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Time dependence of the Skyrme soliton

K. F. Liu, J. S. Zhang,* and G. R. E. Black
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506
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Small oscillations of the static Skyrme soliton are considered. It is found that the Skyrme soliton is stable against small oscillations and there is no bound state. The meson-soliton scattering phase shifts are calculated. A resonance is found at ~ 250–300 MeV above the nucleon, which is only halfway from the Roper resonance $N^*(1440)$. Small oscillations are also considered for spinning Skyrme solitons. The results are similar. Some implications and difficulties associated with spinning solitons are discussed.

It is believed[1,2] that in the large-N_c limit, the low-energy properties of QCD can be reproduced effectively by a weakly coupled field theory of mesons, in which the baryons emerge as the topological solitons. Many years ago, Skyrme[3] pointed out that the nonlinear σ model admits soliton solutions characterized by an integer-valued topological charge, which he suggested identifying as the baryon number. Later investigations[4–7] have confirmed this definition of baryon number as well as the fermion nature of the soliton.

The phenomenological aspects of the Skyrme soliton have been explored. The static properties of the nucleons have been computed.[8,9] The results are within about 30% of the experimental values. The adiabatic nucleon-nucleon potential has been calculated.[10,11] The results are quite successful in comparison with realistic potentials and can be understood in terms of the exchange of π, ρ, and ω between the solitons.

In this Rapid Communication, we shall address some dynamical aspects of the Skyrme soliton through its time dependence, in particular, the vibrations of the static as well as the spinning Skyrme soliton. Other time dependences, such as the zero-frequency translational mode and the boost mode, will not be discussed here. The breathing mode of the static Skyrme soliton has been studied recently by computing the compression modulus,[12] by scaling,[13] and in the strong-coupling approximation.[14] We shall consider vibration in terms of small oscillations around Skyrme's hedgehog solution. This approach differs from the scaling approach in that the small oscillations are not restricted to a shape related to the original soliton solution as in the case of the scaling approach. On the other hand, they are limited to a small amplitude, whereas in the scaling approximation the amplitude can be large in general. The phase shifts of the static Skyrme soliton with the small-oscillation approximation have also been studied by other authors.[15,16] Besides the phase shifts, we would like to study the stability of the Skyrme soliton against small oscillations and spinning Skyrme solitons with projected spins and isospins.

The Skyrme Lagrangian for the $SU(2)\times SU(2)$ chiral theory is

$$L=\frac{F_\pi{}^2}{16}\mathrm{Tr}(\partial_\mu U\partial_\mu U^\dagger)+\frac{1}{32e^2}\mathrm{Tr}[(\partial_\mu U)U^\dagger,(\partial_\nu U)U^\dagger]^2+\tfrac{1}{8}m_\pi{}^2F_\pi{}^2(\mathrm{Tr}\,U-2)\ . \tag{1}$$

The quartic term in U was introduced by Skyrme[3] to stabilize the soliton solution, and the explicit symmetry-breaking term was introduced by Adkins and Nappi[9] to account for a massive-pion theory. We shall first consider small oscillations around the static Skyrme soliton for this Lagrangian with both the massless- and the massive-pion cases. Consider the hedgehog ansatz of the form

$$U=e^{i\vec{\tau}\cdot\hat{r}\theta(r,t)}\ ; \tag{2}$$

the Euler-Lagrange equation is then reduced to a time-dependent equation of motion for the spherical chiral angle $\theta(r,t)$,

$$(\tfrac{1}{4}\tilde{r}^2+2\sin^2\theta)\left(\theta''-\frac{1}{e^2F_\pi{}^2}\ddot{\theta}\right)+\tfrac{1}{2}\tilde{r}\theta'+\sin2\theta\left(\theta'^2-\frac{\dot{\theta}^2}{e^2F_\pi{}^2}\right)-\tfrac{1}{4}\beta^2\tilde{r}^2\sin\theta-\sin2\theta\left(\frac{1}{4}+\frac{\sin^2\theta}{\tilde{r}^2}\right)=0\ , \tag{3}$$

where, following Refs. 8 and 9, $\tilde{r}=F_\pi er$ and $\beta=m_\pi/eF_\pi$ are defined as dimensionless variables. The static solution $\theta_0(r)$ with the boundary conditions $\theta_0(r)=\pi$ at $r=0$ and $\theta_0(r)\rightarrow 0$ as $r\rightarrow\infty$ has been obtained previously.[8–10]

For small oscillations, let $\theta(r,t)=\theta_0(r)+\delta\theta(r)e^{-i\omega t}$, where $\delta\theta \ll \theta_0$. Upon keeping only terms linear in $\delta\theta$, Eq. (3) leads to an eigenvalue equation for $\delta\theta$,

$$(\tfrac{1}{4}\tilde{r}^2+2\sin^2\theta_0)\delta\theta''+[\tfrac{1}{2}\tilde{r}+2(\sin2\theta_0)\theta_0']\delta\theta'+\left[2(\sin2\theta_0)\theta_0''+2(\cos2\theta_0)\theta_0'^2-\tfrac{1}{2}\cos2\theta_0-\frac{1}{\tilde{r}^2}(\sin^22\theta_0+2\sin^2\theta_0\cos2\theta_0)-\tfrac{1}{4}\beta^2\tilde{r}^2\cos\theta_0\right]\delta\theta+(\tfrac{1}{4}\tilde{r}^2+2\sin^2\theta_0)\frac{\omega^2}{e^2F_\pi{}^2}\delta\theta=0\ . \tag{4}$$

Defining

$$E=\omega^2/e^2F_\pi{}^2,\quad u=\left(\frac{\tilde{r}^2}{8}+\sin^2\theta_0\right)^{1/2}\delta\theta\ , \tag{5}$$

Eq. (4) reduces to a Schrödinger equation

$$\frac{d^2u}{d\tilde{r}^2}+[E-V(\tilde{r})]u=0 \ , \tag{6}$$

where the dimensionless potential $V(\tilde{r})$ is

$$V(\tilde{r})=\frac{1}{(\tilde{r}^2/8+\sin^2\theta_0)^2}\left[\left(\frac{1}{\tilde{r}^2}\sin^4\theta_0+\tfrac{1}{4}\sin^2\theta_0\right)(2-3\sin^2\theta_0)+\frac{\tilde{r}^2}{32}\cos2\theta_0+\tfrac{1}{64}\beta^2\tilde{r}^4\cos\theta_0\right] \ . \tag{7}$$

This potential for the cases of $m_\pi=0$ and $m_\pi=138$ MeV with F_π and e from Refs. 8 and 9, respectively, is plotted in Fig 1. Since it includes the centrifugal potential $l(l+1)/\tilde{r}^2$, it goes like $2/\tilde{r}^2$ at $\tilde{r}\rightarrow 0$ for a pion in the P wave and it approaches $m_\pi{}^2/e^2F_\pi{}^2$ asymptotically. Having solved Eq. (7) subject to the boundary conditions that $u=0$ at $r=0$ and ∞ so that the winding number of the soliton is not changed, we first notice that there is no solution for $\omega^2<0$, which means that the soliton is *stable* against small oscillations. Otherwise, it would introduce an exponentially growing term in $\theta(r,t)$. Furthermore, there is no bound-state solution for $\omega^2>0$ either. This can be checked by the Bohr-Sommerfeld quantization rule; i.e.,

$$\frac{4\pi}{3}\int_a^b p^3d^3r\geqslant h^3 \tag{8}$$

holds for having at least one bound state below a chosen energy with a and b being the classical turning points for this energy. To check whether there is any bound state at all, we consider the integral for the maximum allowable bound-state energy $m_\pi{}^2/e^2F_\pi{}^2$. In this case, the rule becomes

$$S=\frac{2}{3\pi}\int_a^b[-V(\tilde{r})]^{3/2}\tilde{r}^2d\tilde{r}\geqslant 1 \ . \tag{9}$$

The integrals S for both the $m_\pi=0$ and $m_\pi=138$ MeV cases are tabulated in Table I. They are much smaller than one, which confirms that the potentials are not deep enough to hold any bound states.

The phase shifts for the scattering states are also calculated. The boundary conditions for the scattering are $u\sim\tilde{r}^2$ at $\tilde{r}\rightarrow 0$ and $u\sim\tilde{r}[a(k)j_1(k\tilde{r})+b(k)n_1(k\tilde{r})]$ asymptotically with $k=(\omega^2-m_\pi{}^2)^{1/2}/eF_\pi$. The results of the phase shifts are plotted in Fig. 2. They rise fairly rapidly, giving rise to a resonance which is reminiscent of the initial rise in the πN phase shift $P11$. But they fall off after $\sim$300 MeV and do not yield a second resonance as does the experimental data. The resonance energies are determined by locating the S-matrix poles at $\omega=\omega_R-i\Gamma/2$. The results of the resonance energies ω_R and the widths Γ are tabulated in Table I. Since there are no bound states, we interpret these resonance energies as radial excitation energies above the N and Δ assuming rigid-body rotation[8] and no coupling between vibration and rotation. Comparing with the Roper resonance $N^*(1440)$, which is $\sim$500 MeV above the nucleon, and $\Delta^*(1600)$, which is $\sim$370 MeV above Δ, we see that the calculated ω_R at 313 MeV (for $m_\pi=0$) and 241 MeV (for $m_\pi=138$ MeV) are lower than the experimental excitation energies by $\sim$200 MeV. This is contrary to the quark-model calculations[17–19] of the Roper resonance which are higher than the experimental value by $\sim$100–200 MeV. It would be interesting to sort out the difference between these two approaches. The calculated widths (Table I) are smaller than the experimental width of $N^*(1440)$, which is $\sim$200 MeV, and that of $\Delta^*(1600)$, which is $\sim$250 MeV. This is all right in the sense that since the resonance energies are lower than the experimental values, the widths are expected to be narrower due to a smaller phase space. Furthermore, the total width of $N^*(1440)$ includes $\sim$40% of other decay modes (e.g., $N\pi\pi$) which are not present in this calculation.

It has been argued by Witten[2,7] that in the large-N_c limit, baryon masses are proportional to N_c. Hence, the vibrational energy ω, which is proportional to eF_π, is of the order 1

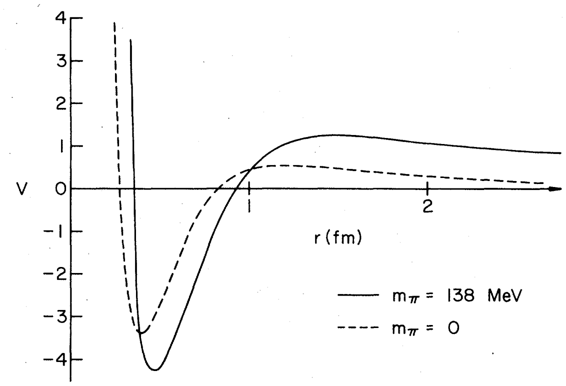

FIG. 1. The potential defined in Eq. (7) for $m_\pi=0$ and $m_\pi=138$ MeV.

TABLE I. Masses and radii of Δ and N for the Skyrme soliton with rigid-body and self-consistent rotations are listed. The resonance energies and widths for the radial excitations of N and Δ are tabulated. The numbers in parentheses are for the cases of Δ. S is defined in Eq. (9).

	Rigid-body rotation		Self-consistent rotation	Experiment
m_π (MeV)	0	138	138	138
F_π (MeV)	129	108	105.3	186
e	5.45	4.84	4.32	
M_Δ (MeV)	1255	1230	1165	1232
M_N (MeV)	942	938	1008	939
$\langle r^2\rangle^{1/2}_{I=0}$ (fm)	0.59	0.68	0.81	0.72
$\langle r^2\rangle^{1/2}_{I=1}$ (fm)	∞	1.04	1.27	0.88
ω_R (MeV)	313	241	223 (206)	500 (370)
Γ (MeV)	172	94	60 (113)	200 (250)
S	0.14	0.18	0.16 (0.11)	

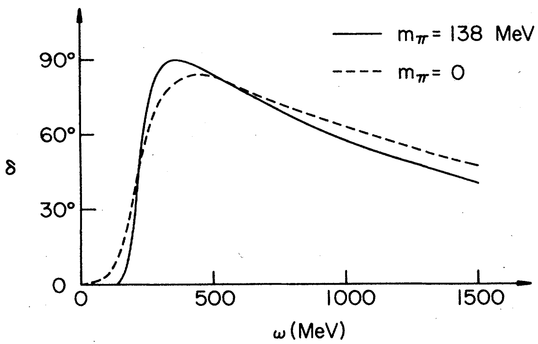

FIG. 2. Phase shifts δ vs pion energy ω for $m_\pi=0$ and $m_\pi=138$ MeV cases.

and the rotational energy, being proportional to $1/F_\pi^2$, is of the order $1/N_c$. Therefore, in the semiclassical approximation, one would in general consider the soliton field energy, the vibrational energy, and the rotational energy, in that order. However, we just learned that there is no bound state in small oscillations which leaves rotation as the most collective low-frequency mode. In this case, it makes sense to consider the rotational degree of freedom first before the pion scattering.

Now, consider the spinning hedgehog solution

$$U=e^{i\vec\tau\cdot\hat r'\theta(r,t)} , \quad (10)$$

where $\hat r'$ is the unit vector in the rotating frame. The Lagrangian takes the form

$$L=L_2+L_4+L_\pi+\tfrac12\lambda\dot\phi^2 , \quad (11)$$

where L_2 and L_4 are due to the quadratic and the quartic terms and L_π is the mass term. λ is the moment of inertia

$$\lambda=\frac{4\pi}{6}\left(\frac{1}{e^3F_\pi}\right)\int \tilde r^2\sin^2\theta\left[1+4\left(\theta'^2+\frac{\sin^2\theta}{\tilde r^2}\right)\right]d\tilde r , \quad (12)$$

and $\dot\phi$ is the angular velocity.

Defining the canonical momentum $J=\partial L/\partial\dot\phi=\lambda\dot\phi$, and upon quantization, we obtain the energy for a spinning soliton

$$E=M+J(J+1)/2\lambda . \quad (13)$$

This has been derived by Adkins, Nappi, and Witten[9] by projecting out the spins and isospins of the N and Δ through adiabatic rotation.

The Euler-Lagrange equation with the quantized angular momentum is then

$$\left(\tfrac14\tilde r^2+2\sin^2\theta\right)\left(\theta''-\frac{\ddot\theta}{e^2F_\pi^2}\right)+\tfrac12\tilde r\theta'+\sin2\theta\left(\theta'^2-\frac{\dot\theta^2}{e^2F_\pi^2}\right)-\sin2\theta\left(\frac14+\frac{\sin^2\theta}{\tilde r^2}\right)-\tfrac14\beta^2\tilde r^2\sin\theta$$
$$+\frac{3J(J+1)e^4}{(4\pi)^2\Lambda^2}\left[\tilde r^2\sin2\theta(1-4\theta'^2)-8\tilde r^2\sin^2\theta\left(\theta''+\frac{2\theta'}{\tilde r}\right)+8\sin^2\theta\sin2\theta\right]=0 , \quad (14)$$

where $\Lambda=(6e^3F_\pi/4\pi)\lambda$.

It has recently been pointed out by Braaten and Ralston[20] that, for massless pion, there is no finite static solution for Eq. (14). This can be seen from the asymptotic behavior of the differential equation. Asymptotically, it is like

$$\tfrac14\tilde r^2\theta''+\tfrac12\tilde r\theta'-\tfrac12\theta-\tfrac14\beta^2\tilde r^2\theta+\frac{6J(J+1)e^4}{(4\pi)^2\Lambda^2}\tilde r^2\theta=0 . \quad (15)$$

For $m_\pi=0$ (i.e., $\beta=0$), the last term in Eq. (15) introduces a negative constant potential at infinity for positive J. Therefore, there is no bound solution for θ. This is analogous to the classical treatment of the hydrogen atom. When one considers that the electron is coupled to the electromagnetic field classically and is forced to orbit with a definite angular momentum, there is no stationary solution and the space will eventually be filled up with radiation. For the massless-pion case, this is what happens, namely, the soliton disperses to infinity with emitted pions. This instability can be avoided to a certain extent with the massive pion. One notices from Eq. (15) that there will be a finite-energy stationary soliton solution as long as

$$\frac{6J(J+1)e^4}{(4\pi)^2\Lambda^2}\leq\tfrac14\beta^2 , \quad (16)$$

so that the spinning soliton is stable against pion emission. With this stability condition, an upper bound for the Δ and N mass difference is derived,[20]

$$M_\Delta-M_N\leq1.31m_\pi=181\text{ MeV} , \quad (17)$$

which is less than the observed mass difference.

While solving the integrodifferential equation (14) self-consistently, we find that, within the range of the parameter space we are searching, the left-hand side of Eq. (16) increases monotonically with J so that the bound in Eq. (16) is always saturated at a certain finite J value. Beyond this critical value of $J=I$, there is no finite-energy solution. The parameters F_π and e are so chosen that self-consistent solutions of Eq. (14) with the bound in Eq. (16) satisfied for $J=\frac32$ are obtained which yield a maximum Δ-N mass difference of 157 MeV. These parameters and the radii for N are listed in Table I.

Now we consider small oscillations around these spinning Skyrme solitons. Following the previous procedure, we obtain again a Schrödinger equation. In this case, u is defined as

$$u=[\tilde r^2/8+\sin^2\theta_0(1-\eta\tilde r^2/2)]^{1/2}\delta\theta , \quad (18)$$

where $\eta=24J(J+1)e^4/(4\pi)^2\Lambda^2$. The potential in this case is

$$V(\tilde r)=\frac{1}{[\tilde r^2/8+\sin^2\theta_0(1-\eta\tilde r^2/2)]^2}\left\{\frac{1}{\tilde r^2}\sin^4\theta_0(2-3\sin^2\theta_0)(1-\eta\tilde r^2)(1-\tfrac12\eta\tilde r^2)+\frac{\sin^2\theta_0}{4}(2-3\sin^2\theta_0-2\eta\sin^2\theta_0)\right.$$
$$+\frac{\tilde r^2}{32}\cos2\theta_0\left(1-12\eta\sin^2\theta_0-\eta\frac{\tilde r^2}{2}+2\eta^2\tilde r^2\sin^2\theta_0\right)-\tfrac{1}{64}\eta^2\tilde r^4\sin^22\theta_0$$
$$\left.+\tfrac{1}{64}\beta^2\tilde r^4\cos\theta_0-\frac{E}{16}\eta\tilde r^2\sin^2\theta_0\left[\frac{\tilde r^2}{8}+\sin^2\theta_0\left(1-\eta\frac{\tilde r^2}{2}\right)\right]\right\} . \quad (19)$$

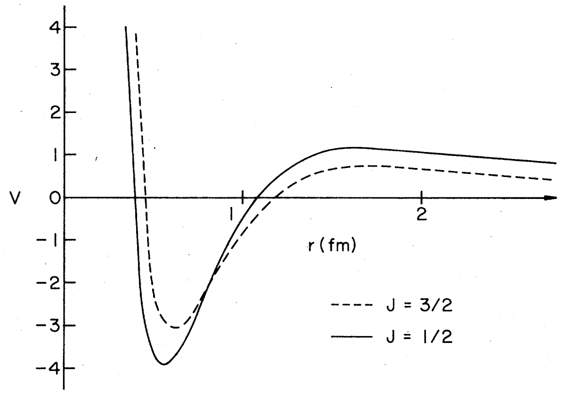

FIG. 3. The potential (at $E=0$) in the N and Δ channels as defined in Eq. (19).

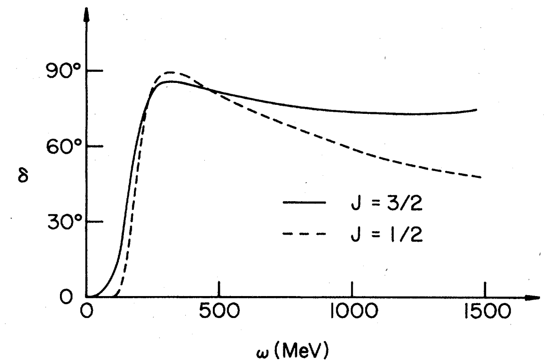

FIG. 4. Phase shifts δ vs pion energy ω in the N and Δ channels with the self-consistent approach to rotation.

Notice that since the moment of inertia is time independent, the potential becomes E dependent through the transformation in Eq. (18).

Again, there are no $\omega^2 < 0$ solutions for $J=\frac{1}{2}$ and $J=\frac{3}{2}$, which implies that they are stable against small oscillations. The integrals S are listed in Table I. They are not large enough to generate bound states. The potentials at $E=0$ are plotted in Fig. 3. It is worthwhile to note that the potential for $J=\frac{1}{2}$ is different from that of $J=\frac{3}{2}$. Despite the fact that the nonlinear σ model is not renormalizable (we do not know how to do quantum corrections properly), it would still be tempting to conjecture that part of the observed Δ-N mass difference is due to rotation, whereas another part may be due to the quantum-oscillation corrections. The large width of Δ (~ 115 MeV) itself is suggestive that a Δ mass shift of comparable magnitude is conceivable. The phase shifts in the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ channels are computed and plotted in Fig. 4. They are similar to the cases for solitons with rigid-body rotation. The resonance energies and widths are reported in Table I. The resonance energies are again about halfway to the Roper resonance $N^*(1440)$ and $\Delta^*(1600)$.

Another point worth mentioning is that since we used the equal condition for $J=\frac{3}{2}$ in Eq. (16) to attain the maximum Δ-N mass difference, the rotational band is cut off after $I=J=\frac{3}{2}$. This is in conformity with the fact that no $I=J=\frac{5}{2}$ baryon has been observed experimentally. Furthermore, in the quark-model language, a baryon with $I=J=\frac{5}{2}$ will require at least four quarks and one antiquark as the valence quarks. Even if such a resonance exists, it is not obvious why it should belong to the rotational band of N and Δ.

In conclusion, we find from the study of the small oscillations around the Skyrme solitons with either rigid-body or self-consistent rotations that they are stable against small oscillations, that there are no bound oscillations, and that the phase-shift analysis indicates resonances at ~ 200–300 MeV excitation, which are much lower than the Roper resonance $N^*(1440)$ and $\Delta^*(1600)$. It is learned from the self-consistent variation for the spinning Skyrme soliton with projected spin and isospin that there is no finite soliton solution for $m_\pi=0$. With $m_\pi=138$ MeV, it yields a maximum of 157 MeV for the Δ-N mass difference. In this case, the rotational band terminates after $I=J=\frac{3}{2}$.

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*On leave from Institute of Atomic Energy, China.

[1]G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).

[2]E. Witten, Nucl. Phys. **B160**, 57 (1979).

[3]T. H. R. Skyrme, Proc. R. Soc. London, Ser. A **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

[4]D. Finkelstein and J. Rubinstein, J. Math. Phys. **9**, 1762 (1968); J. G. Williams, *ibid.* **11**, 2611 (1970).

[5]N. K. Pak and H. Ch. Tze, Ann. Phys. (N.Y.) **117**, 164 (1979).

[6]A. P. Balachandran, V. P. Nair, S. G. Rajeev, and A. Stern, Phys. Rev. Lett. **49**, 1124 (1982); Phys. Rev. D **27**, 1153 (1983).

[7]E. Witten, Nucl. Phys. **B223**, 422 (1983); **B223**, 433 (1983).

[8]G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).

[9]G. S. Adkins and C. R. Nappi, Nucl. Phys. **B233**, 109 (1984).

[10]A. D. Jackson and M. Rho, Phys. Rev. Lett. **51**, 751 (1983).

[11]A. Jackson, A. D. Jackson, and V. Pasquier, Stony Brook report, 1984 (unpublished).

[12]J. Dey and J. Le Tourneux, University of Montreal report, 1984 (unpublished).

[13]Ch. Hajduk and B. Schwesinger, Stony Brook report, 1984 (unpublished).

[14]J. A. Parmentola, Phys. Rev. D **30**, 685 (1984).

[15]J. Breit and C. R. Nappi, Institute for Advanced Study report, 1984 (unpublished).

[16]H. Walliser and G. Eckart, Siegen report, 1984 (unpublished).

[17]K. F. Liu and C. W. Wong, Phys. Rev. D **28**, 170 (1983).

[18]T. DeGrand and C. Rebbi, Phys. Rev. D **17**, 2358 (1978).

[19]F. E. Close and R. R. Horgan, Nucl. Phys. **B164**, 413 (1980).

[20]E. Braaten and J. P. Ralston, Report No. ANL-HEP-PR-84-12 (unpublished).