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Comment on heavy mesons in the two-step potential

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(Received 2 May 1983)

Reanalyzing a recent paper on the two-step potential model, we have arrived at the conclusion that the related nonrelativistic equation cannot be solved analytically. However, a description of quarkonium comparable with descriptions of other potential models may be obtained with standard numerical methods.

In a recent paper, Kulshreshtha and Kaushal[1] applied the so-called two-step potential to an analysis of the S states in the b-quarkonium family. They claimed that (i) an analytic solution of the appropriate Schrödinger equation might be found by using a two-step potential and (ii) the parameters given in the paper successfully described the Υ spectra. We disagree with both of these statements. In our opinion, the parameters given by Eq. (13) in Ref. 1 cannot be used for a reasonable description of b-quarkonium, and the analytic solutions proposed are not solutions of the related nonrelativistic problem. In the following we identify the incorrect assumption used in Ref. 1, outline a method for solving the problems, and give a set of new parameters which, even with the two-step potential, enable one to obtain results comparable with the results predicted by other successful potentials.

The potential investigated in Ref. 1 had the form[2]

$$V(r)=\begin{cases}-\beta/r, & r\leq B\ ,\\ -V_0+Kr, & r>B\ .\end{cases} \tag{1}$$

When the continuity of the potential and of its derivative are used to eliminate the parameters V_0 and K, the Schrödinger equation takes the form

$$\left[-\frac{1}{r}\frac{d^2}{dr^2}r-\frac{m\beta}{r}+\frac{l(l+1)}{r^2}\right]R(r)=mBR(r),\quad r\leq B\ , \tag{2a}$$

$$\left[-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{m\beta}{B}\left[-2+\frac{r}{B}\right]+\frac{l(l+1)}{r^2}\right]R(r)=mBR(r),\quad r>B\ . \tag{2b}$$

There exist several methods for solving the radial equation (2). The authors of Ref. 1 followed a method which looked for general solutions in each of the two regions $(0,B)$ and (B,∞). It is well known that eigensolutions are determined by the condition of regularity at the origin and at infinity and the condition of continuity of these functions and of their logarithmic derivatives at $r=B$.

After a straightforward transformation, the solution of Eq. (2a) can be written in terms of a confluent hypergeometric (infinite) series whose coefficients depend on the energy parameter E. However, in Ref. 1, Laguerre polynomials were given as solutions of this equation, which is incorrect. Laguerre polynomials emerge as solutions of an equation with the Coulomb potential only when the regularity at infinity is required, which is not the case in our problem. As already said, to find a general solution in the region $r\leq B$, the only condition is the regularity at the origin, while the behavior at infinity may be arbitrary. Similarly, the solution of Eq. (2b) can be expressed as an infinite series in r. As noted in Ref. 1, this series is related to Airy functions for $l=0$. However, the conclusion[1] that the zeros of Airy functions thus determine some of the $l=0$ eigenvalues of Eq. (2) is not correct. In no way can one learn eigenvalues of this equation by considering the solution only in one region of space. The correct procedure would be the following: (a) Express general solutions in both regions in terms of appropriate infinite series,

$$R(r\leq B)=\sum_k A_k(E)r^k$$

and

$$R(r>B)=\sum_n B_n(E)r^n\ .$$

(b) Determine the values of E for which the functions $R(r\leq B)$ and $R(r>B)$ and their derivatives at $r=B$ are equal. It is clear that step (b) can be performed only numerically even for $l=0$ and that analytic solutions of the problems cannot be found.[3]

We have described the above method just to locate the source of the mistake in Ref. 1. In practical calculations, more efficient and simpler numerical analyses can be used for any particular form of potential. The method is based on the diagonalization of a matrix related to the Schrödinger equation. (For a simple algorithm, see, e.g., Ref. 4.) In Table I we give excitation energies calculated by this method with the parameters from Ref. 1 ($m_b=5.09$ GeV, $\beta=0.765$, and $B=2.690$ GeV^{-1}). The values obtained poorly match experimental data. In particular, the P level is too close to the S level.

Still, this does not mean that one should abandon the two-step potential as a tool for the investigation of quar-

TABLE I. Excitation energies (in GeV) for some lowest-lying levels of b-quarkonium, calculated with the parameters of Ref. 1. (The incorrect values as quoted in Ref. 1 are given in parentheses.) Measured values are also included.

Theory			Experiment	
$(1S)$	0	(0)	(1^3S_1)	0
$(2S)$	0.574	(0.559)[a]	(2^3S_1)	0.559
$(3S)$	0.792	(0.764)	(3^3S_1)	0.889
$(4S)$	0.972	(0.992)	(4^3S_1)	1.112
$(1P)$	0.565	$(\cdots)$	(1^3P)	0.46[b]

[a]Input quantity. [b]See Ref. 5.

konium. As shown in Ref. 6, the properties of both charmonium and the b-quarkonium can be described with another, flavor-independent set of parameters, namely,

$$m_c = 1.83 \text{ GeV}, \quad m_b = 4.35 \text{ GeV} ,$$
$$\beta = 0.728, \quad B = 1.707 \text{ GeV}^{-1} . \tag{3}$$

With the parameters (3), the two-step potential leads to results similar to those obtained with other commonly used potentials,[7] and should therefore be included in a list of "successful" potentials. We present set (3) just for a reader's orientation. One may expect some changes in these parameters once the relativistic corrections are included in the analyses. The model-independent relations

$$S_{\text{nonrel}} = \tfrac{1}{4}[(^1S_0) + 3(^3S_1)]_{\text{expt}} ,$$
$$P_{\text{nonrel}} = \tfrac{1}{12}[(^3P_0) + 3(^3P_1) + 5(^3P_2) + 3(^1P_1)]_{\text{expt}} , \tag{4}$$

which are valid under the assumption that the relativistic spin-independent forces are negligible, may help to fix parameters (3) more precisely once the lowest $^1P_1$ b-quarkonium level is found.

In conclusion, we have reanalyzed the two-step potential proposed in Ref. 1. It was claimed[1] that this potential has an attractive feature: the possibility of analytic solutions in $l=0$ sector. We exclude this possibility. However, even without that bonus, the two-step potential seems to be as good a candidate for the description of quarkonium as any other potential currently used in literature.

This work was supported by Self-Managing Community of Interest for Science of the Socialist Republic of Croatia and by the National Science Foundation under Grant No. YOR 82/051.

[1]D. S. Kulshreshtha and R. S. Kaushal, Phys. Rev. D **26**, 2331 (1982).

[2]For previous attempts with the two-step potential, see, e.g., R. S. Kaushal, Phys. Lett. **60B**, 81 (1975); R. S. Kaushal, D. S. Kulshreshtha, and D. Parashar, Nuovo Cimento A **37**, 55 (1977); D. Parashar, Can. J. Phys. **59**, 1944 (1981); D. S. Kulshreshtha and R. S. Kaushal, Lett. Nuovo Cimento **35**, 323 (1982).

[3]The authors of Ref. 1 investigated such a numerical solution (Sec. III) in addition to the "analytic solution," but again with Laguerre polynomials in the region $r \leq B$, which is not correct.

[4]K. J. Miller and M. G. Olsson, Phys. Rev. D **25**, 2383 (1982).

[5]E. D. Bloom, in *Proceedings of the 21st International Conference on High Energy Physics, Paris, 1982,* edited by P. Petiau and M. Pornuef [J. Phys. (Paris) Colloq. **43**, C3 (1982)].

[6]H. Galić, Can. J. Phys. **61**, 264 (1983).

[7]A partial list is the following: E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D **21**, 203 (1980); G. Bhanot and S. Rudaz, Phys. Lett. **78B**, 119 (1978); A. Martin, *ibid.* **100B**, 511 (1981); C. Quigg and J. L. Rosner, *ibid.* **71B**, 153 (1977); H. Krasemann and S. Ono, Nucl. Phys. **B154**, 283 (1979); J. L. Richardson, Phys. Lett. **82B**, 272 (1979); R. Y. Levine and Y. Tomozawa, Phys. Rev. D **21**, 840 (1980).