Formation and evolution of cosmic strings

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Statistical properties of the system of strings produced at a phase transition in the early universe are studied using a Monte Carlo simulation. The fractal dimension of the strings and the size distribution of closed loops are determined, and it is shown that infinite strings contribute about 80% to the total length of strings in the universe. The results are used to analyze the cosmological evolution of strings. Formation of domain walls and of walls bounded by strings is also discussed, and it is shown that in both cases the system is dominated by one infinite cluster of very complicated topology. The results of the simulations may also be of interest in condensed-matter systems where similar topological defects can be formed.

I. INTRODUCTION

Phase transitions in the early universe can give rise to macroscopic topologically stable defects—vacuum domain walls and strings.[1] Hybrid topological objects, such as walls bounded by strings, can also be formed.[2] Topologically stable domain walls are disastrous for cosmological models and should be avoided. Walls bounded by strings (as well as monopoles connected by strings) are presumably harmless: they rapidly break into pieces and decay practically without trace. Strings, on the other hand, are potentially very interesting: they can generate cosmological density fluctuations sufficient to explain the galaxy formation[3-7] and can even lead to some observable effects at present.[8]

The simplest model leading to the formation of strings is the Abelian Higgs model,

$$L=D_\mu\phi^\dagger D^\mu\phi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{4}\lambda(\phi^\dagger\phi-\eta^2)^2\ . \tag{1.1}$$

Here, ϕ is a complex scalar field coupled to the Abelian gauge field A_μ, $D_\mu=\partial_\mu+ieA_\mu$, and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Below the critical temperature, $T_c\sim\eta$, the Higgs field develops a vacuum expectation value, $\langle\Phi\rangle=\eta e^{i\theta}$. The magnitude of $\langle\phi\rangle$ is fixed by the model, but the phase θ is arbitrary and can be different in different regions of space. One can introduce the correlation length ξ such that the values of θ are uncorrelated for points separated by a distance greater than ξ. In the early universe, causality requires that ξ cannot exceed the causal horizon: $\xi\lesssim t$. Chaotic spatial variation of the phase leads to the formation of strings. Whenever θ changes by $2\pi n$ around a closed loop, where n is a nonzero integer, thin tubes of the false vacuum ($\langle\phi\rangle=0$) must be caught somewhere inside the loop. Elementary strings have $\Delta\theta=\pm2\pi$. Strings with $|\Delta\theta|>2\pi$ are probably unstable and decay into elementary ones. In the Higgs model of Eq. (1.1) it is possible to assign a direction to the strings. It can be taken as the direction of the magnetic gauge field trapped inside the string. An important property of topologically stable strings is that they cannot have free ends: the strings can be either infinite or closed.

In the general case, let G and H be the symmetry groups before and after the phase transition, respectively, $H\subset G$. Strings are formed if the first homotopy group $\pi_1(G/H)$ is nontrivial, or equivalently, if the manifold of equivalent vacuum states after the symmetry breaking is not simply connected.[1] For the Higgs model (1.1), $G/H=\mathrm{U}(1)$ and $\pi_1(G/H)=Z$. If the group G is connected and simply connected $[\pi_0(G)=\pi_1(G)=I]$, then $\pi_1(G/H)=\pi_0(H)$, and so strings are formed if the new symmetry group H has disconnected components.[9] A fairly common example of this sort is when the group H is a product of a continuous group and Z_2, e.g., $\mathrm{SO}(10)\to\mathrm{SU}(5)\times Z_2$. In this case the strings have no direction and any two parallel strings can be annihilated by one another. This analysis applies to local, as well as global symmetry breakings. The simplest example of a model with global strings is the Goldstone model, which is given by Eq. (1.1) with A_μ set equal to zero.

To analyze the cosmological evolution of strings, it is important to know the statistical properties of the system of strings at formation. In particular, one has to know the number density and size distribution of closed loops and the fractal dimension of the strings. A curve is said to have fractal dimension d if the average length along the curve l between the points separated by a distance R is given by (in the limit of large R and l)

$$l\sim\mathrm{const}\times R^d\ , \tag{1.2}$$

$d=1$ for a straight line, and $d=2$ for a Brownian curve. Since the phases of the Higgs field are uncorrelated on scales greater than the correlation length ξ, it is natural to assume that the strings are Brownian, so that

$$l\sim R^2/\xi \tag{1.3}$$

and $d=2$. Then the same relation is expected to hold for large closed loops with R now characterizing the overall extent of the loop. The "size" R can be defined as $R=\Delta x_1+\Delta x_2+\Delta x_3$, where Δx_i are the extents of the loop in three perpendicular directions. Then R is roughly the length the loop would have if all small-scale irregularities were smoothed out.

The idea of scale invariance has proved to be very fruitful in the physics of phase transitions, and one can try to assume as a working hypothesis that the system of strings is scale invariant.[4,6,10] Mathematically, this can be formulated as follows. If we smooth out all irregularities of the strings on scales smaller than $\xi' > \xi$ and remove all closed loops of size smaller than ξ', we would get a system of strings statistically identical to the original one, up to an overall scale transformation $\xi \rightarrow \xi'$. To put it differently, one can say[6] that increasing ξ corresponds to viewing the system at a lower resolution. Then the scale invariance means that the system looks the same (in terms of its statistical properties) at all resolutions. With this assumption, the size distribution of closed loops is easily derived.[10] From dimensionality, the number of closed loops with size from R to $R+dR$ per unit volume can be written as

$$dn = \frac{dR}{R^4} f\left[\frac{R}{\xi}\right] . \tag{1.4}$$

Smoothing out small-scale irregularities does not change the size of the loops, and so if the system is scale invariant, the size distribution should be independent of ξ. Hence, one expects

$$dn \sim R^{-4} dR . \tag{1.5}$$

The length distribution of loops can be derived from (1.3) and (1.5):

$$dn \sim \xi^{-3/2} l^{-5/2} dl . \tag{1.6}$$

Another prediction following from the assumption of scale invariance is that there should be no infinite strings. All strings must be closed loops with a size distribution (1.5), so that the probability that an arbitrarily chosen segment belongs to an infinite string is zero.

It should be emphasized that the assumptions of scale invariance and of the Brownian character of strings have no firm theoretical foundation and should be subjected to careful checks. For example, the Brownian character of the strings was conjectured on the basis that the phases of the Higgs field are uncorrelated on scales greater than ξ. However, the absence of correlations in the Higgs field does not necessarily mean that there are no correlations in the system of strings. To give an example of a situation where such arguments can be misleading, consider the strings passing through a large contour of size $L \gg \xi$. The integral

$$\delta N = \frac{1}{2\pi} \oint_L d\theta \tag{1.7}$$

gives the difference between the numbers of strings passing through the contour in the two opposite directions. Breaking the contour into $\sim L/\xi$ pieces of size $\sim \xi$ and using the randomness of the Higgs phases, we obtain an estimate

$$\delta N \sim (L/\xi)^{1/2} . \tag{1.8}$$

On the other hand, the total number of strings passing through the contour is $N \sim (L/\xi)^2$, and if one assumes that the system is totally random, one would obtain $\delta N \sim N^{1/2} \sim L/\xi$, which is incorrect.

The main purpose of the present paper is to study the string formation using a Monte Carlo simulation. Our method and the results (which are somewhat surprising) are presented in Sec. II. In Sec. III, we review the cosmological evolution of strings in the light of the results obtained in Sec. II. We have also done simulations of the formation of domain walls (Appendix A) and of walls bounded by strings (Appendix B). The results are summarized in Sec. IV.

Before we turn to our analysis, we would like to note that cosmological strings have condensed-matter analogs.[11] U(1) strings are analogous to magnetic flux tubes in type-II superconductors, Z_2 strings are similar to line defects in neumatics, and global U(1) strings are similar to vortex lines in superfluids. It is possible that our results can also be applied to the analysis of nonequilibrium phase transitions in these systems.

II. MONTE CARLO SIMULATION[12]

We consider a large cubic volume of size $N\xi$ divided into N^3 cubic cells of size ξ and choose our units so that $\xi=1$ (here N is an integer). The Higgs field is randomly assigned a phase at each vertex and is assumed to vary smoothly between the vertices.

As a simple approximation, we allow the phase at the vertices to take only three values: $\theta=0$, $2\pi/3$, $4\pi/3$, which we designate by numbers 0, 1, 2. A string passes through the face of a cubic cell if the Higgs field rotates through 2π when traced around the face in real space. The sense of rotation determines the direction of the string. A continuous dependence of the Higgs field on position is mimicked by assuming that at each step in real space, the Higgs field rotates to its new direction in group space in the shorter of the two possible senses. This is illustrated in Fig. 1. With the conventions of Fig. 1, a rotation from direction 0 to direction 1 is in the clockwise sense, while a rotation from 0 to 2 is in the counterclockwise sense. With Higgs field directions assigned randomly with equal probabilities, one finds that the probability for a string to pass through a particular face is 0.29.

Applying the prescription above to the faces of all cells, we identify all the string segments coming into and out of each cell. It is readily verified that the construction is

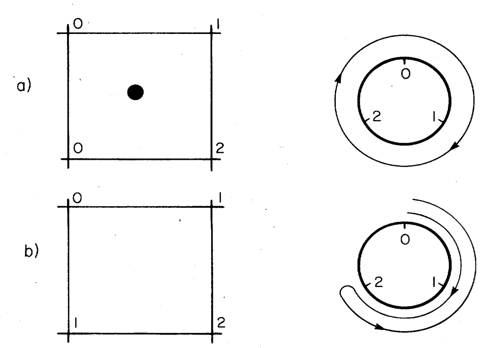

FIG. 1. As we go around the face of a cube in real space, the Higgs phase describes a certain trajectory in the group space (shown in the right column). A string passes through the face if that trajectory has a nonzero winding number (as in case a).

such that a particular string must either be closed, forming a loop, or must terminate on the boundary of the system. An ambiguity arises when two string segments are found to enter (and exit) a particular cell. Then each incoming string is assigned randomly to an outgoing string, mimicking the behavior expected in the absence of a regular cubic lattice.

The probability for no strings to pass through a cell is 0.23, and the probabilities for one and two strings to enter (and leave) a cell are 0.66 and 0.11, respectively. (In our model it is impossible for more than two strings to meet at the same cell.) The average number of strings per unit cell is thus 0.88. When a string passes through a particular face, one finds that the probability for it to continue in a straight line without deflection is $\simeq 0.13$, while the probability for it to be deflected to one of the four orthogonal directions is $\simeq 0.22$. The ratio of these probabilities is $\simeq 1.7$, while for a random walk they would be equal. This shows the presence of short-range correlations in the shape of the strings. Our main interest, however, is in the properties of very long strings.

Most of our numerical simulations were done on a lattice of size $N=40$. To determine the fractal dimension of the strings, we have found the average distance R between the points of the strings separated by the length of string l. For each value of l, the averaging was done over a large number of segments (no less than a hundred) and over three runs of the simulation with different realizations of the random phases. The results are shown in Fig. 2, where $\log R$ is plotted versus $\log l$. The least-squares fit of the data by a straight line gives

$$l = AR^d \tag{2.1}$$

with

$$A = 0.97 \pm 0.03, \quad d = 2.00 \pm 0.07 \; . \tag{2.2}$$

(The standard deviations were determined by repeating the procedure five times, so that we had 15 runs of the simulation together.) We see that $d=2$ with a good accuracy, and so the large-scale behavior of the strings is indeed Brownian.

We define the size (or "perimeter") of a closed loop as $R = n_1 + n_2 + n_3$, where n_1, n_2, and n_3 are the extents of the loop in the x, y, and z directions, respectively. The extents are taken in terms of numbers of cells, not in terms of coordinate differences between the points of the loop. For example, for the smallest possible loop, like the one shown in Fig. 3, $n_1 = n_2 = 2$, $n_3 = 1$, and $R = 5$. This definition of R may seem artificial for small loops; however, we expect that for large loops all reasonable definitions should give similar results. (We have checked this using the definition $R = \max\{n_1, n_2, n_3\}$.) Figure 4 shows the plot of $\log l$ vs $\log R$ for closed loops. The least-squares fit gives Eq. (2.1) with

$$A = 0.21 \pm 0.02, \quad d = 1.90 \pm 0.05 \; . \tag{2.3}$$

For this and all the following fits, we have included only the values of R greater than 5 and smaller than the cutoff value, R_c (shown in the figure). R_c is the first value of R for which less than four loops are present in the sample. (For $R > R_c$, statistical fluctuations in n are expected to be greater than $\sim 4^{-1/2} = 50\%$.) In Eq. (2.3) the fractal dimension of the loops is slightly smaller than 2. This is probably due to the relatively small size of the lattice: the fractal dimension of the loops is expected to approach that for arbitrary segments of string only in the asymptotic limit.

To investigate the overall shape of the loops, we have calculated the average volume-to-surface ratio,

$$\frac{V}{S} = \frac{n_1 n_2 n_3}{2(n_1 n_2 + n_1 n_3 + n_2 n_3)} \; , \tag{2.4}$$

as a function of size R. The result is

$$V/S = KR^{\nu} \; ,$$
$$K = (5.2 \pm 0.1) \times 10^{-2}, \quad \nu = 1.00 \pm 0.01 \; . \tag{2.5}$$

For cubical volumes with $n_1 = n_2 = n_3$ the corresponding relation is $V/S = 0.056\, R$. This shows that the loops typically have more or less spherical shapes and are not, on average, substantially flattened or elongated.

Figure 5 shows our results for the size distribution of the loops, $\log n$ vs $\log R$. Here n is the number of loops of size R divided by the volume of the lattice, N^3. Since R changes in steps of 1, n is $\sim dn/dR$ for large values of R. The fit to the data of Fig. 5 gives

$$dn = BR^{-\beta} dR \tag{2.6}$$

with

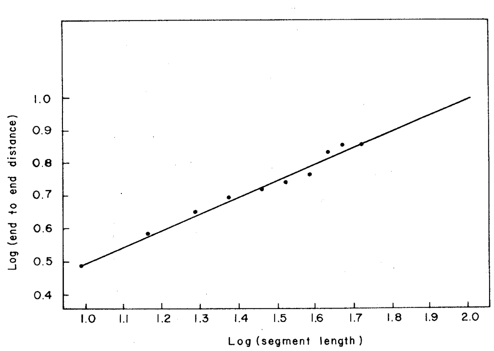

FIG. 2. Logarithmic plot of average displacement as a function of length along the string.

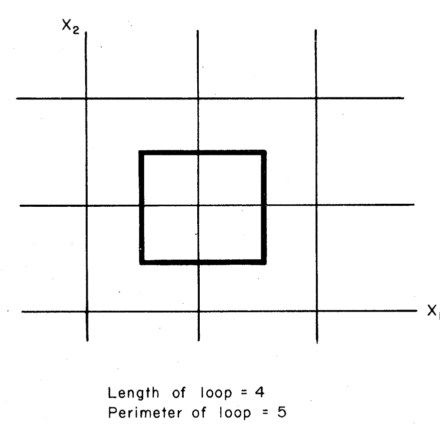

FIG. 3. This loop has length $l=4$ and size $R=5$.

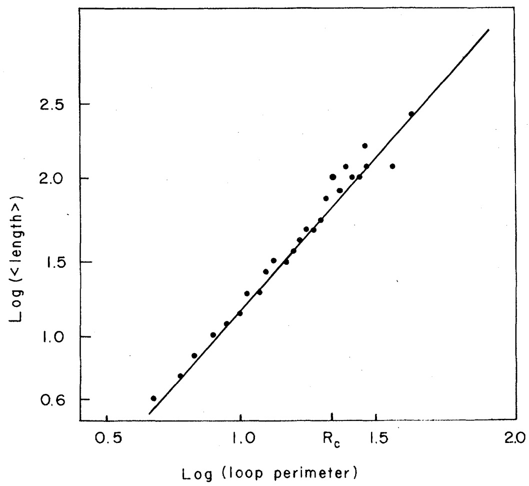

FIG. 4. Average length of the loops as a function of their size (perimeter).

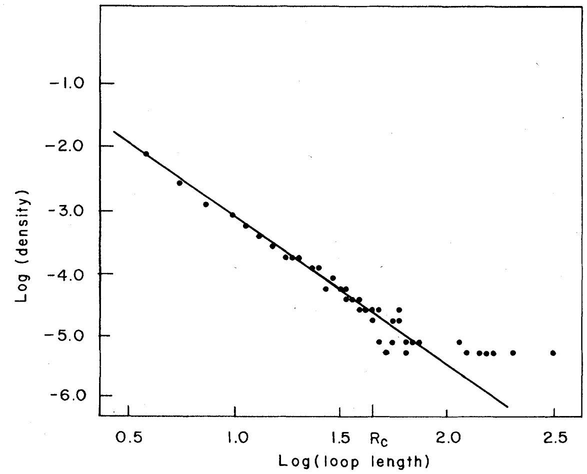

FIG. 6. Length distribution of closed loops.

$$B=6\pm2,\quad \beta=4.0\pm0.2\ . \tag{2.7}$$

The number density of loops n as a function of the loop length l is plotted in Fig. 6, and the corresponding fit is

$$dn=Cl^{-\gamma}dl\ , \tag{2.8}$$

$$C=0.4\pm0.1,\quad \gamma=2.6\pm0.1\ . \tag{2.9}$$

Equations (2.6)–(2.9) are in good agreement with Eqs. (1.5) and (1.6) obtained in the Introduction using the assumption of scale invariance.

This agreement leads one to expect that the other predictions of scale invariance will also be confirmed. This, however, turns out not to be the case. For a lattice of size $N=40$ we have found that only 19% of the total length of strings is in the form of closed loops. All the rest belongs to open strings (that is, to strings terminating at the boundaries of the lattice). If the assumption of scale invariance is correct, then all the strings must be closed, and the presence of open strings is just a surface effect due to the fact that some closed loops are cut by the boundaries

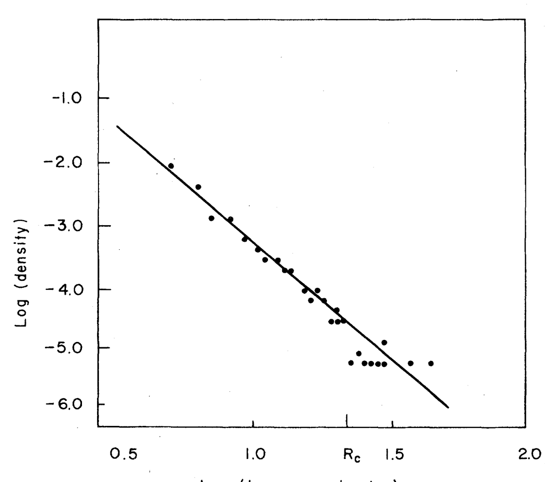

FIG. 5. Size distribution of closed loops.

of the volume. Then it can be shown using Eq. (1.5) that the fraction of the length in the open strings, $f_{\rm open}$, should scale like

$$f_{\rm open}\sim N^{-1}\ln N \tag{2.10}$$

with the size of the lattice N. Simulations for $N=20$, 30, and 40 give $f_{\rm open}\simeq0.86$, 0.83, and 0.81, respectively, in obvious disagreement with (2.10). Instead of rapidly decreasing with size, $f_{\rm open}$ seems to saturate at ~0.8. In fact, it is clear that large loops from the tail of the distribution (1.5) cannot account for the 80% of length found in the open strings, since according to (1.5) the dominant contribution to the length density of strings is given by the smallest closed loops. The conclusion seems to be that, in addition to the scale-invariant system of closed loops, there is a system of infinite strings which is not scale invariant and which contributes about 80% to the total length of strings in the universe. As an additional check of this picture, we made a simulation on a lattice with periodic boundary conditions, which corresponds to a closed universe having the topology of a 3-torus. In a closed universe, all strings are closed, and we expect that the place of open strings is taken by very large loops traversing the whole universe or even running around it several times. We find that introducing periodic boundary conditions does not substantially change the length distribution of loops up to $l\sim200$ (which is about the maximum length of the loop present with nonperiodic boundary conditions). In addition, we observe the appearance of a few very long loops of length 800 up to 39 000, which make up 75–80 % of the total length of strings. (Here, all numbers, correspond to $N=40$.)

That the system of infinite strings is not scale invariant can be seen as follows. The length density due to infinite strings is $\lambda_{\rm inf}\sim\xi^{-2}$. If we smooth all irregularities of strings on scales $\xi'>\xi$, the length density becomes

$$\lambda'_{\rm inf}\sim(\xi\xi')^{-1}\ , \tag{2.11}$$

while for a scale-invariant system we would have $\lambda'_{\rm inf}\sim\xi'^{-2}$. [The simplest way to derive Eq. (2.11) is to consider an infinite universe as a limiting case of a closed

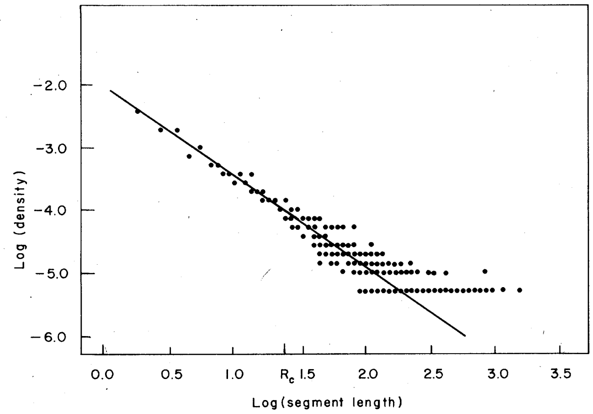

FIG. 7. Length distribution of open strings.

universe. A scale transformation $\xi \to \xi'$ changes the length of long strings by a factor ξ/ξ', while the number of such strings remains unchanged.]

Having in mind condensed matter applications, we show, in Fig. 7, the length distribution of strings terminating at the boundaries of the lattice. The least-squares fit of the data gives

$$dn = Dl^{-\delta}dl \ , \tag{2.12}$$

$$D = (5.5 \pm 0.5) \times 10^{-3}, \quad \delta = 1.41 \pm 0.03 \ . \tag{2.13}$$

III. COSMOLOGICAL EVOLUTION OF STRINGS

In this section we shall review what is known about the cosmological evolution of strings in the light of our findings in Sec. II. Most of the results presented here are not new. Still, we think such a review may be useful since the knowledge of initial conditions removes some ambiguity from the picture and thus reduces the number of possibilities to be dealt with.

As we have found, at the time of formation, t_0, the strings have the shape of Brownian trajectories with a persistence length $\sim\xi$, where ξ is the correlation length of the Higgs field. Kibble[1] has shown that $\xi \sim \eta^{-1} \ll t_0$, where η is the symmetry-breaking scale. In the very early stages the motion of strings is damped by friction due to their interaction with surrounding matter. Friction becomes negligible at time t_* which is roughly given by[1,13] $t_* \sim m_P{}^3/\eta^4$, where m_P is the Planck mass. Thereafter there is little damping and the strings reach relativistic speeds. The persistence length of strings at t_* is $\sim t_*$. Closed loops of size $R_* \ll t_*$ have disappeared and the size distribution of loops with $R_* > t_*$ is $dn \sim R_*{}^{-4}dR_*$. The fact that the distribution of large loops does not change with time is due to the scale invariance of the loop distribution (1.5). The major effects determining the string evolution at $t > t_*$ are expansion of the universe, gravitational radiation, and intercommuting of intersecting strings. We shall first discuss how the strings evolve assuming that intercommuting is improbable.

For $t > t_*$, expansion of the universe straightens out long strings on scales smaller than the horizon (so that the persistence length at time t is $\sim t$) and conformally stretches them on scales greater than the horizon. This result was first obtained by perturbative analysis in Ref. 4 and then confirmed by computer calculations in Ref. 14. Closed loops smaller than the horizon are unaffected by expansion. They oscillate, lose energy by gravitational radiation and disappear with a typical lifetime[4,15]

$$\tau \sim R/G\mu \ , \tag{3.1}$$

where R is the size of the loop and $\mu \sim \eta^2$ is the mass per unit length of string ($G\mu \ll 1$). Consider a loop of size $R_* \gg t_*$ at $t \sim t_*$. In the course of expansion the size of the loop is conformally stretched,

$$R(t) \sim (t/t_*)^{1/2} R_* \ , \tag{3.2}$$

until the loop comes within the horizon. This happens at $t \sim t_h$ such that $R(t_h) \sim t_h$:

$$t_h \sim R_*{}^2/t_* \ . \tag{3.3}$$

At any time $t < t_h$ the loop has the shape of a Brownian trajectory with persistence length $\sim t$, and its length is

$$l(t) \sim R^2(t)/t \sim R_*{}^2/t_* = \text{const} \ . \tag{3.4}$$

When the loop comes within the horizon, its length also remains practically unchanged for a period of time τ given by Eq. (3.1). The loop decays at

$$t \sim R_*{}^2/G\mu t_* \ . \tag{3.5}$$

From this equation we see that the loops surviving at time t are those which have $R_* \gtrsim (G\mu t_* t)^{1/2}$. The total mass density due to the loops at time t is

$$\rho_{\text{loop}} \sim \mu \left[\frac{t_*}{t} \right]^{3/2} \int_{(G\mu t_* t)^{1/2}}^{\infty} \frac{dR_*}{R_*{}^4} \frac{R_*{}^2}{t_*}$$
$$\sim (G\mu)^{-1/2} \mu t^{-2} \tag{3.6}$$

and is independent of t_*.

The density of radiation is $\rho_{\text{rad}} \sim 1/30Gt^2$ and

$$\rho_{\text{loop}}/\rho_{\text{rad}} \sim 30(G\mu)^{1/2} = \text{const} \ . \tag{3.7}$$

This picture may be oversimplified since it assumes complete straightening of strings on scales smaller than the horizon. Turok and Bhattacharjee have argued[14] that the straightening process may not be 100% efficient and that the energy of the loops may slowly grow with the expansion. Probably the only convincing way to resolve this issue is to do a numerical analysis of the evolution of large Brownian loops. In any event, expansion cannot *decrease* the energy of a Brownian loop, since causality requires that the persistence length of the strings cannot grow faster than t. We can therefore regard Eq. (3.4) and (3.7) as lower bounds.

We now turn to the infinite strings, where the situation is quite different. At $t \sim t_0$ the mass density due to infinite strings is $\rho_{\text{inf}}(t_0) \sim \mu\xi^{-2}$. During the time interval $t_0 \lesssim t \lesssim t_*$ the persistence length changes from $\sim\xi$ to $\sim t_*$, the length of infinite strings per comoving volume changes by a factor ξ/t_0, and the volume itself changes by a factor $(t_*/t_0)^{3/2}$. As a result, $\rho_{\rm inf}(t_*)\sim\mu\xi^{-1}t_0^{1/2}t_*^{-3/2}$ and, omitting numerical factors,

$$\rho_{\rm inf}/\rho_{\rm rad}(t_*)\sim G\mu\xi^{-1}(t_0t_*)^{1/2}\sim 1 \ . \tag{3.8}$$

Here we have used the relations $\mu\sim\eta^2$, $\xi\sim\eta^{-1}$, $t_0\sim m_p/\eta^2$, and $t_*\sim m_p{}^3/\eta^4$. Things do not get better for $t>t_*$. Even if we manage to keep $\rho_{\rm inf}/\rho_{\rm rad}$ small up to t_* (say, by calculating t_* more accurately) and assume complete straightening of strings on scales smaller than the horizon, the mass of the strings remains constant and

$$\rho_{\rm inf}/\rho_{\rm rad}(t)\sim(t/t_*)^{1/2}\rho_{\rm inf}/\rho_{\rm rad}(t_*) \ , \tag{3.9}$$

so that the universe rapidly becomes string dominated. If the straightening is incomplete, $\rho_{\rm inf}/\rho_{\rm rad}$ grows even faster. In a string-dominated universe the scale factor grows like[4] $a(t)\propto t$, and one can easily convince oneself that such a universe has little in common with the one we live in.

There is, however, one important effect not taken into account in this analysis. It is the intercommuting of intersecting strings, first discussed by Kibble.[1] As illustrated in Fig. 8(a), pairs of strings may intersect at two points and form closed loops. Loops can also be formed by self-intersection of individual strings [Fig. 8(b)]. Since the loops eventually radiate energy away, it is clear that the loop formation reduces the overall energy of the strings. We note that there exists another potentially important intercommuting effect (also discussed independently by Preskill and Wise).[16] It is illustrated in Fig. 9. Two nearly straight pieces of string intercommute at one point and form two curved pieces of string which are then straightened out by expansion. The energy-loss mechanism here can be understood as follows: potential energy of curved strings is transformed into the kinetic energy, which is then red-shifted by expansion. (To estimate the efficiency of this mechanism one probably has to use numerical solutions of the string equations of motion in the expanding universe.)

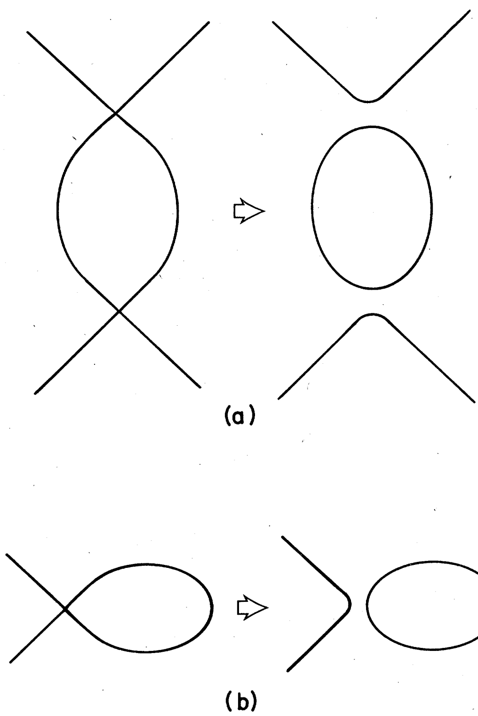

FIG. 8. Closed-loop formation by intercommuting strings.

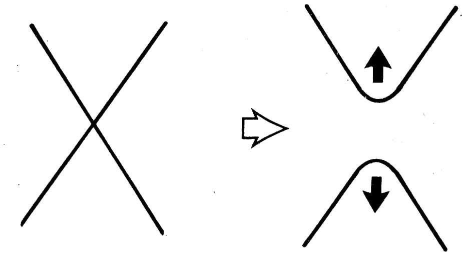

FIG. 9. Intercommuting of two straight strings results in two curved strings which are then straightened out by expansion.

If ν is the typical number of segments of infinite strings per horizon volume, t^3, then each segment has about ν intersections per Hubble time t. If p is the intercommuting probability, we expect that intercommuting processes will reduce ν to $\sim p^{-1}$ in about one expansion time. Then the density of infinite strings at time t is $\rho_{\rm inf}\sim p^{-1}\mu t^{-2}$ and[16]

$$\rho_{\rm inf}/\rho_{\rm rad}\sim 30G\mu/p \ . \tag{3.10}$$

We see that if p is very small, $p<30G\mu$, then intercommutings do not prevent strings from dominating the universe.

Strings are described by classical solutions of the field equations, and their intercommuting probably occurs at the classical level as well. Then the intercommuting process is totally deterministic, and the word "probability" refers to averaging over collision angles and relative velocities of the strings. One's best guess in this case is probably that $p\sim1$. We cannot exclude the possibility that for certain types of strings intercommuting is classically forbidden and can only occur due to quantum tunneling. Then p can be extremely small. According to (3.10), the existence of strings with $p<30G\mu$ is ruled out. From now on, we shall concentrate on the case of $p\sim1$. The evolution of strings with $p\ll1$ has been discussed by Preskill and Wise.[16]

For $p\sim1$ there is one or a few segments of infinite strings per horizon volume at each time, so that $\rho_{\rm inf}/\rho_{\rm rad}\sim30G\mu\ll1$. The typical curvature of infinite strings at time t is $\sim t$, and the loops formed by their intercommuting will also have size $\sim t$. Let dn/dt be the number of loops produced per unit volume per unit time. Assuming that a substantial part of infinite string energy goes into the loop formation, we can write $\mu t\, dn/dt\sim d\rho_{\rm inf}/dt$, which gives

$$dn/dt\sim t^{-4} \ . \tag{3.11}$$

This means that about one loop is formed per horizon volume per expansion time[4] (as one would expect intuitively, since the only characteristic scale of infinite strings is t).

Intercommuting of strings can also destroy the existing closed loops. If a long segment of string intercommutes at one point with a closed loop, the loop gets absorbed into the segment. However, the probability for a loop smaller than the horizon to be hit by a string decreases with time like t^{-1}, and we expect a large fraction of loops to survive. Loops can also be destroyed by multiple self-intercommutings. Here, an important result has been obtained by Kibble and Turok[5] who have demonstrated the existence of a large class of loop trajectories which never self-intersect. Turok[6] has analyzed a representative class of loop trajectories and has shown that a large portion of the parameter space corresponds to non-self-intersecting loops. These results indicate that a substantial fraction of loops is not destroyed by intercommutings. Such loops gradually lose their energy by gravitational radiation, with a typical lifetime given by Eq. (3.1). The mass density of the loops at time t is[4]

$$\rho_{\text{loop}} \sim \int_{G\mu t}^{t} \left[\frac{R}{t}\right]^{3/2} \mu R \frac{dR}{R^4}$$

$$\sim (G\mu)^{-1/2} \mu t^{-2} , \tag{3.12}$$

so that $\rho_{\text{loop}}/\rho_{\text{rad}} \sim 30(G\mu)^{1/2}$. Curiously, these equations are the same as Eqs. (3.6) and (3.7), which were obtained with different assumptions about the evolution of closed loops. In writing the integral in Eq. (3.12), we have used the fact that loops of size R were formed at $t \sim R$ at the rate (3.11) and that the loops surviving at time t were formed in the time interval from $G\mu t$ to t. The size distribution of loops at time t is

$$dn \sim \left[\frac{R}{t}\right]^{3/2} \frac{dR}{R^4} \sim t^{-3/2} R^{-5/2} dR . \tag{3.13}$$

All the above equations are written for a radiation-dominated universe [the scale factor $a(t) \propto t^{1/2}$]; an extension to a matter-dominated universe is straightforward.

Long-lived closed loops can serve as seed for galaxies and clusters. This scenario of galaxy formation has been studied in Refs. 4 and 7. A discussion of this and other astrophysical effects of strings is beyond the scope of this paper, and we refer the reader to the original literature. Black-hole formation due to accretion of matter onto the loops and wake formation behind rapidly moving strings are discussed in Ref. 17. The gravitational lens effect of the strings is discussed in Ref. 8. The spectrum of gravitational waves produced by oscillating loops is obtained in Ref. 18. Some other speculations about the possible role of the strings can be found in Ref. 6.

IV. CONCLUSIONS

We have shown that at the time of formation the system of strings consists of two components: (a) a system of Brownian closed loops with a scale-invariant size distribution given by Eq. (1.5) and (b) a system of infinite Brownian strings, which is not scale-invariant and which contributes about 80% to the total length of strings in the universe. (In a closed universe the place of infinite strings is taken by very long loops traversing the whole universe or even running around it several times.)

The following evolution of strings depends on the probability p for intersecting strings to intercommute. If $p=0$ or very small ($p < 30G\mu$), then the universe rapidly becomes string dominated. In the opposite limit of $p \sim 1$ the evolution of strings is largely insensitive to the initial conditions. In this case each horizon volume ($\sim t^3$) is expected to contain one or a few segments of infinite string and a large number of closed loops of sizes $G\mu t < R < t$ with a size distribution given by Eq. (3.13).

While our results on the string formation were obtained by a direct computer simulation, the conclusions concerning the evolution of strings rest on a weaker foundation of plausible assumptions and order-of-magnitude estimates. To obtain a more quantitative and reliable picture, one has to run simulations for the evolution of large Brownian loops, for intercommuting effects, and, ideally, for the evolution of the whole system of strings using the results of this paper as the initial conditions.

We have also done simulations for the formation of domain walls and of walls bounded by strings. In both cases the system is dominated by one infinite cluster of very complicated topology. In the case of domain walls, the infinite wall comprises about 90% of the total wall area. In the case of walls bounded by strings, the infinite cluster contains 93% of the wall area and 87% of the string perimeter. An intersection of the infinite cluster with a plane gives a large number of short pieces, indicating that the cluster is very "holey." Small finite closed walls and walls bounded by strings are also formed, in numbers rapidly decreasing with their size.

Note added in proof. A number of interesting results on strings have been obtained after this paper was submitted.

(i) P. Shellard (unpublished) has done a numerical analysis of intercommuting for global strings. His results indicate that the intercommuting probability is $p \sim 1$ (note, however, that p can be model dependent).

(ii) A. Albrecht and N. Turok (unpublished) have done computer simulations of the evolution of strings with $p \sim 1$. Their results agree with our conclusions and, in fact, put them on a firmer foundation.

(iii) The present authors have calculated numerically the rate of gravitational radiation from oscillating loops. The resulting lifetime of the loops is $\tau \sim R/\gamma G\mu$ with $\gamma \sim 100$. The large numerical factor γ does not alter the qualitative nature of the string evolution, but it does affect some of the quantitative predictions of the string scenario (e.g., the spectrum of density fluctuations and the power of the stochastic gravitational wave background).

(iv) Finally, a number of papers have appeared on cosmological and observational effects of strings: M. J. Rees and C. J. Hogan, Nature **311**, 109 (1984); N. Kaiser and A. Stebbins, *ibid.* **310**, 391 (1984); A. Vilenkin, Phys. Rev. Lett. **53**, 1016 (1984), Astrophys. J. Lett. **282**, L51 (1984).

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APPENDIX A: FORMATION OF DOMAIN WALLS

Domain walls form when a discrete symmetry is broken. The simplest example is a real scalar field ϕ with a

TABLE I. Size distribution of plus clusters.

Cluster size	1	2	3	4	6	10	31 082
Number	462	84	14	13	1	1	1

potential $U(\phi)=\lambda(\phi^2-\eta^2)^2$. At low temperatures ϕ develops a vacuum expectation value $\langle\phi\rangle=\pm\eta$, breaking the symmetry $\phi\to-\phi$. Domain walls form at the boundaries separating regions of vacua with $\langle\phi\rangle=+\eta$ and $\langle\phi\rangle=-\eta$.

Again we take a volume of size $N\xi$ divided into N^3 cubic cells and assign to each cell a number $+1$ or -1 (corresponding to $\langle\phi\rangle=\pm\eta$) at random with equal probability. We call the corresponding cells plus cells and minus cells, respectively. To characterize the system of domain walls, one can look for the size distribution of clusters of connected plus cells. (Two cells are connected if they have a common face.) Of course, the distribution of minus-cell clusters will be similar. This is a typical problem of percolation theory.[18] Most of the work on percolation has centered on determining the critical concentration, p_c, of plus cells at which an infinite plus cluster first appears (in an infinite lattice) and on studying the properties of the system at or near $p=p_c$. The value of p_c is different for different lattices, but in all three-dimensional cases it is smaller than 0.5. In our case $p=0.5$, and thus the system is above the percolation threshold. (For a cubic lattice $p_c=0.31$.)

The following facts are known[19] about the properties of percolating systems at $p>p_c$: (1) there is only one infinite plus cluster (and one infinite minus cluster if $p<1-p_c$) and (2) the number density of finite clusters, n_s, decreases exponentially with their size, s:

$$n_s \propto s^{-\tau}\exp(-\alpha s^{2/3}) . \tag{A1}$$

Here the numerical coefficients τ and α depend on p, and s is the number of cells in a cluster.

The implications of these results for the system of domain walls are straightforward. The system is dominated by one infinite (or, for a finite universe, very large) domain wall with a very complicated topology. In addition, there are some finite closed domain walls. Most of them have size $R\sim\xi$. The probability of finding closed walls with $R\gg\xi$ exponentially decreases with size, $\ln n\propto -R^2$.

Harvey *et al.*[20] did a numerical simulation of wall formation and found that all but about 1% of plus cells belong to the "infinite" cluster. We have also run several numerical simulations on a lattice of size $N=40$. To eliminate boundary effects, we used periodic boundary conditions, that is, identified the opposite faces of the lattice. The size distribution of plus clusters in a typical simulation is shown in Table I. In this simulation, 97.6% of all plus cells and $\sim$87% of the total wall area belong to the infinite cluster.

APPENDIX B: FORMATION OF WALLS BOUNDED BY STRINGS

As an example of a model with hybrid topological structures, consider the following sequence of symmetry

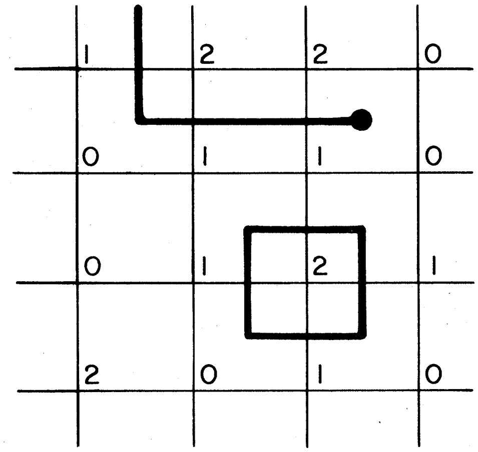

FIG. 10. A slice of the lattice with walls bounded by strings. Walls, here represented by lines, separate sites with phases 1 and 2. Strings, which are represented by dots, are found using the prescription of Sec. II.

breakings: $G\to H\times Z_2\to H$, where G is a simple group. Then strings are formed at the first phase transition and they get connected by domain walls at the second phase transition.[2]

Another example is given by a model with an approximate global U(1) symmetry,[21]

$$L=\partial_\mu\phi^*\partial^\mu\phi-\lambda(|\phi|^2-\eta^2)^2+m^4(\cos\theta-1) , \tag{B1}$$

where $\phi=|\phi|e^{i\theta}$ and it is assumed that $m\ll\eta$. This model is a simplified version of invisible axion models[22] designed to solve the strong *CP* problem. At $T\sim\eta$, ϕ acquires a vacuum expectation value, $|\langle\phi\rangle|=\eta$. This phase transition gives rise to strings. If we neglect the θ-dependent term in (B1), then θ changes uniformly by $\Delta\theta=2\pi$ around a string. However, eventually this term has its effect and forces θ to near zero at all points around the string, except within a wall of thickness $\delta\sim\eta/m^2$, so that θ changes by 2π across the wall. Domain walls can be considered as "formed" when their thickness becomes smaller than the typical distance between the strings. In axion models this happens long after the strings are formed; however, for m/η not too small, the walls can

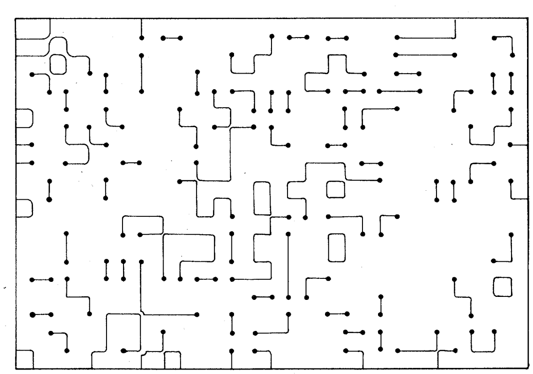

FIG. 11. A computer simulation for a slice of the lattice of size 20×30 with walls bounded by strings.

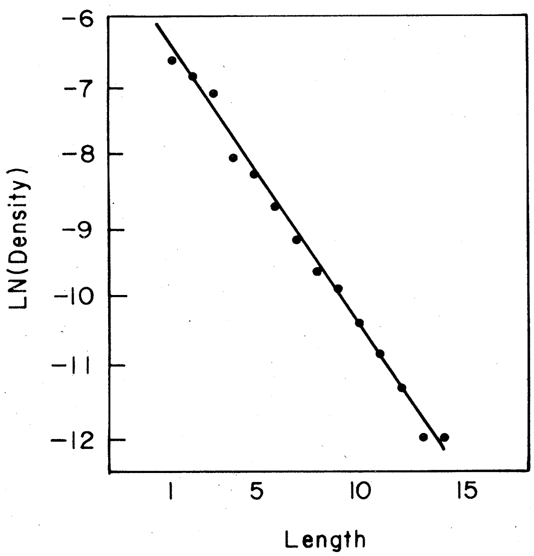

FIG. 12. Length distribution of wall sections for walls bounded by strings.

appear soon after the string formation. The walls appear on the surfaces of $\theta=\pi$ corresponding to the maximum of the θ dependent part of the potential. Note that in addition to walls bounded by strings, some closed walls can also be formed.

The model we used to simulate the formation of walls bounded by strings is basically the same as that of Sec. II. We assign phases designated by numbers, 0, 1, and 2 at random on a cubic lattice and find the strings exactly as in Sec. II.

Domain walls separate sites of the lattice with phases 1 and 2, so that 1's and 2's lie on opposite sides of the wall. With this prescription, the walls can be either closed or bounded by strings (see Fig. 10). When two wall surfaces meet along a line, we have to decide how to identify the pieces. There are three possible identifications, one of them corresponding to intersecting walls. We do not allow walls to intersect and assign the two remaining identifications equal probabilities.

Figure 11 shows a slice of the lattice of size 20×30. In this two-dimensional slice the walls are represented by lines and the strings by points. A similar two-dimensional simulation has been recently done by Sikivie.[23] The length distribution of open-wall sections is shown in Fig. 12. The best fit is given by the exponential distribution

$$dn=\alpha\exp(-\beta l)dl \tag{B2}$$

with $\alpha=2.6\times10^{-3}$ and $\beta=0.44$. The exponential character of the distribution can be understood if we note that, as we go along the wall section, at each step there is a certain probability of getting the combination of phases corresponding to a string and terminating the wall. (This argument is due to Sikivie.[24])

TABLE II. Area and perimeter distribution for connected clusters of walls bounded by strings.

Area	1	2	3	8	654
Perimeter	4	6	8	12	834
Number	5	4	3	1	1

The number of closed wall sections turns out to be rather small. A typical simulation on a square lattice of size $N=30$ gives three or four closed sections of length 4 and no more than one closed section of greater length (6 or 8).

The results of a typical three-dimensional simulation are given in Table II for a lattice of size $N=10$. We see that, as in the case of domain walls, the system is dominated by one huge cluster comprising $\sim98\%$ of the total wall area and $\sim96\%$ of the total string perimeter. (The corresponding numbers for a lattice of size $N=20$ are $\sim93\%$ and $\sim88\%$.) Equation (B2) shows that the domain wall dominating the system is very holey, so that its intersection with a plane gives a large number of short pieces. This is also evident from the large perimeter of the dominant wall.

In addition to the infinite cluster, there are some finite walls bounded by strings, in numbers decreasing with their size. Closed domain walls are rare. We saw none of them in simulations with $N=10$ and only a few closed walls of minimum size (perimeter$=6$) for $N=20$.

Strictly speaking, our simulation is directly applicable only to the case when domain walls are formed soon after the string formation. (Otherwise, one has to take into account dynamical effects of the string evolution on scales smaller than the horizon.) However, we expect that in axion models, where this is not the case, our results correctly represent the qualitative properties of the system on large scales at the time of wall formation. The evolution of walls bounded by strings has been discussed in Ref. 2 assuming intercommuting probabilities $p\sim1$. The case of $p\ll1$ has been analyzed in Ref. 16.

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