Numerical solution of Salpeter's equation

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A method for solving Salpeter's relativistic bound-state equation is presented. The interaction can be given in either momentum space or configuration space and may have various Lorentz-Dirac properties. The operators of the equation are represented as matrices in a basis of nonrelativistic harmonic-oscillator states. The resulting non-Hermitian matrix is diagonalized for various values of the oscillator frequency, a variational parameter. To reduce the size of the matrices a two-particle Foldy-Wouthuysen transformation is applied. As an example, the charmonium and b-quarkonium mass spectra are calculated using a linear confining potential plus one-gluon exchange. The effects of the coupling between the positive- and negative-energy components are examined and found to be important for the light mesons.

I. INTRODUCTION

A common model of a meson is a relativistic bound state of two fermions. It is therefore important to have a method for solving relativistic bound-state equations. Numerical solutions of the Schrodinger equation have given satisfactory fits to the meson mass spectrum and other measured meson properties, but the spin-dependent effects have to be obtained by fitting several parameters to the data or by the use of perturbation expansions. A more satisfying approach is to solve Salpeter's instantaneous approximation to the covariant Bethe-Salpeter equation and thereby include spin-dependent effects without introducing extra parameters. In this treatment, the coupling between positive- and negative-energy components is fully taken into account. Until now, no exact method of solution of Salpeter's equation with a general potential and unequal masses was known.[1] A method of solution is given here in sufficient detail to allow someone who is familiar with numerical solutions of Schrodinger's equation to proceed with a minimum amount of difficulty.

II. THE EQUATION

Salpeter derives his equation by starting from the covariant Bethe-Salpeter equation and assuming that the interaction kernel is independent of the zeroth component of the four-momentum transfer. This kernel, and hence the equation, is no longer covariant, but the equation is soluble and contains some of the relativistic effects of the covariant equation. We start from Salpeter's equation written as four coupled equations in the center-of-momentum frame:[2]

$$[E-E_a(\vec{p})-E_b(\vec{p})]\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})\phi(\vec{p}) = +\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})\int d^3k\, G(\vec{k})\phi(\vec{p}+\vec{k}) , \quad (1a)$$

$$[E+E_a(\vec{p})+E_b(\vec{p})]\Lambda_a^-(\vec{p})\Lambda_b^-(\vec{p})\phi(\vec{p}) = -\Lambda_a^-(\vec{p})\Lambda_b^-(\vec{p})\int d^3k\, G(\vec{k})\phi(\vec{p}+\vec{k}) , \quad (1b)$$

$$\Lambda_a^+(\vec{p})\Lambda_b^-(\vec{p})\phi(\vec{p})=0 , \quad (1c)$$

$$\Lambda_a^-(\vec{p})\Lambda_b^+(\vec{p})\phi(\vec{p})=0 . \quad (1d)$$

In these equations, E is the total rest energy of the bound composite, $\vec{p}$ is the relative momentum, $\phi(\vec{p})$ is the amplitude (referred to henceforth as the wave function), and G represents the interaction between particles a and b. The $\Lambda^{\pm}$ symbols are projection operators that project free particle states of positive or negative energy. They are defined by

$$\Lambda_a^{\pm}(\vec{p})=\frac{1}{2}\left[1\pm\frac{\vec{\alpha}_a\cdot\vec{p}+\beta_a m_a}{E_a(\vec{p})}\right] , \quad (2a)$$

$$\Lambda_b^{\pm}(\vec{p})=\frac{1}{2}\left[1\pm\frac{-\vec{\alpha}_b\cdot\vec{p}+\beta_b m_b}{E_b(\vec{p})}\right] , \quad (2b)$$

$$E_a(\vec{p})=+(m_a{}^2+\vec{p}^{\,2})^{1/2} , \quad (3a)$$

and

$$E_b(\vec{p})=+(m_b{}^2+\vec{p}^{\,2})^{1/2} . \quad (3b)$$

III. A MATRIX REPRESENTATION OF THE TWO-PARTICLE DIRAC MATRICES

Salpeter's equations involve two sets of Dirac matrices, one for each particle. Each set must have the anticommutation properties of the Dirac matrices, but each member of one set must commute with each member of the other set. Using the notation $\alpha^0=\beta$, we require

$$\alpha_a^\mu\alpha_a^\nu+\alpha_a^\nu\alpha_a^\mu=2\delta^{\mu\nu} ,$$

$$\alpha_b^\mu\alpha_b^\nu+\alpha_b^\nu\alpha_b^\mu=2\delta^{\mu\nu} ,$$

and

$$\alpha_a^\mu\alpha_b^\nu-\alpha_b^\nu\alpha_a^\mu=0 .$$

This can be achieved by the following choices:

$$\beta_a=\alpha_a^0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},$$

$$\beta_b=\alpha_b^0=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix},$$

$$\vec{\alpha}_a=\begin{bmatrix}0&0&\vec{\sigma}_a&0\\0&0&0&\vec{\sigma}_a\\\vec{\sigma}_a&0&0&0\\0&\vec{\sigma}_a&0&0\end{bmatrix},\tag{4}$$

and

$$\vec{\alpha}_b=\begin{bmatrix}0&\vec{\sigma}_b&0&0\\\vec{\sigma}_b&0&0&0\\0&0&0&\vec{\sigma}_b\\0&0&\vec{\sigma}_b&0\end{bmatrix}$$

The above 4×4 supermatrices really represent 16×16 matrices because each position acts in the 4×4 space of the two-particle Pauli matrices.

IV. THE BASIS STATES

A convenient set of basis states can be chosen from the solutions of the Schrodinger equation for a spherical harmonic oscillator. Oscillator states have the welcome property that the set of discrete states is complete, that is, there is no continuum as there is in the Coulomb spectrum. Furthermore, since the oscillator potential is confining, its solutions are appropriate for confined quarks. The transformation of oscillator states between momentum and configuration space is very simple. We can find the oscillator matrix elements of an operator which is given in either space with equal ease. For example, we calculate the matrix elements of $E_a(\vec{p})=(m_a{}^2+\vec{p}^{\,2})^{1/2}$ between oscillator states in momentum space while we calculate the matrix elements of the phenomenological confining potential $V(\vec{r})=c\,|\,\vec{r}\,|$ between oscillator states in configuration space.

It is convenient to write the interaction in configuration space where it is local. It is necessary to calculate the interaction in configuration space when it does not have a well-behaved momentum-space representation, as in the potential above. A general method for solving equations containing operators in both momentum and configuration space has been given by Stanley and Robson[3] and methods for solving equations containing the operators

$$(m^2+p^2)^{1/2}+V(r)$$

have been given by Nickisch, Durand, and Durand.[4]

We include the Pauli spin of two particles by including the total spin quantum number S and the total angular momentum quantum number J. The basis states are then described by the quantum numbers n,S,l,J,M, where n is the principal oscillator quantum number, l is the orbital angular momentum, and M is the magnetic quantum number. We expect the interaction to conserve total angular momentum and parity. When this is the case, the equation can be solved separately for each different total angular momentum and parity.

When the interaction has spherical symmetry, its operators are of rank zero in angular momentum space. In this common case, the Wigner-Eckart theorem tells us that the matrix elements are equal to the reduced matrix elements and the dependence on M gives simply a $(2J+1)$-fold degeneracy. Thus we need to calculate only reduced matrix elements, and our basis states are described by n, S, and l for a given J.

We begin by changing the variable of integration in Eq. (1) to $p'=p+k$ and writing the interaction as a more general nonlocal interaction,

$$G(\vec{k})=G(\vec{p}\,'-\vec{p})=G(\vec{p},\vec{p}\,')\,.$$

Then the integral in Eqs. (1a) and (1b) becomes

$$\int d^3k\,G(\vec{k})\phi(\vec{p}+\vec{k})=\int d^3p'G(\vec{p},\vec{p}\,')\phi(\vec{p}\,')\,.\tag{5}$$

V. THE FOLDY-WOUTHUYSEN TRANSFORMATION

Equations (1c) and (1d) can be used to eliminate half of the basis by transforming to the Foldy-Wouthuysen representation. This reduces the number of matrix elements by a factor of four.

Noting that

$$\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})+\Lambda_a^+(\vec{p})\Lambda_b^-(\vec{p})+\Lambda_a^-(\vec{p})\Lambda_b^+(\vec{p})+\Lambda_a^-(\vec{p})\Lambda_b^-(\vec{p})=1$$

and using (1c) and (1d), Eq. (1a) becomes

$$[E-E_a(\vec{p})-E_b(\vec{p})]\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})\phi(\vec{p})=+\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})\int d^3p'G(\vec{p},\vec{p}\,')[\Lambda_a^+(\vec{p}\,')\Lambda_b^+(\vec{p}\,')+\Lambda_a^-(\vec{p}\,')\Lambda_b^-(\vec{p}\,')]\phi(\vec{p}\,')\,.\tag{6}$$

The transformation is implemented by the operator

$$U(\vec{p})=[A_a(\vec{p})+B_a(\vec{p})\beta_a\vec{\alpha}_a\cdot\hat{p}][A_b(\vec{p})-B_b(\vec{p})\beta_b\vec{\alpha}_b\cdot\hat{p}]\,,$$

where $\hat{p}=\vec{p}/|\,\vec{p}\,|$,

$$A_a(\vec{p})=\left[\frac{E_a(\vec{p})+m_a}{2E_a(\vec{p})}\right]^{1/2},\quad B_a(\vec{p})=\left[\frac{E_a(\vec{p})-m_a}{2E_a(\vec{p})}\right]^{1/2},\ \text{etc.}$$

This operator satisfies

$$U(\vec{p})U^\dagger(\vec{p})=U^\dagger(\vec{p})U(\vec{p})=1\ ,$$

$$U(\vec{p})\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})U^\dagger(\vec{p})=\tfrac{1}{2}(1+\beta_a)\tfrac{1}{2}(1+\beta_b)=\begin{bmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{bmatrix}=B^{++}\ ,$$

and

$$U(\vec{p})\Lambda_a^-(\vec{p})\Lambda_b^-(\vec{p})U^\dagger(\vec{p})=\tfrac{1}{2}(1-\beta_a)\tfrac{1}{2}(1-\beta_b)=\begin{bmatrix}0&&&\\&0&&\\&&0&\\&&&1\end{bmatrix}=B^{--}\ .$$

The projection operators B^{++} and B^{--} are independent of $\vec{p}$. We then multiply Eq. (6) by $U(\vec{p})$ from the left and insert $U^\dagger(\vec{p})U(\vec{p})=1$ as required to obtain

$$[E-E_a(\vec{p})-E_b(\vec{p})]B^{++}U(\vec{p})\phi(\vec{p})=B^{++}U(\vec{p})\int d^3p'G(\vec{p},\vec{p}\,')U^\dagger(\vec{p}\,')[B^{++}+B^{--}]U(\vec{p}\,')\phi(\vec{p}\,')\ . \tag{7}$$

To get Eq. (7) we used the fact that $U(\vec{p})$ commutes with both $E_a(\vec{p})$ and $E_b(\vec{p})$. Defining $X(\vec{p})=B^{++}U(\vec{p})\phi(\vec{p})$ and $Z(\vec{p})=B^{--}U(\vec{p})\phi(\vec{p})$, and using $B^{++}B^{++}=B^{++}$ and $B^{--}B^{--}=B^{--}$, Salpeter's equations are reduced to a set of two coupled equations:

$$B^{++}[E-E_a(\vec{p})-E_b(\vec{p})]B^{++}X(\vec{p})=B^{++}U(\vec{p})\int d^3p'G(\vec{p},\vec{p}\,')U^\dagger(\vec{p}\,')[B^{++}X(\vec{p}\,')+B^{--}Z(\vec{p}\,')] \tag{8a}$$

and

$$B^{--}[E+E_a(\vec{p})+E_b(\vec{p})]B^{--}Z(\vec{p})=-B^{--}U(\vec{p})\int d^3p'G(\vec{p},\vec{p}\,')U^\dagger(\vec{p}\,')[B^{++}X(\vec{p}\,')+B^{--}Z(\vec{p}\,')]\ . \tag{8b}$$

Since the operators of Eqs. (8) are sandwiched between B^{++} and B^{--}, we need only one fourth of the number of matrix elements needed in Eqs. (1). This is illustrated below, where the positions marked $\times$ denote the matrix elements which are needed:

$$\begin{bmatrix}\times&\cdot&\cdot&\times\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\times&\cdot&\cdot&\times\end{bmatrix}.$$

VI. MATRIX REPRESENTATION IN OSCILLATOR STATES

Equations (8) are put into a form suitable for diagonalization by first expanding in an infinite, complete set of basis states and then truncating to a finite set that will fit into a real computer. The accuracy of this method is limited only by the amount of computer power available. The ground-state energy and wave function are the most accurate, the second lowest state is less accurate, etc., with the states of highest excitation incorrect. It is necessary to include the higher excitation states because they mix with the lower states. This situation is the same for a solution of the Schrodinger equation by expansion in oscillator states followed by diagonalization of the resulting matrix.

The accuracy of a computational result can be estimated by changing the truncation point and therefore the size of the basis. An accurate result changes very little when the size is increased.

We next show how to write Eq. (8a) as a matrix equation in oscillator states. The development of Eq. (8b) is similar. First, the momentum dependence of each of the factors on the right-hand side is isolated by introducing Dirac δ functions and integrating. Equation (8a) is written

$$B^{++}[E-E_a(\vec{p})-E_b(\vec{p})]B^{++}X(\vec{p})$$
$$=B^{++}U(\vec{p})\int d^3q\,\delta(\vec{p}-\vec{q})\int d^3p'G(\vec{q},\vec{p}\,')\int d^3q'\delta(\vec{p}\,'-\vec{q}\,')U^\dagger(\vec{q}\,')[B^{++}X(\vec{q}\,')+B^{--}Z(\vec{q}\,')]\ . \tag{9}$$

Next, the wave functions are expanded in orthonormal basis states, $f_\alpha(\vec{p})$, as

$$X(\vec{p})=X_\alpha f_\alpha(\vec{p}) ,$$

where the greek subscript is a collective label for a particular combination of the quantum numbers n,S,l,J,M, and

$$X_\alpha=\int d^3p\, f^*_\alpha(\vec{p})X(\vec{p})$$

is a coefficient. A sum over repeated subscripts is implied throughout this paper.

The δ functions are expressed as

$$\delta(\vec{p}-\vec{q})=f_\alpha(\vec{p})f^*_\alpha(\vec{q}) .$$

Making these substitutions and multiplying Eq. (9) by $f^*_\alpha(\vec{p})$ and integrating over $\vec{p}$, we obtain

$$B^{++}\int d^3p\, f^*_\alpha(\vec{p})[E-E_a(\vec{p})-E_b(\vec{p})]f_\beta(\vec{p})B^{++}X_\beta$$
$$=B^{++}\int d^3p\, f^*_\alpha(\vec{p})U(\vec{p})$$
$$\times\int d^3q\, f_\beta(\vec{p})f^*_\beta(\vec{q})\int d^3p' G(\vec{q},\vec{p}\,')\int d^3q' f_\gamma(\vec{p}\,')f^*_\gamma(\vec{q}\,')U^\dagger(\vec{q}\,')f_\delta(\vec{q}\,')[B^{++}X_\delta+B^{--}Z_\delta] . \quad (10)$$

Using the orthonormality of the basis functions

$$\int d^3p\, f^*_\alpha(\vec{p})f_\beta(\vec{p})=\delta_{\alpha\beta} ,$$

we obtain the matrix equation

$$B^{++}[E\delta_{\alpha\beta}-(E_a+E_b)_{\alpha\beta}]B^{++}X_\beta$$
$$=B^{++}U_{\alpha\beta}G_{\beta\gamma}U^\dagger_{\gamma\delta}(B^{++}X_\delta+B^{--}Z_\delta) , \quad (11)$$

where

$$(E_a+E_b)_{\alpha\beta}=\int d^3p\, f^*_\alpha(\vec{p})[E_a(\vec{p})+E_b(\vec{p})]f_\beta(\vec{p}) , \quad (12)$$

$$U_{\alpha\beta}=\int d^3p\, f^*_\alpha(\vec{p})U(\vec{p})f_\beta(\vec{p}) , \quad (13)$$

and

$$G_{\beta\gamma}=\int d^3p\, d^3p' f^*_\beta(\vec{p})G(\vec{p},\vec{p}\,')f_\gamma(\vec{p}\,') . \quad (14)$$

Using the same procedure, we reduce Eq. (8b) to

$$B^{--}[E\delta_{\alpha\beta}+(E_a+E_b)_{\alpha\beta}]B^{--}Z_\beta$$
$$=-B^{--}U_{\alpha\beta}G_{\beta\gamma}U^\dagger_{\gamma\delta}(B^{++}X_\delta+B^{--}Z_\delta) . \quad (15)$$

Equations (11) and (15) form a set of coupled equations in X_α and Z_α.

These equations can be solved by diagonalization, but since the matrix to be diagonalized is not Hermitian, complex eigenvalues are expected. This will be the case when the interaction contains a part which gives the states a finite lifetime, such as an annihilation term.

In many cases of interest, the interaction G is Hermitian and has the property that it is unchanged when the signs of all energies are changed. In this case,

$$B^{++}U_{\alpha\beta}G_{\beta\gamma}U^\dagger_{\gamma\delta}B^{++}=B^{--}U_{\alpha\beta}G_{\beta\gamma}U^\dagger_{\gamma\delta}B^{--}\equiv W^\epsilon_{\alpha\delta} \quad (16)$$

and

$$B^{++}U_{\alpha\beta}G_{\beta\gamma}U^\dagger_{\gamma\delta}B^{--}=B^{--}U_{\alpha\beta}G_{\beta\gamma}U^\dagger_{\gamma\delta}B^{++}\equiv W^\sigma_{\alpha\delta} \quad (17)$$

with $W^\epsilon_{\alpha\delta}$ and $W^\sigma_{\alpha\delta}$ Hermitian matrices.

VII. RANDOM-PHASE-APPROXIMATION DIAGONALIZATION

Under conditions Eqs. (16) and (17), and with phases chosen so that the matrix elements are real, Eqs. (11) and (15) take the form of equations encountered in the random-phase approximation (RPA) in nuclear physics.[5]

Defining two real square symmetric matrices R and T by

$$(R)_{\alpha\beta}=(E_a+E_b)_{\alpha\beta}+W^\epsilon_{\alpha\beta}$$

and

$$(T)_{\alpha\beta}=W^\sigma_{\alpha\beta} ,$$

Salpeter's equations are written in supermatrix form as

$$\begin{bmatrix} R & T \\ -T & -R \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}=E\begin{bmatrix} X \\ Z \end{bmatrix} . \quad (18)$$

This nonsymmetric matrix is diagonalized using the method given by Ullah and Rowe[6] in which the difference $R-T$ is first diagonalized and the resulting unitary transformation is applied to the sum $R+T$ which is then diagonalized. Since the sum and difference matrices are symmetric, standard methods can be used to diagonalize them.

The eigenvalues of this RPA matrix are either real or pure imaginary, and occur in pairs with one member of the pair having the opposite sign of the other. For real eigenvalues, this situation is acceptable, since we expect each positive energy solution to have its negative-energy counterpart. We have no interpretation of the solutions which have imaginary eigenvalues, so we discard them. Experience shows that interactions whose characteristic energy is small compared to the rest masses of the constituent particles give only real eigenvalues.

VIII. CALCULATION OF THE INTERACTION MATRICES

Using the method given here, we can solve Salpeter's equations with many different kinds of interactions. The interaction can be given in momentum space or configuration space. It can be phenomenological or derivable from

TABLE I. Matrices that determine the Lorentz-Dirac properties of the interaction. Zeros are implied in the blank positions.

Scalar:	$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$	
Vector:	$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$	$-\vec{\sigma}_a\cdot\vec{\sigma}_b \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$
Tensor:	$-\vec{\sigma}_a\cdot\vec{\sigma}_b \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}$	$-\vec{\sigma}_a\cdot\vec{\sigma}_b \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$
Axial vector:	$- \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$	$+\vec{\sigma}_a\cdot\vec{\sigma}_b \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$
Pseudoscalar:	$- \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}$	

fundamentals. It can have various Lorentz-Dirac properties. We can even use a combination of the interactions mentioned above.

The interaction is specified by a functional form in either momentum or configuration space, and a Lorentz-Dirac property which specifies the Dirac matrices that multiply the functional form. The matrices which determine the Lorentz-Dirac properties are given in Table I. In this method, the Dirac matrices are separated into two parts which operate in separate spaces, the space of two-particle Pauli spin and the two-particle energy-sign space. The Pauli spin part is treated by the familiar methods of angular momentum coupling in which the two Pauli spin-$\frac{1}{2}$ particles are coupled to a total spin S, and then S is coupled to the orbital angular momentum l to form the total angular momentum J. Each basis state has a definite J, S, and l, and matrix elements of operators containing the Pauli matrices $\vec{\sigma}_a$ and $\vec{\sigma}_b$ are computed according to the formulas of angular momentum theory.

The energy-sign part is treated by multiplying out the operators $UGU^\dagger$ and picking out the parts that contribute to R and T. The process of picking out the parts is facilitated by a schematic calculation that identifies operators that connect states of the same energy sign as even (ϵ) and operators that connect states of the opposite energy sign as odd (σ). Thus the unit Dirac matrix 1 and β are indicated ϵ, while $\vec{\alpha}$ and γ^5 are indicated σ. When even and odd Dirac matrices are multiplied, the product is even or odd like the products of symmetric and antisymmetric wave functions:

$$\epsilon\times\epsilon=\epsilon\,,$$
$$\sigma\times\sigma=\epsilon\,,$$
$$\sigma\times\epsilon=\sigma\,,$$
$$\epsilon\times\sigma=\sigma\,.$$

The properties of the two-particle Dirac matrices are indicated by $\epsilon\epsilon$, $\epsilon\sigma$, etc., where the left-hand symbol refers to particle a. Because of the B^{++} and B^{--} projection operators in Eqs. (16) and (17), W^ϵ contains only $\epsilon\epsilon$ and W^σ contains only $\sigma\sigma$. An example of the schematic calculation is given in Appendix A.

IX. PARITY-CHANGING OPERATORS

In order to calculate $\vec{\alpha}_a\cdot\hat{p}$ and $\vec{\alpha}_b\cdot\hat{p}$, we use reduced matrix elements of $\vec{\sigma}_a\cdot\hat{p}$ and $\vec{\sigma}_b\cdot\hat{p}$. They are given in terms of Clebsch-Gordan and Wigner coefficients as

$$\langle SlJ||\vec{\sigma}_a\cdot\hat{p}||S'l'J'\rangle\equiv K=\delta(JJ')(-1)^{J+l'}[6(2S+1)(2l+1)(2S'+1)]^{1/2}W(SS'll';1J)W(SS'\tfrac{1}{2}\tfrac{1}{2};1\tfrac{1}{2})\langle l100\,|\,l'0\rangle$$

and

$$\langle SlJ||\vec{\sigma}_b\cdot\hat{p}||S'l'J'\rangle=(-1)^{S+S'}K\ ,$$

using the conventions of Brink and Satchler.[7] This formula is obtained by writing $\hat{p}$ in terms of $l=1$ spherical harmonics. Then $\langle SlJ||\sigma_a\cdot\hat{p}||S'l'J\rangle$ is written as a function of $\langle S||\vec{\sigma}_a||S'\rangle$ and $\langle l||\hat{p}||l'\rangle$. These operators connect states whose l values differ by 1, that is, they connect states of different parity. This introduces a complication that is not encountered in calculations of the Schrodinger equation. Even when the interaction does not connect states of different parity, matrix elements of the interaction between states of both parities are used in the calculation. As a concrete example, suppose we wish to calculate states of $J=3$ and even l parity and our potential does not connect opposite parities. We include only $l=2$ and 4 in our expansion of the wave functions, but because of the presence of parity-changing operators in U, we need matrix elements of G between states of odd as well as even l. In order to ensure that the matrix $\vec{\sigma}_a\cdot\hat{p}$ multiplied by itself approximately equals the unit matrix, care must be taken to include all l values that can be coupled according to the rules of angular momentum. Thus we need matrix elements of the interaction between states of $l=2$, 3, and 4.

X. CALCULATION OF THE FREE-PARTICLE ENERGIES

The only matrix elements that have not been discussed are those of E_a+E_b. They are spherically symmetric and are easily calculated by numerical integration with oscillator radial functions $R_{nl}(p)$ in momentum space.

$$\langle nSlJ||E_a+E_b||n'S'l'J\rangle=\delta_{SS'}\delta_{ll'}\int_0^\infty p^2dp\,R^*_{nl}(p)R_{n'l}(p)\times[(m_a{}^2+p^2)^{1/2}+(m_b{}^2+p^2)^{1/2}]\ .$$

XI. EXAMPLE OF THE METHOD

As an example, a calculation of the charmonium and b-quarkonium mass spectra is included. The interaction is specified by a local, spherically symmetric potential in configuration space. It contains a linear scalar confining potential plus a Coulomb Lorentz vector potential that is meant to represent the one-gluon-exchange interaction. The spacelike as well as timelike parts of the vector potential are included. The potential is flavor independent: it is the same for charmonium and b-quarkonium. It depends on two strength parameters that are adjusted to fit the data. With the charmed-quark mass m_c and the bottom-quark mass m_b as additional parameters, the calculation uses four parameters to fit 21 measured meson masses.

The potential is written

$$V(r)=C_l r\beta_a\beta_b+C_c r^{-1}(1-\vec{\alpha}_a\cdot\vec{\alpha}_b)\ .$$

Matrix elements of this potential are substituted for $G_{\beta\gamma}$ in Eqs. (16) and (17). Matrix elements of r and r^{-1} are calculated by integrating with oscillator radial functions in configuration space. For example,

$$\langle nSlJ||r^{-1}||n'S'l'J\rangle=\delta_{SS'}\delta_{ll'}\int_0^\infty r^2dr\,R^*_{nl}(r)r^{-1}R_{n'l}(r)\ ,$$

where $R_{nl}(r)$ are solutions of the radial part of the Schrodinger equation for the isotropic harmonic oscillator.[8]

TABLE II. Results of a calculation of the charmonium and b-quarkonium spectra. Masses are in MeV.

Meson	J^P	M(expt)	M(calc)	M(calc) $-M$(expt)
η_c	0^-	2981	2966	−15
J/ψ	1^-	3097	3095	−2
χ_0	0^+	3415	3434	+19
χ_1	1^+	3510	3475	−35
χ_2	2^+	3556	3447	−109
η_c'	0^-	3594	3622	+28
ψ	1^-	3686	3682	−4
ψ	1^-	3770	3735	−35
ψ	1^-	4029	4085	+56
ψ	1^-	4159	4119	−40
ψ	1^-	4415	4405	−10
Υ	1^-	9460	9471	+11
$1\,^3P_0$	0^+	9873	9822	−51
$1\,^3P_1$	1^+	9895	9837	−58
$1\,^3P_2$	2^+	9915	9843	−72
Υ	1^-	10023	9997	−26
$2\,^3P_0$	0^+	10233	10225	−8
$2\,^3P_1$	1^+	10254	10237	−17
$2\,^3P_2$	2^+	10271	10244	−27
Υ	1^-	10356	10376	+20
Υ	1^-	10573	10693	+120

Since the interaction of this example is even under parity and conserves total angular momentum, we solve separately for each J^P and need to consider only reduced matrix elements. Therefore our basis states are described by n, S, and l only.

The results for charmonium and b-quarkonium are presented in Table II. The average of the absolute differences between these results and experiment is 36 MeV. The values of the potential strength parameters that were used are $C_l=0.29$ GeV^2 and $C_c=-0.25$, and the masses were $m_c=1.25$ GeV and $m_b=4.58$ GeV. No attempt was made to optimize the fit by attaching a minimizer search routine.

XII. EFFECT OF THE ODD-ODD COUPLINGS

In order to assess the importance of the odd-odd couplings, the $J^P=0^-$ part of the charmonium spectrum was recalculated with the matrix W^σ set to zero. The result was that the masses of the states increased by 5–10 MeV. This result can be understood in terms of second-order perturbation theory, where the energy shift due to the odd-odd terms is a level-level *attraction* between the positive-energy and negative-energy states, and the energy denominator is two times the energy. This effect is small for the charmonium system and negligible for the b-quarkonium system. Since this effect should be important for light mesons, it was investigated as a function of the quark mass. Using the same potential that fitted the heavy mesons, the ground state of the $J^P=0^-$ meson was calculated with the full equation and recalculated with the odd-odd coupling turned off. The results are presented in Fig. 1. The effect on the π was about 400 MeV, but we interpret this result with caution because it is precisely this low-mass region where Salpeter's instantaneous approximation is most suspect. It is amusing that this coupling between positive- and negative-energy components produces the effect of a running coupling constant when the coupling constant is held fixed. In this calculation, the vector-to-pseudoscalar mass difference is mostly due to the $r^{-1}\vec{\alpha}_a\cdot\vec{\alpha}_b$ term in the potential and it is a large effect in the light mesons due to the small energy denominator.

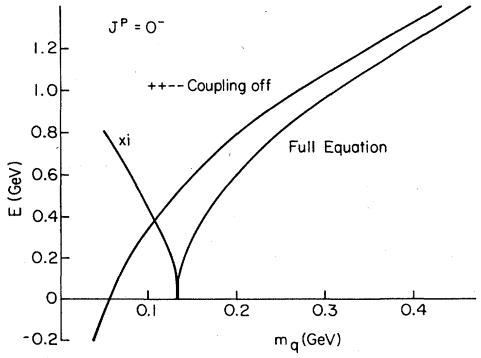

FIG. 1. The effects of the odd-odd coupling in Salpeter's equation. The rest energy of the lightest pseudoscalar meson is plotted as a function of the quark mass calculated with the full equation and compared to the results calculated with the coupling turned off. For each eigenvalue plotted here, there is another with opposite sign.

We are currently attempting to solve the full Bethe-Salpeter equation with a general covariant interaction.

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APPENDIX A

In this appendix we give an example of the schematic calculation for picking out the terms that contribute to the interaction part of the matrices W^ϵ and W^σ.

The schematic calculation can be illustrated by considering an interaction with vector Lorentz-Dirac properties:

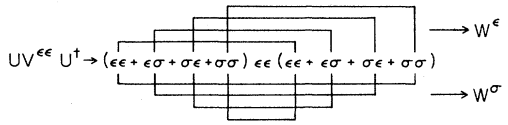
$$V(1-\vec{\alpha}_a\cdot\vec{\alpha}_b)=V^{\epsilon\epsilon}+V^{\sigma\sigma}\rightarrow\epsilon\epsilon+\sigma\sigma\ .$$

Here V represents a function of p or r, but does not contain Dirac matrices. Multiplying out the operator $U(\vec{p})$ we get

$$U=A_aA_b-A_aB_b\beta_b\vec{\alpha}_b\cdot\hat{p}+B_aA_b\beta_a\vec{\alpha}_a\cdot\hat{p}$$
$$-B_aB_b\beta_a\vec{\alpha}_a\cdot\hat{p}\beta_b\vec{\alpha}_b\cdot\hat{p}$$

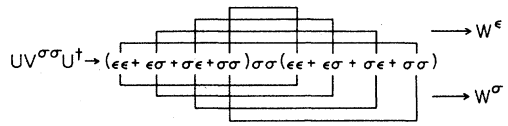
and $U^\dagger$ is almost the same, but has the signs in front of the second and third terms changed.

Studying the $\epsilon\epsilon$ part first, we write



where the connecting lines above the product show the four terms that contribute to W^ϵ, and the connecting lines below show the four terms that contribute to W^σ. The other eight terms do not contribute and need not be calculated.

Similarly, for the $\sigma\sigma$ part,



A typical term contributing to W^ϵ, $\epsilon\epsilon\times\sigma\sigma\times\sigma\sigma$, is the matrix product

$$B^{++}(A_aA_b)(-V\vec{\alpha}_a\cdot\vec{\alpha}_b)(-B_aB_b\beta_a\vec{\alpha}_a\cdot\hat{p}\beta_b\vec{\sigma}_b\cdot\hat{p})B^{++}$$
$$=(A_aA_b)(-V\vec{\sigma}_a\cdot\vec{\sigma}_b)(-B_aB_b\vec{\sigma}_a\cdot\hat{p}\vec{\sigma}_b\cdot\hat{p})\ .$$

We also use

$$\langle nSlJ||\vec{\sigma}_a\cdot\vec{\sigma}_b||n'S'l'J\rangle=\delta_{nn'}\delta_{SS'}\delta_{ll'}(4S-3)$$
$$\text{for } S=0 \text{ or } 1\ .$$

APPENDIX B

This appendix gives the relationship between momentum-space and configuration-space matrix elements as well as practical suggestions to save computer time and job size.

It is convenient to use $\hbar = c = 1$ and dimensionless variables for r and p, choosing the length scale so that the oscillator parameter $mk = 1$, where m and k are the mass and spring constant in the Schrodinger equation for the spherical harmonic oscillator. Then the length scale becomes the variational parameter and the oscillator radial functions in configuration space are related to those in momentum space by simply a phase. We calculate configuration-space matrix elements exactly as if they were the analogous matrix elements in momentum space, and then multiply by the appropriate power of the length scale and the phase

$$(i)^{(2n+1)-(2n'+l')} .$$

In most cases of interest, the potential is even under parity inversion, so that its nonzero matrix elements have $l-l'$ even. In these cases, we can choose our phases so that all matrix elements are real.

[1]R. H. Dalitz, in *Progress in Particle and Nuclear Physics,* edited by D. H. Wilkinson (Pergamon, New York, 1981), Vol. 8.
[2]E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
[3]D. P. Stanley and D. Robson, Phys. Rev. D **21**, 3180 (1980).
[4]L. J. Nickisch, Loyal Durand, and Bernice Durand, Phys. Rev. D **30**, 660 (1984).
[5]D. Thouless, Nucl. Phys. **22**, 78 (1961).
[6]N. Ullah and D. J. Rowe, Nucl. Phys. **A163**, 257 (1971).
[7]D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon, Oxford, 1968).
[8]A. deShalit and I. Talmi, *Nuclear Shell Theory* (Academic, New York, 1963), p. 512.