Relativistic corrections to radiative transitions and spectra of quarkonia

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We calculate the energy spectrum and the one-photon $E1$ and $M1$ decays of charmonium and b-quarkonium including their first-order relativistic corrections, within the context of the Buchmüller-Tye potential. Results seem to suggest that the confining potential is predominantly scalar since this assumption gives good agreement for spectra and for $E1$ decay rates of $c\bar{c}$ and gives reasonable results for $b\bar{b}$. For $M1$ decays discrepancies persist unless the quarks have large anomalous moments, but the resolution of such discrepancies can probably be achieved by other means. As an interesting by-product of our investigations we find that for the nonrelativistically inhibited $E1$ decay $\Upsilon'' \to \chi_J^b + \gamma$ of the $b\bar{b}$ system, some rates are enhanced by a huge factor by first-order relativistic effects. When more data are available these decays should provide an additional testing ground for models of relativistic corrections.

I. INTRODUCTION

The study of electric dipole ($E1$) and magnetic dipole ($M1$) one-photon transitions between the bound states of quarkonia such as the $c\bar{c}$ and $b\bar{b}$ systems is useful in the search for a phenomenological potential describing the flavor-independent interaction between the quark and the antiquark. Although the potential is primarily determined by comparing the calculated and the experimental energy spectrum, it is clear that the choice of a potential may be constrained further by the comparison of the theoretical and experimental one-photon radiative transitions of quarkonia. In recent years many authors have attempted to calculate the $E1$ and the $M1$ transitions of charmonium from several different points of view.[1–11] The common feature of the $E1$ calculations is that the predicted nonrelativistic rate for the $\psi' \to \chi_j + \gamma$ decay is too high by a factor of two to three, no matter what potential one uses. This suggests that the relativistic corrections may be extremely important in $E1$ decays. In fact, several recent calculations support this point of view.[6,7,9,10] Relativistic corrections are also crucial for the calculation of certain $M1$ decays such as $\psi' \to \eta_c + \gamma$ and $\eta_c' \to \psi + \gamma$, since these decays are forbidden in the nonrelativistic limit. Two of us have recently carried out calculations of the $M1$ decays of the $q\bar{q}$ system, and have concluded that some important recoil terms were neglected in previous works.[8] These neglected terms turned out to be as important numerically as the terms which had already been calculated.

In this paper we present a comprehensive approach to both the $E1$ and the $M1$ transitions of massive $q\bar{q}$ systems, including their leading relativistic corrections. We present formulas for the decay rates in a way which largely avoids commitment to a specific choice of the potential. We find that the predicted results for the $M1$ decays depend crucially on whether or not the potential contains a scalar part and on whether the quark has an anomalous magnetic moment. We present numerical results for the Buchmüller-Tye potential[12] for which the nonrelativistic predictions on the energy spectrum of $q\bar{q}$ systems are very good. The confining part of the Buchmüller-Tye (BT) potential, namely, the linear potential, is assumed to be a mixture of two terms, one of which transforms as a vector and the other as a scalar in the covariant limit.[8] The part of the BT potential derived from perturbative quantum chromodynamics is always assumed to be the fourth component of a four-vector. We have calculated the energy spectrum of both the $c\bar{c}$ and the $b\bar{b}$ systems in two cases: (1) when the confining linear potential is purely scalar ($\eta_S = 1$) and (2) when the confining potential is also purely vector ($\eta_S = 0$). In the second case the entire potential can be thought of as arising from the exchange of a vector particle. We have obtained numerical results for $M1$ decays in terms of the parameter η_S and a presumed quark anomalous moment a. We have also calculated the $E1$ decays of both the $c\bar{c}$ and the $b\bar{b}$ systems in terms of the parameters η_S and a.

The format of the rest of the paper is as follows. In Sec. II we derive the formulas for the $E1$ and the $M1$ decay rates. In Sec. III we introduce our Hamiltonian correct to order v^2/c^2 based on the Buchmüller-Tye potential and present numerical results for the energy spectrum of the $c\bar{c}$ and the $b\bar{b}$ bound states including their fine and hyperfine structure splittings for two cases, namely, (a) when the confining part of the Buchmüller-Tye potential is a Lorentz scalar and the rest of the potential is vector and (b) when the entire Buchmüller-Tye potential including the confining linear potential is vector. In Sec. IV we present numerical results for the $M1$ decay rates of the $b\bar{b}$ and the $c\bar{c}$ bound states for the cases mentioned above. In Sec. V we give the numerical results for the $E1$ decay rates of the $c\bar{c}$ and the $b\bar{b}$ bound states. Finally, in Sec. VI we make some concluding remarks.

II. DERIVATION OF $E1$ AND $M1$ DECAY RATES

Let H be the Hamiltonian for an isolated composite system consisting of two particles. It is assumed that H is correct to order v^2/c^2 and in this section we will not choose a specific form. As shown in a previous paper[13] by one of the authors (K.J.S.), if $\vec{A}$ is the quantized radiation field, the linear terms coupling the composite system to the radiation field are given by

$$H_I=\sum_{j=1}^{2}\left[\frac{ie_j}{2c}[\vec{A}_j,\cdot[\vec{r}_j,H]]_+-\frac{e_j}{m_jc}\vec{S}_j\cdot\vec{B}_j-\frac{e_j}{2m_j^2c^2}\vec{S}_j\cdot(\vec{E}_j\times\vec{p}_j)-\frac{ie_j}{4m_j^2c^2}\vec{S}_j\cdot(\vec{\nabla}_j\times\vec{E}_j)+\frac{e_j}{4m_j^3c^3}[\vec{p}_j^{\,2},\vec{S}_j\cdot\vec{B}_j]_+ +\frac{e_j}{m_j^2c^3}V_S\vec{S}_j\cdot\vec{B}_j-\frac{e_j}{m_jc}a_j\vec{S}_j\cdot\vec{B}_j+\frac{e_ja_j}{4m_j^3c^3}[\vec{B}_j\cdot\vec{p}_j,\vec{S}_j\cdot\vec{p}_j]_+-\frac{e_ja_j}{m_j^2c^2}\vec{S}_j\cdot(\vec{E}_j\times\vec{p}_j)+\frac{ie_ja_j}{2m_j^2c^3}\vec{S}_j\cdot\vec{B}_j\right], \quad (1)$$

where

$$\vec{A}_j\equiv\vec{A}(\vec{r}_j,t)\,,\qquad \vec{B}_j=\vec{\nabla}_j\times\vec{A}(\vec{r}_j,t)\,,\qquad \vec{E}_j=-\dot{\vec{A}}(\vec{r}_j,t)\,, \quad (2)$$

and $\vec{r}_j$ and $\vec{p}_j$ are, respectively, the position and momentum operators of the jth particle. In Eq. (1), the anomalous moment parameter a_j of the jth particle is defined by

$$\vec{\mu}_j=e_j\vec{S}_j(1+a_j)/mc$$

where $\vec{\mu}_j$ is the magnetic-moment operator for the jth particle. For a system such as quarkonium, made up of a particle and its antiparticle, we have $a_1=a_2=a$. We shall assume $\hbar=1$, throughout. In Eq. (1) we have included possible anomalous moments a_j for the constituent particles. The terms involving the anomalous moments were absent in the previous paper[13] because there we assumed the constituent particles to be point Dirac particles. These terms can be derived on the basis that when the internal interaction among the constituent particles goes to zero, the interaction Hamiltonian should be a simple sum of single-particle Hamiltonians of Dirac particles (with anomalous moments) in an external electromagnetic field, each single-particle Hamiltonian being reduced to order v^2/c^2 by the Foldy-Wouthuysen technique. The V_S term of Eq. (1) cannot be derived by the principles stated in Ref. 13. However, if we start from a covariant Breit-type two-particle Hamiltonian with scalar and vector interactions (see Ref. 8), introduce the electromagnetic field by minimal coupling, carry out a Barker-Glover reduction to order v^2/c^2, then we obtain this scalar term and also verify the other terms of Eq. (1). This equation can also be derived by other methods.[4,11,14] The symbol V_S stands for the part of the potential which transforms as a scalar in the covariant Hamiltonian mentioned above.

We now proceed to the derivation of radiative decay rates. Let $T(t_0)$ denote the probability amplitude for observing the composite system at t_0 in state $|A\rangle$ with the simultaneous presence of a photon of energy ω, momentum $\vec{k}$, and polarization vector $\hat{\epsilon}_\alpha$, if at time $t=0$ the system is known to be in state $|B\rangle$ with no photon present. Using the interaction Hamiltonian H_I in first-order perturbation theory we find that

$$\begin{aligned}T(t_0)=\frac{c}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}\sum_{j=1}^{2}\Bigg[\Bigg\langle A\Bigg|\Bigg[&\frac{e_j}{2c}[[\vec{r}_j,H]\cdot,\hat{\epsilon}_\alpha e^{-i\vec{k}\cdot\vec{r}_j}]_+ +\frac{e_j}{m_jc}\vec{S}_j\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_j}-\frac{e_jk}{2m_j^2c^2}\vec{S}_j\cdot e^{-i\vec{k}\cdot\vec{r}_j}(\vec{p}_j\times\hat{\epsilon}_\alpha)\\ &+\frac{e_jk}{4m_j^2c^2}\vec{S}_j\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_j}-\frac{e_j}{4m_j^3c^3}[\vec{p}_j^{\,2},\vec{S}_j\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_j}]_+\\ &-\frac{e_j}{m_j^2c^3}V_S\vec{S}_j\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_j}+\frac{e_ja_j}{m_jc}\vec{S}_j\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_j}\\ &-\frac{e_ja_j}{4m_j^3c^3}[e^{-i\vec{k}\cdot\vec{r}_j}(\vec{k}\times\hat{\epsilon}_j)\cdot\vec{p}_j,\vec{S}_j\cdot\vec{p}_j]_+ +\frac{e_ja_j\omega}{m_j^2c^3}e^{-i\vec{k}\cdot\vec{r}_j}\vec{S}_j\cdot(\hat{\epsilon}_\alpha\times\vec{p}_j)\\ &+\frac{e_ja_j\omega}{2m_j^2c^3}\vec{S}_j\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_j}\Bigg]\Bigg|B\Bigg\rangle\Bigg]\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt'\,,\end{aligned} \quad (3)$$

where $\omega=|\vec{k}|=k$. In Eq. (3) the states $|A\rangle$ and $|B\rangle$ are direct products of the internal states and the center-of-mass states with energy eigenvalues ω_A and ω_B. Thus, assuming the initial state to have zero center-of-mass momentum, it follows that

$$|B\rangle=|B\rangle_I\otimes|0\rangle_{\text{c.m.}};\quad |A\rangle=|A\rangle_I\otimes|-\vec{k}\rangle_{\text{c.m.}}\,, \quad (4)$$

where $|A\rangle_I$ and $|B\rangle_I$ are eigenstates of the internal Hamiltonian h, which is related to the total Hamiltonian H of the composite system by

$$H=(h^2+c^2p^2)^{1/2} \tag{5}$$

to order v^2/c^2. The state vectors $|0\rangle_{\text{c.m.}}$ and $|-\vec{k}\rangle_{\text{c.m.}}$ are, respectively, eigenvectors of the total-momentum operator $\vec{p}=\vec{p}_1+\vec{p}_2$ with eigenvalues of zero and the recoil momentum $-\vec{k}$.

Next we express the constituent variables $\vec{r}_j$, $\vec{p}_j$, and $\vec{S}_j$ $(j=1,2)$ in Eq. (3) in terms of the relativistic internal and center-of-mass variables by means of the relations given by Krajcik and Foldy[15] and others.[16] After extracting the parity-odd and parity-even portions of the vector part of the transition operator, we obtain the electric dipole ($E1$) and the magnetic dipole ($M1$) transition amplitudes in the form[13,17]

$$T_{E1}(t_0)=\frac{c}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2} k_0\hat{\epsilon}_\alpha\cdot{}_I\langle A\,|\vec{X}_0+\vec{X}_1|\,B\rangle_I\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt' , \tag{6}$$

where $ck_0=E_B^I-E_A^I$, and

$$\vec{X}_0=e_Q\vec{r} , \tag{7}$$

$$\vec{X}_1=-\frac{ike_Q}{20mc}\{(\vec{r}^{\,2}\vec{\pi}+\vec{\pi}\,\vec{r}^{\,2})-\tfrac{1}{2}[\vec{r}(\vec{r}\cdot\vec{\pi})+(\vec{\pi}\cdot\vec{r})\vec{r}]\}-\frac{ike_Qa}{4mc}(\vec{r}\times\vec{\Sigma}) , \tag{8}$$

and

$$\vec{\Sigma}=\tfrac{1}{2}(\vec{\sigma}_1+\vec{\sigma}_2) . \tag{9}$$

In the preceding expressions e_Q is assumed to be the charge of the particle (e.g., quark, electron, etc.) rather than the antiparticle, a is the anomalous moment parameter, $\vec{\sigma}_{1,2}$ are the Pauli matrices for particles 1 and 2, respectively, $\vec{r}$ is the difference $\vec{\rho}_1-\vec{\rho}_2$ between the relativistic internal position variables of the two particles,[6] and $\vec{\pi}$ is the corresponding relativistic relative momentum:

$$T_{M1}(t_0)=\frac{c}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2} k(\hat{k}\times\hat{\epsilon}_\alpha)\cdot{}_I\langle A\,|\vec{Y}_0+\vec{Y}_1|\,B\rangle_I\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt' \tag{10}$$

with

$$\vec{Y}_0=\frac{e_Q}{2mc}(\vec{\sigma}_1-\vec{\sigma}_2)(1+a) \tag{11}$$

and

$$\begin{aligned}\vec{Y}_1=&\frac{e_Q}{2mc}\left[\frac{k}{4mc}-\frac{\vec{\pi}^{\,2}}{2m^2c^2}\right](\vec{\sigma}_1-\vec{\sigma}_2)-\frac{e_Qa}{4m^3c^3}[(\vec{\sigma}_1-\vec{\sigma}_2)\cdot\vec{\pi}]\vec{\pi}\\&+\frac{ie_Qk}{16m^2c^2}\vec{r}\times[(\vec{\sigma}_1-\vec{\sigma}_2)\times\vec{\pi}](1+2a)+\frac{e_Qa}{4mc}\frac{k}{mc}(\vec{\sigma}_1-\vec{\sigma}_2)-\frac{e_Qk^2}{40mc}\{\vec{r}^{\,2}(\vec{\sigma}_1-\vec{\sigma}_2)-\tfrac{1}{2}[\vec{r}\cdot(\vec{\sigma}_1-\vec{\sigma}_2)]\vec{r}\}(1+a)\\&-\frac{e_Q}{16m^2c^3}\frac{1}{r}\frac{\partial U^0}{\partial r}\vec{r}\times[(\vec{\sigma}_1-\vec{\sigma}_2)\times\vec{r}]+\frac{ie_Q}{2Mc}\{\vec{r}\times[\vec{W}_S^{(1)},h^{(0)}]+\vec{W}_S^{(1)}\times[\vec{r},h^{(0)}]\}-\frac{e_Q}{2m^2c^3}V_S(\vec{\sigma}_1-\vec{\sigma}_2)\\&+\frac{e_Q}{8m^3c^3}[(\vec{\sigma}_1-\vec{\sigma}_2)\times\vec{\pi}]\times\vec{\pi} .\end{aligned} \tag{12}$$

In Eq. (12) U^0 is the nonrelativistic limit of the internal interaction (i.e., the nonrelativistic potential), M is the sum of the rest masses comprising the composite system, $\vec{W}_S^{(1)}$ is the spin-dependent part of $\vec{W}^{(1)}$ (the interaction-dependent part of the Lorentz boost operator) as defined by Krajcik and Foldy,[15] and $h^{(0)}$ is the nonrelativistic part of the internal Hamiltonian.

In Eq. (12) only the spin-dependent operators have been retained, since only these can connect the initial and final states of interest, namely, states in which the total spin changes by one unit (i.e., singlet-triplet or vice versa). We wish to point out that the last term of Eq. (12) is due to the nonzero recoil momentum of the composite system, and its correct form emerges only through the use of relativistic center-of-mass variables, or by carrying out a Lorentz boost of the final composite wave function, as discussed in Ref. 8. As we will see later, this term makes an important contribution to the decay rate of the "relativistic" $M1$ transition between ψ' and η_c and also between η_c' and ψ.

Equations (6) and (10) give rise to the following decay rates:

$$W_{BA}^{E1}=\tfrac{4}{3}k_0{}^2k\,|\,{}_I\langle A\,|\,\vec{X}_0+\vec{X}_1\,|\,B\rangle_I\,|^2\,, \tag{13}$$

$$W_{BA}^{M1}=\tfrac{4}{3}k^3\,|\,{}_I\langle A\,|\,\vec{Y}_0+\vec{Y}_1\,|\,B\rangle_I\,|^2\,. \tag{14}$$

For $E1$ decays of charmonium the relevant decays are

$$2\,^3S_1\rightarrow 1\,^3P_j+\gamma \text{ and } 1\,^3P_j\rightarrow 1\,^3S_1+\gamma\,, \tag{15}$$

where $j=0$, 1, or 2. When Eq. (13) is applied to the first of these decays the decay rate becomes[18]

$$W^{E1}_{2\,^3S_1\rightarrow 1\,^3P_j}=\Gamma_{\rm NR}(1+r_{ij}+r_{2j}+r_{3j})\,, \tag{16}$$

where $\Gamma_{\rm NR}$ is the nonrelativistic decay rate, r_{1j} is the relativistic correction due to the modification of the nonrelativistic wave function, r_{2j} is the relativistic correction originating from the relativistic modification of the transition operator due to terms involving the anomalous magnetic moment of the quark in the interaction Hamiltonian, and r_{3j} is also a relativistic correction but it originates from the relativistic modification of the transition operator, coming from the difference of $e^{i\vec{k}\cdot\vec{r}}$ and 1 in the plane-wave expansion of the vector potential $\vec{A}$. We find that

$$\Gamma_{\rm NR}=\tfrac{4}{27}k_0{}^2ke_q{}^2(2j+1)G_1{}^2\,, \tag{17a}$$

$$r_{1j}=2\frac{(G_2+G_3^j)}{G_1}+\frac{(G_2+G_3^j)^2}{G_1{}^2}\,, \tag{17b}$$

$$r_{2j}=\frac{ak}{2mc}\eta_j\,, \tag{17c}$$

where

$$\eta_j=2,1,-1 \text{ for } j=0,1,2 \tag{17d}$$

and

$$r_{3j}=-\frac{1}{10}k_0{}^2\frac{G_4}{G_1}+\frac{1}{8}\frac{\omega_0}{mc^2}\frac{(G_5-G_6)}{G_1}\,. \tag{17e}$$

In Eqs. (17), the integrals G_i ($i=1,2,\ldots,6$) are defined by the following equations:

$$G_1=\int_0^\infty R_{1P}^{(0)}R_{2S}^{(0)}r^3dr\,, \tag{18a}$$

$$G_2=\int_0^\infty R_{1P}^{(0)}R_{2S}^{(1)}r^3dr\,, \tag{18b}$$

$$G_3^j=\int_0^\infty R_{1P_j}^{(1)}R_{2S}^{(0)}r^3dr\,, \tag{18c}$$

$$G_4=\int_0^\infty R_{1P}^{(0)}R_{2S}^{(0)}r^5dr\,, \tag{18d}$$

$$G_5=\int_0^\infty R_{1P}^{(0)}R_{2S}^{(0)\prime}r^3dr\,, \tag{18e}$$

$$G_6=\int_0^\infty R_{1P}^{(0)\prime}R_{2S}^{(0)}r^3dr\,, \tag{18f}$$

where $R^{(0)}$ is the radial part of the zeroth order or nonrelativistic wave function (chosen to be real) and

$$R^{(0)\prime}=2r\frac{dR^{(0)}}{dr}+R^{(0)}\,. \tag{19}$$

The quantity $R^{(1)}$ is the first-order relativistic correction to the radial wave function $R^{(0)}$. For example, if an unperturbed $2S$ state has a state vector $\psi_{2S}^{(0)}$ while the corrected state is

$$\psi_{2S}=\sum(a_n\psi_{nS}^{(0)}+b_n\psi_{nD}^{(0)})\,,$$

then the correction to the $2S$ radial wave function is

$$R_{2S}^{(1)}=\sum(a_nR_{nS}^{(0)}+b_nR_{nD}^{(0)}-R_{2S}^{(0)})\,. \tag{20}$$

Similarly for the $1P$ states we find

$$R_{1P_j}^{(1)}=\sum C_n^jR_{nP}^{(0)}-R_{1P}^{(0)},\quad j=0,\ 1,\text{ or } 2\,. \tag{21}$$

The expression for r_{1j} is now given by

$$r_{1j}=\frac{\left|\int_0^\infty R_{1P_j}R_{2S}r^3dr\right|^2-\left|\int_0^\infty R_{1P}^{(0)}R_{2S}^{(0)}r^3dr\right|^2}{\left|\int_0^\infty R_{1P}^{(0)}R_{2S}^{(0)}r^3d^3r\right|^2}\,, \tag{22}$$

where

$$R_{1P_j}=R_{1P}^{(0)}+R_{1P_j}^{(1)} \text{ and } R_{2S}=R_{2S}^{(0)}+R_{2S}^{(1)}\,. \tag{23}$$

Now let us turn to the expressions for the rates of the $E1$ decays $1\,^3P_j\rightarrow 1\,^3S_1+\gamma$. They are given by

$$W^{E1}_{1\,^3P_j\rightarrow 1\,^3S_1}=\Gamma_{\rm NR}(1+r_{1j}+r_{2j}+r_{3j})\,, \tag{24a}$$

where $\Gamma_{\rm NR}$, r_{1j}, r_{2j}, and r_{3j} have the same physical significance as in Eq. (16) but they are now given by the expressions

$$\Gamma_{\rm NR}=\tfrac{4}{9}k_0{}^2ke_q{}^2F_1{}^2\,, \tag{24b}$$

$$r_{1j}=2\frac{(F_2^j+F_3)}{F_1}+\frac{(F_2^j+F_3)^2}{F_1{}^2}\,, \tag{24c}$$

$$r_{2j}=-\frac{ak}{2mc}\eta_j\quad(\eta_j=2,1,-1 \text{ for } j=0,1,2)\,, \tag{24d}$$

and

$$r_{3j}=-\frac{1}{10}k_0{}^2\frac{F_4}{F_1}+\frac{1}{8}\frac{\omega_0}{mc^2}\frac{(F_5-F_6)}{F_1}\,, \tag{24e}$$

where the integral F_i ($i=1,2,\ldots,6$) is obtained by replacing $1P$ by $1S$ and $2S$ by $1P$ in Eqs. (18) for the integrals G_i.

Equations (16)—(18) are applicable to any potential acting between the quark and the antiquark. They can be easily generalized to any $E1$ decays $n\,^3S_1\rightarrow n'\,^3P_j+\gamma$ and $n\,^3P_j\rightarrow n'\,^3S_1+\gamma$ of the $q\bar{q}$ system.

Let us now turn to the magnetic dipole decays of $q\bar{q}$. There are four $M1$ decays of interest in charmonium. We have

(a) $2\,^3S_1\rightarrow 1\,^1S_0+\gamma$ or $\psi'\rightarrow\eta_c+\gamma$,

(b) $2\,^1S_0\rightarrow 1\,^3S_1+\gamma$ or $\eta_c'\rightarrow\psi+\gamma$,

(c) $2\,^3S_1\rightarrow 2\,^1S_0+\gamma$ or $\psi'\rightarrow\eta_c'+\gamma$,

(d) $1\,^3S_1\rightarrow 1\,^1S_0+\gamma$ or $\psi\rightarrow\eta_c+\gamma$.

The first two are "nonrelativistically forbidden" and can occur only due to relativistic effects, because the leading $M1$ operator $\vec{Y}_0$ will have zero matrix elements between the nonrelativistic $n=1$ and $n=2$ states. The last two decays (c) and (d) are allowed even in the nonrelativistic limit.

For any of the above cases we can write the decay rate as

$$W_{fi}^{M1}=\frac{16}{9}\frac{1}{(2S_i+1)}k^3\frac{\alpha}{m^2c}\,|I_1+I_2+I_3+I_4|^2\ , \tag{25}$$

where S_i is the initial spin, α is the fine-structure constant, and

$$I_1=\left\langle\phi_f\left|(1+a)(1-\tfrac{1}{24}k^2r^2)+\frac{k}{4mc}(1+2a)\right|\phi_i\right\rangle\ , \tag{26a}$$

$$I_2=\left\langle\phi_f\left|-\tfrac{1}{2}(1+a)\frac{\pi^2}{m^2c^2}-\frac{1}{3}\frac{\pi^2}{m^2c^2}\right|\phi_i\right\rangle\ , \tag{26b}$$

$$I_3=\left\langle\phi_f\left|\frac{a}{6}\frac{r}{mc}\frac{\partial U^{(0)}}{\partial r}\right|\phi_i\right\rangle\ , \tag{26c}$$

$$I_4=\left\langle\phi_f\left|-\frac{V_S}{mc^2}\right|\phi_i\right\rangle\ . \tag{26d}$$

In Eqs. (26) $|\phi_i\rangle$ and $|\phi_f\rangle$ are the spatial parts of the initial and the final-state wave functions of quarkonium. Since we are calculating the matrix elements correct only to order v^2/c^2 we can use nonrelativistic wave functions everywhere except when we calculate the matrix element of $1+a$ between $|\phi_f\rangle$ and $|\phi_i\rangle$ in the expression for I_1. As Sucher[4] has previously demonstrated, the matrix element of the unity operator in Eq. (26a) can be written as

$$\langle\phi_f|\phi_i\rangle=\frac{1}{(E_i-E_f)}\langle\phi_f|U_{\rm eff}^{SS}|\phi_i\rangle\ , \tag{27}$$

where $U_{\rm eff}^{SS}$ is the coefficient of $\vec{S}_1\cdot\vec{S}_2$ in the interaction terms of the internal Hamiltonian $h^{(1)}$. Since $U_{\rm eff}^{SS}$ is already of order v^2/c^2 in most models, on the right-hand side of Eq. (27) we can take $|\phi_i\rangle$ and $|\phi_f\rangle$ to be the nonrelativistic spatial wave functions. If the spin-spin interaction is due *only* to the one-gluon exchange between the quark and the antiquark, then

$$U_{\rm eff}^{SS}=-\frac{8}{3}\frac{K\pi}{m^2c^2}\delta^{(3)}(\vec{r})\ . \tag{28}$$

In our case the operator has a different form dictated by the reduction. It is important to point out that a contribution of $-\pi^2/6m^2c^2$ to Eq. (26b) came from the last term of Eq. (12). This originated from the simultaneous presence of the recoil of quarkonium and of the v^2/c^2 terms in the relativistic relation between the constituent and the center-of-mass variables.[8] Alternatively, as discussed in Ref. 8, we can think of it as arising from the Lorentz-boosted wave function of the recoiling composite system in the final state. The presence of recoil corrections to magnetic-moment operators of composite systems was also known and studied many years ago in connection with g factors of composite systems.[19]

In Secs. IV and V we will turn to the numerical evaluation of the $M1$ and the $E1$ decay rates of the $c\bar{c}$ and the $b\bar{b}$ systems based on the Buchmüller-Tye potential.[12] In Sec. III we will present the results on the energy spectrum of charmonium ($c\bar{c}$) and b-quarkonium ($b\bar{b}$) based on this potential.

III. NUMERICAL RESULTS ON THE ENERGY SPECTRUM OF CHARMONIUM AND b-QUARKONIUM BASED ON THE BUCHMÜLLER-TYE POTENTIAL

We shall assume as a working model that at the relativistic level the quark-quark interaction is simulated by the exchange of vector and scalar particles. Then the Hamiltonian at the relativistic level can be written as

$$H=(c\vec{\alpha}_1\cdot\vec{p}_1+\beta_1m_1c^2)+(c\vec{\alpha}_2\cdot\vec{p}_2+\beta_2m_2c^2)+\beta_1\beta_2V_S+(1-\tfrac{1}{2}\vec{\alpha}_1\cdot\vec{\alpha}_2)V_V+\tfrac{1}{2}\vec{\alpha}_1\cdot\hat{r}\,\vec{\alpha}_2\cdot\hat{r}\,rV_V'\ , \tag{29}$$

where V_S and V_V are the scalar and the vector potentials. Making a Barker-Glover reduction[8,20] to order v^2/c^2 and writing the Hamiltonian in the c.m. frame we get the internal Hamiltonian

$$h^{(0)}=\frac{\pi^2}{m}+V_S+V_V\ , \tag{30a}$$

$$\begin{aligned}h^{(1)}=&-\frac{\pi^4}{4m^3c^2}+\frac{1}{2m^2c^2}\vec{\pi}\cdot V_V\vec{\pi}-\frac{1}{2m^2c^2}\vec{\pi}\cdot\hat{r}rV_V'\hat{r}\cdot\vec{\pi}+\frac{i}{8m^2c^2}[\vec{\pi}\cdot,\vec{r}\nabla^2V_V]+\frac{1}{2m^2c^2}\frac{1}{r}(3V_V'-V_S')\vec{S}\cdot\vec{L}\\&+\frac{1}{4m^2c^2}\nabla^2(V_V+V_S)-\frac{1}{m^2c^2}\vec{\pi}\cdot V_S\vec{\pi}+\frac{1}{6m^2c^2}(\vec{\sigma}_1\cdot\vec{\sigma}_2)\nabla^2V_V+\frac{1}{12m^2c^2}[\vec{\sigma}_1\cdot\vec{\sigma}_2-3(\vec{\sigma}_1\cdot\hat{r})(\vec{\sigma}_2\cdot\hat{r})]\left[V_V''-\frac{1}{r}V_V'\right]\ .\end{aligned} \tag{30b}$$

For V_V+V_S we will take the Buchmüller-Tye potential[12]

$$\begin{aligned}V_{\rm BT}=V_V+V_S&=kr-\frac{8}{27}\frac{v(\lambda r)}{r}\quad\text{for } r\geq 0.01\text{ fm}\\&=-\frac{16\pi}{25r}\frac{1}{\ln(\Lambda_{\overline{\rm MS}}{}^2r^2)^{-1}}\left[1+(2\gamma_E+\tfrac{53}{75})\frac{1}{\ln(\Lambda_{\overline{\rm MS}}{}^2r^2)^{-1}}-\tfrac{462}{635}\ln\ln(\Lambda_{\overline{\rm MS}}{}^2r^2)^{-1}/\ln(\Lambda_{\overline{\rm MS}}{}^2r^2)^{-1}\right]\quad\text{for } r<0.01\text{ fm}\ ,\end{aligned} \tag{31}$$

TABLE I. Energy spectrum of charmonium ($\eta_S=1$) in GeV. $m_c=1.548$ GeV, $m_{c_0}=1.486$ GeV.

State	$E_{\rm NR}$	ΔE	δK	$M_{\rm th}$	$M_{\rm expt}$[a]
$2\,^3S_1$ (ψ')	0.7272	−0.2424	−0.018	3.563	3.686 ±0.0001
$2\,^1S_0$ (η_c')	0.7272	−0.3137	−0.018	3.491	3.592 ±0.005
$1\,^3S_1$ (ψ)	0.1275	−0.1120	−0.015	3.097	3.0969±0.0001
$1\,^1S_0$ (η_c)	0.1275	−0.2329	−0.015	2.976	2.981 ±0.006
$1\,^3P_2$ (χ_2^c)	0.5504	−0.1719	−0.016	3.459	3.5558±0.0006
$1\,^3P_1$ (χ_1^c)	0.5504	−0.2135	−0.016	3.417	3.5100±0.0006
$1\,^3P_0$ (χ_0^c)	0.5504	−0.2777	−0.016	3.353	3.415 ±0.001
$1\,^1P_1$	0.5504	−0.2037	−0.016	3.427	

[a]See Ref. 10 and references given in that paper.

where

$$\Lambda_{\overline{\rm MS}}=0.509 \text{ GeV}, \quad k=0.153 \text{ GeV}^2 ,$$

$$\lambda=0.406 \text{ GeV}, \quad \gamma_E=0.5772$$

($\overline{\rm MS}$ denotes the modified minimal-subtraction scheme). The function $v(x)$ has been tabulated by Buchmüller and Tye.[12] Except for the confining part $V_c=kr$, the Buchmüller-Tye potential is based on perturbative quantum chromodynamics. The piece that is derivable from perturbative quantum chromodynamics (based on one-gluon-exchange diagrams and their corrections) is vector and the rest of the potential is assumed to be partly vector and partly scalar, i.e.,

$$V_S=\eta_S kr \text{ and } V_V=V_{\rm BT}-\eta_S kr \ . \tag{32}$$

The eigenvalues and eigenfunctions of the nonrelativistic internal Hamiltonian $h^{(0)}$ were solved numerically by the Runge-Kutta method. The nonrelativistic wave functions were evaluated for about 2000 points equally spaced between zero and 2 fm. The corrections to the nonrelativistic energies due to $h^{(1)}$ were calculated in first-order perturbation theory.

The results for the energy spectrum of charmonium and b-quarkonium for the two special cases (a) $\eta_S=1$ (confining potential is purely scalar) and (b) $\eta_S=0$ (confining potential and hence the whole potential is purely vector) are given in Tables I—IV.

The nonrelativistic energies are obtained for the input mass of the charmed quark, $m_c=1.486$ GeV. When the relativistic corrections are added to these nonrelativistic energies, for case (a) above the predicted energies were found to be consistently low. This can be remedied by increasing quark mass to 1.548 to refit the energy of the ψ state. To the desired degree of accuracy, the effect of this mass shift on the rest of the terms making up the energy levels can be obtained perturbatively from the correction to the kinetic energy term due to the increase of the quark mass. In the table on charmonium energy levels this correction to the kinetic energy due to the change of quark mass is called δK. The predicted value for the various masses will then be given by

$$M_{\rm th}=2mc^2+E_{\rm NR}+\Delta E+\delta K \ . \tag{33}$$

Since the relativistic corrections to the $c\bar{c}$ levels are large, higher-order perturbation theory could alter these corrections by as much as 20%. In view of this uncertainty we have not fine-tuned the results to the best possible value of η_S. However our work suggests that a value of η_S near unity gives significantly better overall agreement with experiment than does a value close to zero.[21] For η_S of unity our hyperfine splittings are 0.072 and 0.120 GeV as compared with 0.094 and 0.116 GeV for the $2S$ and $1S$ states, respectively. For the energy separations between the ψ' and the $\chi_{2,1,0}$ states, states of interest for the $E1$ transitions later to be discussed, the energies we obtain are 0.088, 0.130, and 0.194 compared to experimental values of 0.130, 0.176, and 0.271. Although our separations are too small for these states we feel that higher-order relativistic effects could account for this. The fine-structure splittings for $\eta_S=1$ appear to be closer to experiment than for $\eta_S=0$.

Turning now to $b\bar{b}$, we examine the fine-structure splittings of the $1P$ and $2P$ states and find the results to be in

TABLE II. Energy spectrum of charmonium ($\eta_S=0$) in GeV. $m_c=1.478$ GeV, $m_{c_0}=1.486$ GeV.

State	$E_{\rm NR}$	ΔE	δK	$M_{\rm th}$	$M_{\rm expt}$[a]
$2\,^3S_1$ (ψ')	0.7272	−0.097	0.002	3.587	3.686 ±0.0001
$2\,^1S_0$ (η_c')	0.7272	−0.206	0.002	3.478	3.592 ±0.005
$1\,^3S_1$ (ψ)	0.1275	0.0125	0.002	3.097	3.0969±0.0001
$1\,^1S_0$ (η_c)	0.1275	−0.184	0.002	2.901	2.981 ±0.006
$1\,^3P_2$ (χ_2^c)	0.5504	−0.124	0.002	3.383	3.5558±0.0006
$1\,^3P_1$ (χ_1^c)	0.5504	−0.198	0.002	3.309	3.5100±0.0006
$1\,^3P_0$ (χ_0^c)	0.5504	−0.283	0.002	3.224	3.415 ±0.001
$1\,^1P_1$	0.5504	−0.183	0.002	3.324	

[a]See Ref. 10 and references cited in that paper.

TABLE III. Energy spectrum of b-quarkonium ($\eta_S=1$) in GeV. $m_b=4.88$ GeV.

State	E_{NR}	ΔE	M_{th}	M_{expt} (Ref. 22)
$2\,^3S_1$ (Υ')	0.2581	−0.058	9.960	10.0234±0.0007
$2\,^1S_0$ (η_b')	0.2581	−0.089	9.929	
$1\,^3S_1$ (Υ)	−0.2962	−0.031	9.433	9.4600±0.0001
$1\,^1S_0$ (η_b)	−0.2962	−0.098	9.366	
$1\,^3P_2$ (χ_2^b)	0.1319	−0.036	9.856	9.915 ±0.002
$1\,^3P_1$ (χ_1^b)	0.1319	−0.058	9.834	9.894 ±0.003
$1\,^3P_0$ (χ_0^b)	0.1319	−0.086	9.805	9.873 ±0.005
$1\,^1P_1$	0.1319	−0.052	9.840	
$3\,^3S_1$ (Υ'')	0.5940	−0.077	10.302	10.3500±0.0007
$3\,^1S_0$ (η_b'')	0.5940	−0.100	10.305	
$2\,^3P_2$ ($\chi_2^{b'}$)	0.4920	−0.058	10.194	10.266 ±0.002
$2\,^3P_1$ ($\chi_1^{b'}$)	0.4920	−0.074	10.178	10.248 ±0.003
$2\,^3P_0$ ($\chi_0^{b'}$)	0.4920	−0.095	10.157	10.228 ±0.005
$2\,^1P_1$	0.4920	−0.069	10.183	

significantly better agreement with experiment for $\eta_S=1$ than for $\eta_S=0$. With $\eta_S=1$, the $1P$ states $1P_2$, $1P_1$, and $1P_0$ are, respectively, 13.0, −9.0, and −38 MeV from the center of gravity (c.o.g.) while for the $2P$ states we find $2P_2$, $2P_1$, and $2P_0$ to be 9.0, −7.0, and −28 MeV from the c.o.g. On the other hand with $\eta_s=0$ for the $1P$ states we find 21, −17, and −55 while for the $2P$ states we obtain 15, −12, and −40. The experimental numbers are 11.7, −9.3, and −30.3 for $1P$ and 10.2, −7.8, and −27.8 for $2P$. While the $\eta_S=1$ numbers are generally in good agreement with experiment, the $\eta_S=0$ results are quite poor. Since fine-structure splittings are very sensitive to the value of η_S these splittings support an η_S near unity.

Silverman recently reported an average R value,

$$R=(E_1-E_0)/(E_2-E_1)\ ,$$

for the $1P$ states of 1.30±0.20.[23] Our theoretical result yields 1.32 for $\eta_S=1$ and 1.00 for $\eta_S=0$. Thus we see that our $\eta_S=1$ result is in excellent agreement with experiment.

IV. NUMERICAL RESULTS ON THE $M1$ DECAY RATES OF QUARKONIA

We have calculated the $M1$ decay rates of quarkonia using Eqs. (25)—(27). In order to evaluate the matrix elements I_1, I_2, I_3, and I_4 we need to know only the nonrelativistic wave functions ϕ_i and ϕ_f for initial and final states, respectively. For the leading operator, $1+a$, we make use of Sucher's result,[4] given in our Eq. (27), while for the other terms, which are already of order $1/c^2$, nonrelativistic wave functions can be used. With our expression for $h^{(1)}$ given by Eq. (30b), U_{eff}^{SS} of Eq.(28) will be given by

$$U_{\text{eff}}^{SS}=\frac{2}{3m^2c^2}\nabla^2 V_V\ . \tag{34}$$

Since this is of order $1/c^2$, nonrelativistic wave functions may be used.

It is clear that I_1 and I_4 (26a) and (26d) can depend in an important way on the parameter η_S determining the scalar admixture in the potential. We find that the $M1$ widths in keV for $c\bar{c}$ decays are as follows:

TABLE IV. Energy spectrum of b-quarkonium ($\eta_S=0$) in GeV. $m_b=4.88$ GeV.

State	E_{NR}	ΔE	M_{th}	M_{expt} (Ref. 22)
$2\,^3S_1$ (Υ')	0.2581	−0.037	9.981	10.0234±0.0007
$2\,^1S_0$ (η_b')	0.2581	−0.074	9.944	
$1\,^3S_1$ (Υ)	−0.2962	−0.010	9.454	9.4600±0.0001
$1\,^1S_0$ (η_b)	−0.2962	−0.088	9.376	
$1\,^3P_2$ (χ_2^b)	0.1319	−0.011	9.889	9.915 ±0.002
$1\,^3P_1$ (χ_1^b)	0.1319	−0.040	9.851	9.894 ±0.003
$1\,^3P_0$ (χ_0^b)	0.1319	−0.079	9.813	9.873 ±0.005
$1\,^1P_1$	0.1319	−0.032	9.860	
$3\,^3S_1$ (Υ'')	0.5940	−0.048	10.306	10.3500±0.0007
$3\,^1S_0$ (η_b'')	0.5940	−0.075	10.279	
$2\,^3P_2$ ($\chi_2^{b'}$)	0.4920	−0.022	10.230	10.266 ±0.002
$2\,^3P_1$ ($\chi_1^{b'}$)	0.4920	−0.049	10.203	10.248 ±0.003
$2\,^3P_0$ ($\chi_0^{b'}$)	0.4920	−0.077	10.175	10.228 ±0.005
$2\,^1P_1$	0.4920	−0.043	10.209	

TABLE V. $M1$ decay rates of charmonium.

Decay (k in GeV)	$\eta_S \backslash a$	I_1 0	I_1 -0.25	I_1 -1.00	I_2 0	I_2 -0.25	I_2 -1.00	I_3 0	I_3 -0.25	I_3 -1.00	I_4	0	-0.25	-1.00	Experimental rate (keV) (Ref. 24)
$\psi' \to \eta_c + \gamma$	1	0.202	0.151	0	-0.156	-0.133	-0.060	0	0.0009	0.0036	0.087	8.40	5.33	0.44	0.6±0.2
$k = 0.622$	0	0.235	0.176	0							0	2.96	0.92	1.51	
$\eta_c' \to \psi + \gamma$	1	0.224	0.168	0	-0.156	-0.133	-0.060	0	0.0009	0.0036	0.087	13.4	8.43	0.52	Unknown
$k = 0.455$	0	0.271	0.203	0							0	7.38	2.80	1.77	
$\psi' \to \eta_c' + \gamma$	1	1.01	0.753	-0.016	-0.243	-0.207	-0.097	0	-0.024	-0.097	-0.409	0.20	0.02	0.59	0.4−2.8
$k = 0.092$	0										0	0.91	0.42	0.07	
$\psi \to \eta_c + \gamma$	1	1.02	0.758	-0.019	-0.196	-0.166	-0.078	0	-0.020	-0.079	-0.200	1.14	0.41	0.41	0.5−1
$k = 0.114$	0										0	1.99	0.96	0.91	

$$W^{M1}(\psi' \to \eta_c + \gamma) = 475(0.079 + 0.137a + 0.054\eta_S - 0.033a\eta_S)^2 , \quad (35a)$$

$$W^{M1}(\eta_c' \to \psi + \gamma) = 558(0.115 + 0.173a + 0.040\eta_S - 0.047a\eta_S)^2 , \quad (35b)$$

$$W^{M1}(\psi' \to \eta_c' + \gamma) = 1.54(0.766 + 0.975a - 0.409\eta_S)^2 , \quad (35c)$$

$$W^{M1}(\psi \to \eta_c + \gamma) = 2.93(0.821 + 0.997a - 0.200\eta_S)^2 . \quad (35d)$$

In Table V we present results for $\eta_S = 1$ and $\eta_S = 0$ with three possible values of the anomalous moment, namely, $a = 0$, $a = -0.25$, and $a = -1.00$.

For the $b\bar{b}$ states none of the rates have been measured, but we have calculated the $M1$ decays for the transitions which correspond to the $c\bar{c}$ transitions, namely, $\Upsilon' \to \eta_b + \gamma$, $\eta_b' \to \Upsilon + \gamma$, $\Upsilon' \to \eta_b' + \gamma$, and $\Upsilon \to \eta_b + \gamma$. These results are presented in Table VI. We have not presented the individual contributions I_i as in Table V.

The comparison of the theoretical and experimental rates indicates a substantial discrepancy if the quarks have no anomalous moment. This disagreement occurs in the $\psi' \to \eta_c + \gamma$ decay. It is clearly reduced by assuming a large negative anomalous moment for charmed quarks. The choice of $a = -1.00$ and $\eta_S = 1$ gives quite good agreement for the $M1$ decays, but such a large value does not appear to be reasonable. Even with an anomalous moment of $a \cong -0.25$ with η_S still 1 our result for this decay still disagrees substantially with experiment. On the other hand for $\eta_S = 0$ and $a = -0.25$ satisfactory agreement is found for this decay. All of this suggests that the $M1$ decay rates are very model-dependent and that factors of two or three can occur in the decay rates if the confining potential is changed from pure scalar to pure vector.

Our results for $M1$ decay rates may be compared to those of Zambetakis and Byers (ZB)[11] if a is chosen to be zero and the confining potential is purely scalar. For the so-called "hindered" transition $\psi' \to \eta_c + \gamma$, without coupled channel mixing the authors of Ref. 11 obtain $\sum I = 0.12$ whereas we obtain $\sum I = 0.13$.[25] It should be noted that our I_1 includes both I_1 and I_3 of Ref. 11 while

TABLE VI. Theoretical $M1$ decay rates of b-quarkonium.

Decay and photon energy in GeV	$\eta_S \backslash a$	Predicted rate in eV 0	-0.25	-1.00
$\Upsilon' \to \eta_b + \gamma$	1	33.6	17.5	0.452
0.594	0	23.1	9.55	4.28
$\eta_b' \to \Upsilon + \gamma$	1	70.8	37.1	0.751
0.486	0	57.4	38.0	7.03
$\Upsilon' \to \eta_b' + \gamma$	1	1.72	0.877	0.035
0.037	0	2.03	1.10	0.006
$\Upsilon \to \eta_b + \gamma$	1	17.7	9.38	0.150
0.078	0	19.0	10.3	0.056

TABLE VII. Comparison of $M1$ decays.

	I_1[a]	I_2	I_4	$\sum I$	Experiment (Refs. 24 and 11)
$\psi' \to \eta_c + \gamma$					
This work	0.20	−0.16	0.09	0.13	0.047±0.008
ZB	0.18	−0.14	0.07	0.12 → 0.032 with coupled-channel mixing	
$\psi \to \eta_c + \gamma$					
This work	1.02	−0.20	−0.20	0.62	0.7 ±0.2
ZB	1.00	−0.17	0.25	1.08	
$\psi' \to \eta_c' + \gamma$					
This work	1.01	−0.24	−0.41	0.36	1.1−1.7
ZB	1.00	−0.20	0.07	0.87	

[a]Our value of I_1 above is equivalent to $-I_1 - I_3$ of ZB while I_2 and I_4 are equivalent to $-I_2$ and $-I_4$.

our I_3 is a different contribution, which is proportional to a, and hence vanished when there is no anomalous moment. Also our I's have opposite signs to that of ZB. We see from Table VII that without the coupled-channel mixing used by ZB our results essentially agree with those in Ref. 11, but both are much larger than experiment. Although this problem could be cured with a sufficiently large anomalous moment, we are somewhat skeptical and place more credibility in a resolution based on coupled channel mixing. In the present work we have not attempted to include such effects.

For the other decays in which a comparison is possible, namely, $\psi \to \eta_c + \gamma$ and $\psi' \to \eta_c' + \gamma$ we find that for the former we have $\sum I = 0.62$ as compared to 1.08 in Ref. 11 while for the latter we have 0.36 compared to 0.87. Of these two decays, for the first Table VII shows our result to be closer to experiment than that of Ref. 11, while for the second the reverse is true.

In Table VII a comparison of various terms is provided. For the $\psi \to \eta_c + \gamma$ and $\psi' \to \eta_c' + \gamma$ the main difference occurs for I_4. It is a model-dependent difference which presumably arises because ZB have a constant term in the scalar potential whereas we do not.

V. NUMERICAL RESULTS IN THE $E1$ DECAY RATES OF QUARKONIA AND COMPARISON WITH OTHER WORK

We have calculated the $E1$ decay rates of the $c\bar{c}$ and $b\bar{b}$ states by making use of Eqs. (16)–(24). The integrals G_i and F_i, with i running from one to six were calculated numerically. Since the correction to the decay rate due to the relativistic modification of the wave functions is quite large for some of the $E1$ decays these cases should be handled with care. For example, in Eqs. (17b) and (24c) for r_{1j}, which give the dominant corrections to the rates due to the relativistic modification of the wave functions, the first term on the right-hand side is of order v^2/c^2 whereas the second term is of order v^4/c^4. In cases where this v^4/c^4 term is large we include it since we believe that in such cases it is the most important v^4/c^4 term.[18]

For the states of interest in $c\bar{c}$ we find the following corrections to the nonrelativistic radial wave functions:

$$\begin{aligned} &R_{2S} \simeq R_{2S}^{(0)} + a_{1S} R_{1S}^{(0)} + a_{1D} R_{1D}^{(0)} , \\ &R_{1S} \simeq R_{1S}^{(0)} + a_{2S} R_{2S}^{(0)} + a'_{1D} R_{1D}^{(0)} , \\ &R_{1\,{}^3P_j} \simeq R_{1P}^{(0)} + a_{2P_j} R_{2P}^{(0)} \quad (j = 0,1,2) . \end{aligned} \tag{36}$$

The coefficients a_{1S}, a_{1D}, a_{2S}, a'_{1D}, and a_{2P_j} are calculated in first-order perturbation theory. We find

$$\begin{aligned} &a_{1S} \simeq -0.187 + 0.097(1-\eta_S) , \\ &a_{1D} \simeq -0.0052 + 0.0024(1-\eta_S) , \\ &a_{2S} = -a_{1S} , \\ &a'_{1D} \simeq -0.0153 - 0.005(1-\eta_S) , \\ &a_{2P_2} \simeq = 0.243 - 0.163(1-\eta_S) , \\ &a_{2P_1} \simeq = 0.341 - 0.096(1-\eta_S) , \\ &a_{2P_0} \simeq = 0.461 - 0.052(1-\eta_S) . \end{aligned} \tag{37}$$

It should be noted that other intermediate state corrections to Eq. (36) can be neglected for two reasons: (1) the coefficients of the other intermediate states will be smaller because of larger energy denominators and more importantly (2) when the radial quantum numbers n differ by more than one unit we generally find that the overlap radial integral of the relevant operator r is negligibly small.

In Table VIII we present theoretical decay rates for the $E1$ transitions of $\psi' \to \chi_j + \gamma$ for η_S equal to one and zero for various values of a presumed quark anomalous moment a. In Table IX corresponding results are given for $\chi_j \to \psi + \gamma$. From these tables we find that for the decays $\psi' \to \chi_j + \gamma$ and $\chi_j \to \psi + \gamma$, the overall agreement of the

TABLE VIII. Numerical results of the decay rates for $\psi' \to \chi_j + \gamma$ as given by Eq. (16) for $\eta_S = 1$ and 0 and $a = 0$, -0.25, and -1.00.

Decay (k in GeV)	Nonrelativistic rate Γ_{NR} (keV)	$\eta_S \backslash a$	Correction from relativistic modification of wave function r_{1j}	Correction from anomalous moment r_{2j} 0	−0.25	−1.00	Finite-size correction r_{3j}	Predicted rate $\Gamma = \Gamma_{NR} \times (1 + r_{1j} + r_{2j} + r_{3j})$ (keV) 0	−0.25	−1.00	Experimental rate Γ_{expt} (keV) (Ref. 26)
$\psi' \to \chi_2 + \gamma$	35.7	1	−0.291	0	0.007	0.029	0.020	26.0	26.3	27.1	17±5
$k = 0.128$		0	0.108	0	0.007	0.029		40.3	40.5	41.3	
$\psi' \to \chi_1 + \gamma$	54.5	1	−0.489	0	−0.007	−0.027	0.009	28.3	28.0	26.9	19±5
$k = 0.173$		0	−0.324	0	−0.009	−0.036		37.3	36.8	35.4	
$\psi' \to \chi_0 + \gamma$	64.7	1	−0.686	0	−0.012	−0.047	−0.043	17.5	16.8	14.5	21±6
$k = 0.262$		0	−0.606	0	−0.013	−0.052		22.7	21.9	19.3	

predicted results with experiment is definitely better for the $\eta_S = 1$ case than for the $\eta_S = 0$ case. We have also worked out the decay rates for the $E1$ decays of the $b\bar{b}$ system. In Table X we provide the results of our calculations for the $b\bar{b}$ system for $a = 0$. For the $E1$ decays of b-quarkonium, the experimental data are rather sparse. In general the theoretical predictions for the $E1$ decay rates of the $b\bar{b}$ system are not strongly dependent on either η_S or the anomalous moment. But there is one important exception. This happens for the interesting decays $\Upsilon'' \to \chi_j^b + \gamma$. These decays are strongly inhibited in a nonrelativistic calculation because of the approximate vanishing of the radial integral of the position operator between $3S$ and $1P$ states. But in a first-order relativistic calculation, the decay rates are greatly enhanced by first-order relativistic effects. Moxhay and Rosner[10] have noted that there is some evidence that these decays have been seen experimentally which would indicate that their actual rates are much larger than the nonrelativistic prediction, which is too small to be observed.

Our results for $c\bar{c}$ may readily be compared with the work of McClary and Byers (MB)[9] and Moxhay and Rosner (MR).[10] In Table XI we present relativistically corrected decay rates for $c\bar{c}$ $E1$ decays. Here we find that our corrected rates more nearly agree with those of MB than those of MR. We find very substantial relativistic corrections, which are large enough to suggest that second-order corrections may be significant.

Since our decay rates and those of MB are closer to experiment than those of MR, this suggests that the $E1$ rates are quite sensitive to the treatment of the relativistic corrections, as well as to the nature of the potential. In both the present work (with η_S equal to 1) and that of MB it is presumed that the confining potential is scalar while

TABLE IX. Numerical results of the decay rates for $\chi_j \to \psi + \gamma$ as given by Eq. (24a) for $\eta_S = 1$ and 0 and $a = 0$, -0.25, and -1.00.

Decay (k in GeV)	Nonrelativistic rate Γ_{NR} (keV)	$\eta_S \backslash a$	Correction from relativistic modification of wave function r_{1j}	Correction from anomalous moment r_{2j} 0	−0.25	−1.00	Finite-size correction r_{3j}	Predicted rate $\Gamma = \Gamma_{NR} \times (1 + r_{1j} + r_{2j} + r_{3j})$ (keV) 0	−0.25	−1.00	Experimental rate Γ_{expt} (keV) (Ref. 26)
$\chi_2 \to \psi + \gamma$	579	1	−0.187	0	−0.024	−0.096	−0.100	413	399	357	490±330
$k = 0.425$		0	−0.099	0	−0.031	−0.126		464	446	391	
$\chi_1 \to \psi + \gamma$	425	1	−0.175	0	0.022	0.089	−0.071	340	350	378	<700
$k = 0.385$		0	−0.076	0	0.030	0.122		363	375	414	
$\chi_0 \to \psi + \gamma$	198	1	−0.157	0	0.037	0.149	−0.025	162	176	198	97±38
$k = 0.302$		0	−0.051	0	0.047	0.189		183	192	220	

TABLE X. Numerical results of the $E1$ decay rates in $b\bar{b}$.

Decay	Photon energy (GeV)	Nonrelativistic rate Γ_{NR} (in keV)	Predicted rate Γ in keV $\eta_S=1$	$\eta_S=0$	Experiment
$\Upsilon'\to\chi_2^b+\gamma$	0.107	2.01	1.84	1.93	2.75 ± 1.05[b]
$\Upsilon'\to\chi_1^b+\gamma$	0.127	2.01	1.62	1.65	2.16 ± 0.89[b]
$\Upsilon'\to\chi_0^b+\gamma$	0.148	1.06	0.73	0.72	1.19 ± 0.86[b]
$\chi_2^b\to\Upsilon+\gamma$	0.441	41.0	33.0	34.5	
$\chi_1^b\to\Upsilon+\gamma$	0.422	35.9	29.8	31.2	
$\chi_0^b\to\Upsilon+\gamma$	0.402	30.0	25.7	27.0	
$\Upsilon''\to\chi_2^{b'}+\gamma$	0.0838	2.49	2.29	2.44	total exptl. rate[a] $6.5\ \pm1.4$
$\Upsilon''\to\chi_1^{b'}+\gamma$	0.101	2.63	2.15	2.24	
$\Upsilon''\to\chi_0^{b'}+\gamma$	0.122	1.54	1.09	1.07	
$\chi_2^{b'}\to\Upsilon+\gamma$	0.772	9.98	18.2	18.9	
$\chi_1^{b'}\to\Upsilon+\gamma$	0.755	9.31	11.8	11.0	
$\chi_0^{b'}\to\Upsilon+\gamma$	0.737	8.66	6.50	5.33	
$\chi_2^{b'}\to\Upsilon'+\gamma$	0.243	19.8	12.9	13.1	
$\chi_1^{b'}\to\Upsilon'+\gamma$	0.225	15.7	11.9	12.5	
$\chi_0^{b'}\to\Upsilon'+\gamma$	0.206	12.0	10.6	11.4	
$\Upsilon''\to\chi_2^b+\gamma$	0.428	0.0055	0.194	0.430	
$\Upsilon''\to\chi_1^b+\gamma$	0.447	0.0038	0.0034	0.0003	
$\Upsilon''\to\chi_0^b+\gamma$	0.467	0.0014	0.114	0.130	

[a]See Han *et al.* (Ref. 27).
[b]Several reports of these decays have been given (Ref. 28). We have used the branching ratios given by the CLEO collaboration, along with a total width of 27 keV to obtain these partial rates, with errors due only to errors in the branching ratios.

in the work of MR confinement occurs due to the longitudinal color electric field. Thus while we use $\beta_1\beta_2V_C$ in the "covariant" Hamiltonian they use just V_C.

In Table XII we compare results for the decays $\Upsilon''\to\chi_j^b+\gamma$ with Moxhay and Rosner. As mentioned earlier this decay from $3S$ to $1P$ is quite interesting because nonrelativistically it is extremely small. When wave-function corrections are included, enormous enhancements occur. Moxhay and Rosner also find very large enhancements, but their results differ from ours. This is not surprising in view of the fact that these decays come almost entirely from relativistic effects and our treatment of such effects differs from theirs in the assumptions concerning the nature of the confining potential as well as in the nonrelativistic potentials used.

The bulk of the relativistic correction in most decays come from the relativistic modification of the wave function. The corrections to the rates coming from the anomalous moments (r_{2j}) of the quarks are negligible unless the anomalous moment is exceptionally large as in the case of $a=-1$. The finite-size correction r_{3j} as given by Eq. (17b) is also small in all cases except when the energy of the emitted photon is very high as in the decays $\chi_j^{b'}\to\Upsilon+\gamma$ and $\Upsilon''\to\chi_j^b+\gamma$. For these exceptional decays we have written the finite-size correction as

$$r'_{3j}=r_{3j}\left[1+b_j+\frac{r_{3j}}{4}\right], \tag{38}$$

where

$$b_j=\frac{(F_2^j+F_3)}{F_1}\quad\text{for } \chi_j^{b'}\to\Upsilon+\gamma \text{ decays}$$

and (39)

$$b_j=\frac{(G_2+G_3^j)}{G_1}\quad\text{for } \Upsilon''\to\chi_j^b+\gamma \text{ decays}\,.$$

The terms involving b_j and $r_{3j}/4$ in Eq. (38) are actually of order v^4/c^4. But since the wave-function corrections and the finite-size corrections are huge in these cases it is justifiable to keep these terms in our formulas, for the

TABLE XI. Comparison of $E1$ decay rates in keV for $c\bar{c}$.

	MB[a]	MR	This work ($\eta_S=1$)	Expt. (Ref. 26)
$\psi'\to\chi_2+\gamma$	27 (22)	41	26	17 ± 5
$\psi'\to\chi_1+\gamma$	31 (23)	48	28	19 ± 5
$\psi'\to\chi_0+\gamma$	19 (16)	37	18	21 ± 6
$\chi_2\to\psi+\gamma$	347 (305)	609	413	490 ± 330
$\chi_1\to\psi+\gamma$	270 (240)	460	340	<700
$\chi_0\to\psi+\gamma$	128 (117)	225	162	97 ± 38

[a]Numbers in parentheses give results when coupled-channel mixing is used.

TABLE XII. Comparison of Γ_{E1} in keV for $\Upsilon''\to\chi_j^b+\gamma$.

	$J=2$	$J=1$	$J=0$
Predicted (Moxhay-Rosner)	0.15	0.025	0.025
Predicted (nonrelativistic rate)	5.5×10^{-3}	3.8×10^{-3}	1.4×10^{-3}
Predicted ($\eta_S=0$)	0.43	0.0003	0.130
Predicted ($\eta_S=1$)	0.19	0.003	0.114

same reasons we kept the quadratic term in the formula for r_{1j}. The justification is that even though we cannot calculate all corrections of order v^4/c^4 to the rates, we are including the most important ones in these exceptional situations. Since the quantity b_j depends on η_S, we see that r'_{3j} has different numerical values for the $\eta_S=1$ case and the $\eta_S=0$ case. We would like to point out that the correction r'_{3j} is quite large for the decays $\chi_j^{b'}\to\Upsilon+\gamma$ and $\Upsilon''\to\chi_j^{b}+\gamma$. In the case of $\chi_j^{b'}\to\Upsilon+\gamma$, the finite-size correction even dominates over the correction due to the relativistic modification of the wave function. In the case of the decays $\chi_j^{c}\to\psi+\gamma$ and $\chi_j^{b'}\to\Upsilon+\gamma$, this correction, even though much smaller, is yet significant.

VI. CONCLUSIONS

In conclusion, we find that when the Buchmüller-Tye potential is used in an approximately relativistic formulation of the $c\bar{c}$ bound state problem, results for the spectra and the $E1$ decay rates are in reasonable agreement with experiment if the confining potential is predominantly scalar.[29] The calculation of some of the $M1$ decay rates indicates substantial discrepancy unless a large negative anomalous moment is assumed. Although this assumption seems implausible and it appears more likely that the discrepancy may be remedied by the inclusion of coupled-channel mixing,[11] experimental determination of quark moments can help to resolve this issue.[30]

Calculations are also carried out for $b\bar{b}$, and $\eta_S=1$ provides much better fine-structure separation than $\eta_S=0$. For $b\bar{b}$ states the $E1$ and $M1$ decay rates have been given in Tables X and VI. The sparseness of experimental data does not yet permit a detailed comparison between theory and experiment.

Our results confirm conclusions by others that the relativistic effects in $q\bar{q}$ can be very large, and due to cancellations between various terms decay rates can change dramatically due to fairly small changes in several of the terms.

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$$\langle A|\vec{x}_0|B\rangle = {}_0\langle A|\vec{x}_0|B\rangle_0 + {}_0\langle A|\vec{x}_0|B\rangle_1 + {}_1\langle A|\vec{x}_0|B\rangle_0 + {}_1\langle A|\vec{x}_0|B\rangle_1 .$$

We will neglect the last term on the right-hand side since it is of order v^4/c^4. Then the ratio

$$\frac{|\langle A|\vec{x}_0|B\rangle|^2}{|{}_0\langle A|\vec{x}_0|B\rangle_0|^2} = 1+2\,\mathrm{Re}\frac{({}_0\langle A|\vec{x}_0|B\rangle_1+{}_1\langle A|\vec{x}_0|B\rangle_0)}{|{}_0\langle A|\vec{x}_0|B\rangle_0|^2}\times{}_0\langle A|\vec{x}_0|B\rangle_0 + \frac{|{}_0\langle A|\vec{x}_0|B\rangle_1+{}_1\langle A|\vec{x}_0|B\rangle_0|^2}{|{}_0\langle A|\vec{x}_0|B\rangle_0|^2} .$$

The second term on the right-hand side will give the first term of Eq. (17b) while the third term will give the second term of Eq. (17b). Even though the third term is strictly of order v^4/c^4 we have retained it in the calculation of those rates where this term makes a substantial difference. In these cases we will assume that this term gives the most important corrections of order v^4/c^4.

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