpha a_\beta+a_\alpha a_\beta\approx 0\ . \tag{6.20}$$

(This makes $y^\gamma r_{\alpha\beta\gamma\delta}=0$, not just ≈ 0.) Now the complete small curvature factorizes:

$$r_{\alpha\beta\gamma\delta}=e_\alpha{}^a e_\beta{}^b[\partial_a\bar{\gamma}_{b\gamma\delta}+\bar{\gamma}_{a\gamma}{}^\sigma\bar{\gamma}_{b\sigma\delta}-(a,b)]\ . \tag{6.21}$$

For this to vanish, $(\bar{\gamma}_{\mu\nu\lambda})$ must be an ordinary flat, conformal connection. This is consistent with our secondary constraint. From now on the small curvature is taken to vanish, so that the small covariant derivative is commutative.

D. Derivation of the wave equation

We shall now find the condition for the integrability of Eq. (6.2). The tensor G was reduced to (6.8). With $r=0$ the transverse part is

$${}^tG_{\alpha\beta\gamma\delta}=(q_\alpha+a_\alpha)\pi_{\beta\gamma\delta}-(\alpha,\beta)+(\alpha\beta,\gamma\delta)\ .$$

The operator q_α is commutative, and because of (6.20) so is $q_\alpha+a_\alpha$. Instead of ${}^tG\approx 0$, we can write

$$(q_\alpha+a_\alpha)\pi_{\beta\gamma\delta}-y_\alpha\chi_{\beta\gamma\delta}-(\alpha,\beta)+(\alpha\beta,\gamma\delta)=0\ , \tag{6.22}$$

with χ arbitrary. Our first task is to express χ in terms of π.

To simplify the calculations we shall now introduce yet another constraint, though this seems to be not at all relevant to the final results. We suppose that

$$a^\alpha\pi_{\alpha\beta\gamma}=0\ . \tag{6.23}$$

In the first order this becomes $\tilde{h}_{Bbc}=0$; the need for this was seen already. Now we easily obtain

$$\chi_{\beta\gamma\delta}\approx(q_\gamma+2a_\gamma)\pi_{\delta\sigma\beta}a^\sigma-(\gamma,\delta)\ . \tag{6.24}$$

The condition of integrability of (6.22) is

$$\sum_{\epsilon\alpha\beta}\sum_{\eta\gamma\delta}(q_\epsilon+a_\epsilon)(q_\eta+a_\eta)(y_\alpha\chi_{\beta\gamma\delta}+y_\gamma\chi_{\delta\alpha\beta})\approx 0\ .$$

Each sum is an alternating sum over the permutations of the indices written underneath. We insert our result (6.24) for χ and get after some calculation

$$\sum_{\epsilon\alpha\beta}\sum_{\eta\gamma\delta}\bar{\delta}_{\epsilon\eta}(q_\alpha+2a_\alpha)(q_\gamma+2a_\gamma)h_{\beta\delta}\approx 0\ , \tag{6.25}$$

$$h_{\alpha\gamma}\equiv a^\beta(\pi_{\alpha\beta\gamma}+\pi_{\gamma\beta\alpha}),\quad \bar{\delta}_{\alpha\beta}\equiv\delta_{\alpha\beta}-y_\alpha a_\beta-a_\alpha y_\beta\ . \tag{6.26}$$

The transverse part vanishes and the y terms are undetermined, so this is just a condition on the vector components. We now convert to Minkowski notation.

E. Minkowski notation

It is convenient to make a slight change in our conventions. Recall that, with $a,b\neq B$,

$$e_\alpha{}^a\theta_a{}^\beta=\delta_\alpha{}^\beta-y_\alpha a^\beta,\quad e_\alpha{}^a\theta_b{}^\alpha=\delta_b{}^a\quad (a,b\neq B)\ .$$

Now define

$$e_\alpha{}^B\equiv y_\alpha,\quad \theta_B{}^\alpha\equiv a^\alpha\ , \tag{6.27}$$

to make θ the precise inverse of e. (But $e_\alpha=e_\alpha{}^a\partial_a$, $a\neq B$, as before.) We collect some identities, and a definition:

$$e_\alpha{}^a\theta_a{}^\beta=\delta_\alpha{}^\beta,\quad e_\alpha{}^a\theta_b{}^\alpha=\delta_a{}^b\ , \tag{6.28}$$

$$y^\alpha e_\alpha{}^a=\delta_+{}^a,\quad a^\alpha e_\alpha{}^a=\delta_B{}^a\ , \tag{6.29}$$

$$\theta_a{}^\alpha y_\alpha=\delta_a{}^B,\quad \theta_a{}^\alpha a_\alpha\equiv\check{a}_a\ . \tag{6.30}$$

If $(A_{\alpha\cdots}{}^{\beta\cdots})$ is any internal tensor we define the associated Minkowski complex by

$$\check{A}_{a\cdots}{}^{b\cdots}=\theta_a{}^\alpha\cdots e_\beta{}^b\cdots A_{\alpha\cdots}{}^{\beta\cdots}\ . \tag{6.31}$$

Most of our Dirac tensors are transverse, and it is convenient to define the small covariant derivative as a map between transverse tensors. If A is any transverse tensor field let

$${}^tq_\alpha A\equiv\text{ transverse part of }q_\alpha A\ . \tag{6.32}$$

For example,

$${}^tq_\alpha h_{\beta\gamma}=(q_\alpha+2a_\alpha)h_{\beta\gamma}+a_\beta h_{\alpha\gamma}+a_\gamma h_{\beta\alpha}\ . \tag{6.33}$$

Now define, for A transverse,

$$\nabla_\mu\check{A}_{a\cdots}=\theta_\mu{}^\alpha\theta_a{}^\beta\cdots{}^tq_\alpha A_{\beta\cdots}\ . \tag{6.34}$$

If A has no B components, then this is exactly the same as Dirac's co-covariant derivative.[10]

In this notation (6.25) takes the form

$$\sum_{\mu\nu\lambda}\sum_{\rho\sigma\tau}g_{\mu\rho}\nabla_\nu\nabla_\sigma h_{\lambda\tau}=0\ . \tag{6.35}$$

We have left off the caret on h. The co-covariant derivatives commute and the co-covariant derivative of g is zero. Proceeding as in the linear case we find that this is the same as

$$R(g,h)_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R(g,h)=0\ , \tag{6.36}$$

$$R(g,h)_{\mu\nu}=g^{\lambda\rho}(\nabla_\lambda\nabla_\rho h_{\mu\nu}-\nabla_\mu\nabla_\lambda h_{\nu\rho}-\nabla_\nu\nabla_\lambda h_{\mu\rho}+\nabla_\mu\nabla_\nu h_{\lambda\rho})\ . \tag{6.37}$$

This is the equation for gravitational waves in empty space.

The derivation of (6.36) from an action principal will be taken up in Sec. VII, together with the problem of matter coupling and symmetry breaking.

VII. ACTION PRINCIPLE AND MATTER COUPLING

A. Matter fields

The final wave equation (6.37) involves only the metric g (with the vector field a) and the symmetric tensor field h. The main "connection field" π has disappeared. It is important to understand whether this nonappearance of π is permanent. We now investigate the extent to which π reappears in the coupling of gravity to matter fields. We make no assumptions about flatness of g.

Consider first a scalar field u of degree -1. The suggestion of minimal coupling is that the action density be proportional to $(Q_\alpha u)(Q_\alpha u)$. But $Q_\alpha u = e_\alpha u$ so in this case π decouples. The action is

$$\frac{1}{2}\int \frac{d^4x}{\sqrt{-g}} g^{ab}(\partial_a \check{u})(\partial_b \check{u})$$
$$= \frac{1}{2}\int \frac{d^4x}{\sqrt{-g}} g^{\mu\nu}(\partial_\mu + a_\mu)\check{u}(\partial_\nu + a_\nu)\check{u} \;.$$

The last formula was obtained from the following intermediary result:

$$g^{\mu b}a_b = g^{\mu\nu}a_\nu + g^{\mu +} = 0,\quad g^{++} = a^\mu a_\mu \;.$$

If we modify the minimal coupling by adding $-\frac{1}{6}{}^*R\check{u}^2$ to the action density, then the dependence on a_μ cancels out leaving the familiar

$$\frac{1}{2}\int \frac{d^4x}{\sqrt{-g}}(g^{\mu\nu}\partial_\mu \check{u}\partial_\nu \check{u} - \tfrac{1}{6}R\check{u}^2) \;, \tag{7.1}$$

where R is the ordinary curvature of g.

The next case of interest is electromagnetism, but this requires that we develop an exterior calculus. If u,A,F are internal tensors of ranks 0,1,2 with F antisymmetric, define an operator d by

$$(du)_\alpha = Q_\alpha u \;,$$
$$(dA)_{\alpha\beta} = Q_\alpha A_\beta - Q_\beta A_\alpha - t_{\alpha\beta}{}^\gamma A_\gamma \;,$$
$$(dF)_{\alpha\beta\gamma} = \sum_{\text{cycl}} (Q_\alpha F_{\beta\gamma} - t_{\alpha\beta}{}^\sigma F_{\sigma\gamma}) \;.$$

The terms that involve the torsion are completely determined by the requirement that $d\circ d = 0$. The verification makes use of the Jacobi identity

$$\sum_{\text{cycl}} [Q_\alpha,[Q_\beta,Q_\gamma]] = 0 \;,$$

which is equivalent to the following two equations (identities):

$$\sum_{\text{cycl}} (Q_\alpha t_{\beta\gamma}{}^\sigma + t_{\beta\gamma}{}^\tau t_{\alpha\tau}{}^\sigma - R_{\alpha\beta\gamma}{}^\sigma) = 0 \;, \tag{7.2}$$

$$\sum_{\text{cycl}} (Q_\alpha R_{\beta\gamma}{}^A + t_{\beta\gamma}{}^\tau R_{\alpha\tau}{}^A) = 0 \;. \tag{7.3}$$

But now it is seen that π cancels out; the exterior derivative is just the antisymmetric small covariant derivative.

The conclusion is that π decouples as long as no spinor fields appear in the matter Lagrangian. This is quite analogous to the absence of torsion in ordinary gravity without spinors. Setting aside the problem of spinors, we conclude that matter moves in the fields g,a and in some cases at least, independently of a along the geodesics of g. It is quite clear, therefore, that the conformal flatness of g will have to be replaced by something else.

B. Action principle

In applications without spinors the "connection" π decouples and it is natural to ask whether a self-contained theory can be formulated entirely in terms of g,a and h, and matter fields. Consider, therefore, the action

$$\int \frac{d^4x}{\sqrt{-g}} h^{\mu\nu}{}^*R_{\mu\nu}(g,a) + I_m \;. \tag{7.4}$$

Here I_m stands for the matter action, and ${}^*R_{\mu\nu}(g,a)$ is the contracted curvature tensor of Weyl[3] and Dirac.[10] Variation with respect to $h^{\mu\nu}$, $g^{\mu\nu}$, and a^μ yields, respectively,

$${}^*R_{\mu\nu}(g,a) = 0 \;, \tag{7.5}$$

$$R(g,h)_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R(g,h) = T_{\mu\nu} \;, \tag{7.6}$$

$$g^{\sigma\nu}(\nabla_\sigma h_{\mu\nu} + \nabla_\mu h_{\nu\sigma} + \nabla_\nu h_{\sigma\mu}) = 0 \;. \tag{7.7}$$

The first equation is a weakened form of the constraint on g,a. In the second equation, the left-side is just (6.37) and the right-hand side comes from the variation of I_m with respect to $g^{\mu\nu}$. The last equation is valid if I_m is independent of the field a; it is our reward for tolerating this otherwise dubious field. Note that the action and all the field equations are invariant with respect to the local conformal algebra, and in particular, that Eq. (7.6) is scale invariant.

If the action (7.4) is adopted, then it is seen that the field g is canonically conjugate to h. Evidently, therefore, both fields must be quantized. This is in accord with the insight of the linear theory,[7] where the constraints must be imposed as initial conditions on the states, and not on the field operators.

The next urgent problem is symmetry breaking. Einstein's theory can be obtained by setting $a=0$ and $g=h$ in the action, but such a procedure is of course not soft enough. Certainly, g and h are not to remain totally independent of each other, but their respective roles are nevertheless worthy of notice. Thus h describes the (helicity ± 2, massless) radiated quanta, while g provides the geodesics for matter to move on. This separation of waves and background could turn out to be an important feature of the description of gravitational waves. It may turn out that the field g is (in its action on the states) essentially classical; in this case the conflict between causal propagation and a quantized metric would become less perplexing.

ACKNOWLEDGMENTS

I am happy to acknowledge my indebtedness to Moshe Flato for his help in developing the physical ideas that have been presented here. The program of this paper has been developed jointly with him. During the last two years it has been the subject of numerous discussions between us, without which the paper could not have been completed. Others who have been helpful include Birne Binegar and Gregg Zuckerman. I thank RIMS, and especially Huzihiro Araki, for hospitality. I also thank the Japanese Ministry of Education and the U.S. National Science Foundation for financial support.

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