Connection of relativistic and nonrelativistic wave functions in the calculation of leptonic widths

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We generalize our previous JWKB relations between the relativistic $q\bar{q}$ wave function at the origin and (a) the inverse density of states of the $q\bar{q}$ system and (b) the nonrelativistic $q\bar{q}$ wave function at the origin, to the case of potentials with a Coulomb singularity. We show that the square of the Bethe-Salpeter wave function at the the origin is given approximately for 1^- states by

$$|\chi_{nS}(0,0)|^2 \simeq F(v^{\rm rel}) v^{\rm rel} \frac{M_n^2}{16\pi^2} \frac{dM_n}{dn} \left[1-\frac{16\alpha_s}{3\pi} g(v^{\rm rel})+O(\alpha_s^2)\right]$$

$$\simeq \frac{F(v^{\rm rel})}{F(v^{\rm nonrel})} \frac{M_n^2}{4m_q^2} \frac{v^{\rm rel}}{v^{\rm nonrel}} |\psi_{nS}^{\rm nonrel}(0)|^2 \left[1-\frac{16\alpha}{3\pi} g(v^{\rm rel})+O(\alpha_s^2)\right]$$

for $M_n > 2m_q$, where $F(v)=(4\pi\alpha_s/3v)[1-\exp(-4\pi\alpha_s/3v)]^{-1}$ is the usual Coulomb factor and $g(v)\simeq 1$ is associated with the lowest-order gluonic radiative corrections. We present numerical evidence for the remarkable accuracy of these relations, which have important implications for the use of nonrelativistic potential models to describe quarkonium systems. We also discuss some subtleties in the v and α_s dependence of corrections to leptonic widths.

I. INTRODUCTION

The square of the Bethe-Salpeter wave function at the origin is an important quantity for $q\bar{q}$ systems. For example, it appears in expressions for the leptonic and hadronic widths $\Gamma_{e^+e^-}$ and Γ_{3g} for 1^- states, and the hadronic widths Γ_{2g} for the 0^- states. The leptonic width for the decay of 1^- states, which will appear in this paper, is given by

$$\Gamma_n(e^+e^-)=\frac{16\pi\alpha^2 e_q^2}{M_n^2}|\chi_{nS}(\vec{0},0)|^2(1-\Delta_n)\ . \tag{1}$$

Here α is the fine-structure constant, e_q is the quark charge in units of e, and $M_n=2m_q+E_n=W$ is the total energy of the $q\bar{q}$ or e^+e^- system. $\chi_{nS}(\vec{0},0)$ is the large-large S-state component of the Bethe-Salpeter two-fermion wave function for zero space-time separation of the quarks, and Δ_n corrects for D-state, small-small, and (kinematic) relativistic effects left out in the large-large approximation.[1]

The function $\chi_{nS}(\vec{0},0)$ is frequently treated theoretically by replacing the full Bethe-Salpeter interaction kernel by an appropriate instantaneous interaction (the Salpeter approximation). This reduces the Bethe-Salpeter equation to a relative-time-independent Salpeter equation. $\chi_{nS}(\vec{0},0)$ is then written in terms of a first approximation $\Psi_{nS}^{\rm rel}(\vec{0})$ and a factor which corrects for the retardation and gluonic radiative effects omitted in the instantaneous approximation,

$$|\chi_{nS}(\vec{0},0)|^2=|\Psi_{nS}^{\rm rel}(\vec{0})|^2(1-\Delta_n')\ . \tag{2}$$

The definition of Δ_n' in this expression depends on the scheme used to determine $\Psi_{nS}^{\rm rel}$; only the complete function $\chi_{nS}(\vec{0},0)$ is unique. Because of the difficulties involved in constructing and solving relativistic models for $q\bar{q}$ systems, the relativistic wave function $\Psi_{nS}^{\rm rel}(\vec{0})$ is usually approximated in phenomenological studies of quarkonium by the solution $\psi_{nS}^{\rm nonrel}(\vec{0})$ of a Schrödinger equation with a potential adjusted to fit the observed spectrum. It is therefore of considerable importance to explore the relationships between $\chi_{nS}(\vec{0},0)$ [or an appropriately defined $\Psi_{nS}^{\rm rel}(\vec{0})$], and the Schrödinger wave function $\psi_{nS}^{\rm nonrel}(\vec{0})$ or other better understood quantities.

In two recent papers,[1,2] we discussed the calculation of the leptonic width $\Gamma_n(e^+e^-)$ in detail [Eqs. (1) and (2) appear as Eqs. (57) and (58) in Ref. 1]. We showed that for nonsingular interactions the Salpeter wave function $\Psi_{nS}^{\rm rel}(\vec{0})$ is simply related to the Schrödinger wave function $\psi_{nS}^{\rm nonrel}(\vec{0})$ calculated with a potential which fits the exact relativistic spectrum. We then used a duality argument and a conjectured extension of this result to the case of potentials with a color-Coulomb singularity to estimate the "radiative correction" Δ_n' to the leptonic widths of $^3S_1$ states in charmonium and b-quarkonium to $O(\alpha_s^2)$,

where α_s is the strong coupling constant. In the present paper, we will extend our earlier results to the case of singular potentials, and demonstrate the remarkable accuracy of our relations numerically.

The plan of the paper is as follows. In Sec. II, we review some background: the relation of $|\Psi_{nS}(0)|^2$ to the inverse density of states for either relativistic or nonrelativistic systems with nonsingular interactions, the resulting relation between $|\Psi_{nS}^{\rm rel}(0)|^2$ and $|\psi_{nS}^{\rm nonrel}(0)|^2$, and the extension of the nonrelativistic relation to singular potentials. In Sec. III we present numerical tests of our relations for a relativistic oscillator interaction, and show that they are accurate to a few percent even for highly relativistic particles. In Sec. IV, we extend our results for the relativistic wave function to the physically interesting case of potentials with color-Coulomb singularities, and again demonstrate their validity numerically. In Sec. V, we discuss the calculation of Δ_n in Eq. (1), and the relation of Δ_n and Δ_n' to the QCD perturbation expansion for the e^+e^- annihilation cross section. We summarize our principal results in Sec. VI.

II. BACKGROUND

We begin by recalling that $|\psi_{nS}^{\rm nonrel}(0)|^2$, the square of the Schrödinger wave function at the origin, is related to the inverse density of states dE_n/dn in the JWKB approximation by[3–7]

$$|\psi_{nS}^{\rm nonrel}(0)|^2=\frac{m_q^{\,2}}{4\pi^2}v_n^{\rm nonrel}\frac{dE_n}{dn}\ , \tag{3}$$

where v_n is the velocity of a free quark with kinetic energy $\frac{1}{2}[E_n-V_c(0)]$,

$$v_n^{\rm nonrel}=\{[E_n-V_c(0)]/m_q\}^{1/2}\ , \tag{4}$$

and the confining potential $V_c(r)$ is assumed to be nonsingular at $r=0$. In Ref. 1, we derived a relativistic analog of Eq. (3) for the Salpeter wave function for an instantaneous $q\bar{q}$ interaction,

$$|\Psi_{nS}^{\rm rel}(0)|^2=\frac{M_n'^{\,2}v_n^{\rm rel}}{16\pi^2}\frac{dM_n}{dn}\ , \tag{5}$$

where $v_n^{\rm rel}$ is the velocity of a free quark with total energy $\frac{1}{2}M_n'=\frac{1}{2}[M_n-V_c(0)]$, $M_n=2m_q+E_n$, and

$$v_n^{\rm rel}=\left[1-\frac{4m_q^{\,2}}{M_n'^{\,2}}\right]^{1/2}\ . \tag{6}$$

The existence of such a relation for the relativistic case had been conjectured but not proved by Tainov.[8] Again, the interaction must be nonsingular at the origin for Eq. (5) to hold, as this result was derived by making the JWKB approximation on a "relativistic Schrödinger" reduction of the Salpeter equation correct to $O(v^2/c^2)$.

For nonsingular potentials, then, we can use Eq. (5) to relate $|\Psi_{nS}^{\rm rel}(0)|^2$ directly to the measurable quantity dM_n/dn; or we can combine Eqs. (3) and (5) to relate $|\Psi_{nS}^{\rm rel}(0)|^2$ to the Schrödinger wave function $|\psi_{nS}^{\rm nonrel}(0)|^2$ corresponding to the *same* measured spectrum,

$$|\Psi_{nS}^{\rm rel}(0)|^2=\frac{M_n'^{\,2}}{4m_q^{\,2}}\frac{v_n^{\rm rel}}{v_n^{\rm nonrel}}|\psi_{nS}^{\rm nonrel}(0)|^2\ . \tag{7}$$

We emphasize that a *given spectrum* is generated by *different interactions* in the nonrelativistic and relativistic cases. We are *not* concerned with the more familiar problem (e.g., in QED) of relating relativistic and nonrelativistic wave functions for a *fixed* interaction.

If $V_c(0)=0$ for both the relativistic and nonrelativistic cases, we can write the factor in Eq. (7) as

$$\frac{M_n^{\,2}}{4m_q^{\,2}}\frac{v_n^{\rm rel}}{v_n^{\rm nonrel}}=\left[1+\frac{E_n}{2m_q}\right]\left[1+\frac{E_n}{4m_q}\right]^{1/2}\ . \tag{8}$$

This is always greater than unity for $E_n>0$ (as for a monotonically rising confining potential), so that $|\Psi_{nS}^{\rm rel}(0)|^2$ is always greater than $|\psi_{nS}^{\rm nonrel}(0)|^2$. The physical reason for this is that the relativistic kinetic energy in the Salpeter equation is less than the nonrelativistic kinetic energy,

$$2(p^2+m_q^{\,2})^{1/2}-2m_q<p^2/m_q\ . \tag{9}$$

As a result, for a *fixed* spectrum, the relativistic potential energy must be larger than the nonrelativistic potential energy, and the relativistic wave function is more tightly confined, hence larger at the origin.

In Sec. III, we present numerical tests of Eqs. (5) and (7) for the spinless oscillator potential which demonstrate that these equations are accurate to a few percent even for highly relativistic particles. We also test Eq. (5) for the oscillator interaction with vector coupling (a strongly spin-dependent case). Although spin dependence was not built into our derivation of Eq. (5), the results are again excellent.

Since realistic $q\bar{q}$ potentials involve a singular color-Coulomb component, Eqs. (3) and (5) cannot be used as they stand for the $q\bar{q}$ system. However, Bell and Pasupathy[9] and Fröman and Fröman[10,11] have extended the nonrelativistic relationship to the singular potential

$$V=-\tfrac{4}{3}\alpha_s r^{-1}+V_c(r)\ ,$$

where α_s is the strong coupling constant and the confining potential $V_c(r)$ is again nonsingular at the origin. Their result for $E_n-V_c(0)>0$ is

$$|\psi_{nS}^{\rm nonrel}(0)|^2=F(v_n^{\rm nonrel})\frac{m_q^{\,2}}{4\pi^2}v_n^{\rm nonrel}\frac{dE_n}{dn}\ , \tag{10}$$

where $v_n^{\rm nonrel}$ is given in Eq. (4), and $F(v)$ is the Coulomb factor

$$F(v)=|\phi_S^{\rm Coulomb}(0)|^2=\frac{4\pi\alpha_s}{3v}[1-\exp(-4\pi\alpha_s/3v)]^{-1}\ . \tag{11}$$

Thus, the only explicit effect of the extra color-Coulomb interaction is to multiply the original formula in Eq. (3) by $F(v_n^{\rm nonrel})$, though there is an implicit change through the change in the spectrum and dE_n/dn. We will derive a relativistic generalization of this result in Sec. IV.

For $E-V_c(0)<0$, we can use either the phase integral method of Fröman and Fröman[10] or a modification of the

Bell-Pasupathy procedure to obtain the alternative expression[12]

$$|\psi_{nS}^{\text{nonrel}}(0)|^2=\frac{\alpha_s m_q^2}{3\pi}\frac{dE_n}{dn},\quad E_n-V_c(0)<0\ . \tag{12}$$

This result is exact for a pure Coulomb potential, and connects smoothly with Eq. (10) for $E_n-V_c(0)\to 0$.

III. NUMERICAL TESTS OF THE JWKB RELATIONS FOR OSCILLATOR POTENTIALS

The expression for $|\psi_{nS}^{\text{nonrel}}(0)|^2$ for nonsingular confining potentials given in Eq. (3) has been tested by a number of authors,[13] and is quite accurate. We will therefore concentrate on tests of the relativistic relations in Eqs. (5) and (7).

We first consider a spinless Salpeter equation for a $q\bar{q}$ system with an oscillator interaction,

$$[2(p^2+m_q^2)^{1/2}-M+\tfrac{1}{2}r^2]\tilde{\Psi}^{\text{rel}}(\vec{p})=0\ , \tag{13}$$

where $\tilde{\Psi}(\vec{p})$ is the Salpeter wave function in momentum space, and we have expressed p, m_q, M, and r^{-1} in units of $k^{1/3}$, with k the spring constant of the oscillator. With the substitutions $r^2\to-\nabla_p^2$ and $M=2m_q+E$, we obtain the differential equation

$$\{\tfrac{1}{2}\nabla_p^2+E-[2(p^2+m_q^2)^{1/2}-2m_q]\}\tilde{\Psi}^{\text{rel}}(\vec{p})=0\ . \tag{14}$$

The wave functions are of the form

$$\tilde{\Psi}_{nlm}^{\text{rel}}(\vec{p})=\Phi_{nl}^{\text{rel}}(p)Y_{lm}(\hat{p}) \tag{15}$$

with the normalization

$$\frac{1}{(2\pi)^3}\int|\Phi_{nl}^{\text{rel}}(p)|^2p^2dp=1\ . \tag{16}$$

The S-state wave functions at the origin in coordinate space are given by

$$\Psi_{nS}^{\text{rel}}(0)=\frac{1}{(2\pi)^3}\int d^3p\,\tilde{\Psi}_{nS}^{\text{rel}}(\vec{p})=\frac{1}{4\pi^{5/2}}\int_0^\infty\Phi_{nS}^{\text{rel}}(p)p^2dp\ . \tag{17}$$

We have solved Eq. (14) numerically for $m_q=3.276$ and $m_q=1.310$ (values chosen to permit later comparison with spin-dependent calculations of Hostler and Repko[14]). Our results for the S-state spectrum and the exact wave functions at the origin are given in Table I. We have also calculated the JWKB prediction for $|\Psi_{nS}^{\text{rel}}(0)|^2$ using Eq. (5) with dM_n/dn (or dE_n/dn) calculated from a cubic polynomial fit to the n dependence of the spectrum. The predictions for $|\Psi_{nS}^{\text{rel}}(0)|^2$ given in Table I are in excellent agreement with the exact results even for $n=1$. The uncertainty in dM_n/dn is on the order of 1%, as judged by comparing results for quadratic and cubic fits to the energies, and is essentially as large as the errors in $|\Psi_{nS}^{\text{rel}}(0)|^2$. We note that the case $m_q=1.310$ is quite relativistic with ground-state quark velocities $v=0.79$ at the origin and $v_{\text{rms}}\simeq 0.65$.

In order to test the relation in Eq. (7), we need Schrödinger wave functions for a potential which has the same spectrum as the Salpeter equation, Eq. (14). We can convert Eq. (14) into the desired Schrödinger equation without changing the eigenvalues E_n by the substitution $\vec{p}\to(m_q/2)^{1/2}\vec{r}$, and find that $\psi^{\text{nonrel}}(\vec{r})$ satisfies the equation

$$\left[\frac{1}{m_q}\nabla_r^2+E-V(r)\right]\psi^{\text{nonrel}}(\vec{r})=0\ , \tag{18}$$

where

$$V(r)=(2m_qr^2+4m_q^2)^{1/2}-2m_q \tag{19a}$$

$$\to\begin{cases}\tfrac{1}{2}r^2, & r^2\ll 2m_q\ , \quad (19b)\\ (2m_q)^{1/2}r-2m_q, & r^2\gg 2m_q\ . \quad (19c)\end{cases}$$

The Schrödinger potential $V(r)$ is always less confining than the oscillator potential $\frac{1}{2}r^2$ used in the Salpeter equation, in agreement with the physical argument following Eq. (8).

The potentials and the (identical) spectra for Eqs. (14) and (18) are shown in Fig. 1. The expected trend toward closer spacing of the energy levels as $V(r)$ approaches the

TABLE I. Numerical test of the relativistic JWKB expression for $|\Psi_{nS}^{\text{rel}}(0)|^2$ in terms of the inverse density of states, Eq. (3). The energies and wave functions were calculated for the spinless Salpeter equation with an oscillator potential, Eq. (13). dE_n/dn was calculated from a cubic fit to the spectrum for $n=1$–4. Energies and masses are given in units of $k^{1/3}$, with k the spring constant of the oscillator. $|\Psi_{nS}^{\text{rel}}(0)|^2$ is given in units of k.

m_q	n	E_{nS}	dE_n/dn	$\lvert\Psi_{nS}^{\text{rel}}(0)\rvert^2$ JWKB	$\lvert\Psi_{nS}^{\text{rel}}(0)\rvert^2$ Exact	Error (%)
3.276	1	1.133	1.469	0.2872	0.2931	−2.0
	2	2.553	1.375	0.5012	0.5020	−0.2
	3	3.887	1.307	0.7023	0.6972	+0.7
	4	5.152	1.238	0.8896	0.8871	+0.3
1.310	1	1.660	2.012	0.1846	0.1864	−1.0
	2	3.528	1.739	0.3766	0.3758	+0.2
	3	5.165	1.550	0.5601	0.5616	−0.3
	4	6.654	1.443	0.7545	0.7477	+0.9

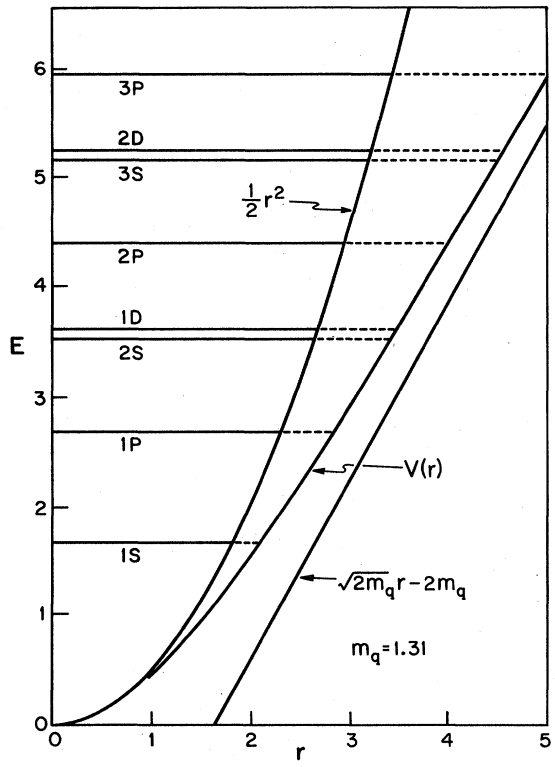

FIG. 1. Illustration of the difference between nonrelativistic and relativistic potentials which give the same energy spectrum, using the potential $\frac{1}{2}r^2$ in the spinless Salpeter equation, and the equivalent nonrelativistic potential $V(r)=(2m_q r^2+4m_q^2)^{1/2}-2m_q$ in the Schrödinger equation. $V(r)$ is quadratic for $r \ll 2m_q$, and approaches the linear potential shown in the figure for $r \gg 2m_q$. This change in the dominant r dependence is reflected in the decreasing spacing between adjacent levels. E, m_q, and r^{-1} are given in units of $k^{1/3}$, with k the spring constant of the relativistic oscillator.

linear potential in Eq. (19c) is clearly evident [$E_{nS} \propto (n-\frac{1}{4})^{2/3}$ for the high states in a linear potential, while $E_{nS} \propto (n-\frac{1}{4})$ for the oscillator].

In Table II, we compare the exact values of the Salpeter wave function at the origin with the values predicted using Eq. (7). The agreement is excellent, with a maximum error of 3.3% for the $n=1$ state in the highly relativistic system with $m_q=1.310$. We note that the conversion factor

$$M_n^2 v_n^{\rm rel}/4m_q^2 v_n^{\rm nonrel}$$

is 1.22 even for the least relativistic state ($n=1$, $m_q=3.276$), and is quite large for the more relativistic states (5.33 for $n=4$, $m_q=1.31$).

In the last column of Table II we give the Schrödinger wave function at the origin for the nonrelativistic oscillator problem

$$\left[\frac{1}{m_q}\nabla_r^2+E-\tfrac{1}{2}r^2\right]\psi^{\rm osc}(\vec{r})=0\,, \tag{20}$$

the nonrelativistic limit of Eq. (14). The Schrödinger wave function is only a reasonable (10%) approximation to the Salpeter wave function for the $n=1$ state with $m_q=3.276$.

As a separate, somewhat more realistic test of our relations in physical problems, we have used Eqs. (5) and (7) to predict $|\Psi_{nS}^{\rm rel}(0)|^2$ for the $^3S_1$ states in the problem studied by Hostler and Repko.[14] Those authors solved the (large-large) Salpeter equation exactly for an oscillator interaction with vector coupling. The results are strongly spin-dependent. (The splitting between the 1^3S_1 and $1\,^1S_0$ states for the case $m_q=1.31$ is 30% of the spin-averaged S-state energy while the splitting between the $1\,^3P_0$ and $1\,^3P_2$ states is 35% of the spin-averaged P-state energy.) Nevertheless, as shown in Table III, Eq. (5) gives excellent values for $|\Psi_{nS}^{\rm rel}(0)|^2$ in terms of the $^3S_1$ spectrum. The predictions for the square of the $^1S_0$ wave function at the origin are of comparable accuracy. This is not a trivial result, since the $^3S_1$ and $^1S_0$ spectra and wave functions differ significantly.[14] We therefore conclude that Eq. (5) is more generally valid than the derivation in Ref. 1 would suggest.

We have also made a rough check of the relation in Eq. (7) between the relativistic and nonrelativistic wave functions for equivalent potentials by obtaining best fits to the

TABLE II. Numerical test of Eq. (5), the JWKB prediction for the square of the spinless Salpeter wave function at the origin for an oscillator potential in terms of the nonrelativistic Schrödinger wave function for a potential which gives the same spectrum, Eqs. (18) and (19). Energies and masses are given in units of $k^{1/3}$, with k the spring constant of the oscillator. $|\Psi_{nS}^{\rm rel}(0)|^2$ is given in units of k.

m_q	n	E_n	$\frac{M_n^2}{4m_q^2}\frac{v_n^{\rm rel}}{v_n^{\rm nonrel}}$	$\lvert\psi_{nS}^{\rm nonrel}(0)\rvert^2$	$\lvert\Psi_{nS}^{\rm rel}(0)\rvert^2$ Exact	$\lvert\Psi_{nS}^{\rm rel}(0)\rvert^2$ Predicted	Error (%)	$\lvert\psi_{ns}^{\rm osc}(0)\rvert^2$ Nonrelativistic oscillator
3.276	1	1.133	1.223	0.2411	0.2931	0.2949	0.6	0.2600
	2	2.553	1.519	0.3316	0.5020	0.5037	0.3	0.3900
	3	3.887	1.814	0.3850	0.6972	0.6984	0.2	0.4875
	4	5.152	2.109	0.4216	0.8871	0.8892	0.2	0.5688
1.310	1	1.660	1.874	0.1027	0.1864	0.1925	3.3	0.1308
	2	3.528	3.035	0.1246	0.3758	0.3782	0.6	0.1962
	3	5.165	4.186	0.1347	0.5616	0.5639	0.4	0.2452
	4	6.654	5.332	0.1406	0.7477	0.7497	0.3	0.2861

TABLE III. Numerical test of the relativistic JWKB expression for $|\Psi^{\rm rel}_{nS}(0)|^2$ in terms of the inverse density of states, Eq. (3), using exact results for the spin-dependent Salpeter equation for an oscillator kernel with vector coupling (Ref. 14). dE_n/dn was calculated from a quadratic fit to the $^3S_1$ spectrum for $n=1$–3. Energies and masses are given in units of $k^{1/3}$, with k the spring constant of the oscillator. $|\Psi^{\rm rel}_{nS}(0)|^2$ is given in units of k.

m_q	n	$E_n(^3S_1)$	dE_n/dn	$\lvert\Psi^{\rm rel}_{nS}(0)\rvert^2$ JWKB	$\lvert\Psi^{\rm rel}_{nS}(0)\rvert^2$ Exact	Error (%)
3.276	1	1.316	1.550	0.3364	0.3271	+2.8
	2	2.851	1.520	0.6106	0.6078	0.5
	3	4.356	1.490	0.8975	0.8924	0.6
1.310	1	2.420	2.101	0.2886	0.2852	1.2
	2	4.457	1.974	0.5816	0.5825	0.02
	3	6.368	1.848	0.9042	0.9003	0.5

spin-averaged Hostler-Repko spectra using Schrödinger potentials of the form $V(r)=Kr^\nu$ and similar forms. Although we could not fit the relativistic spectrum precisely, the results were reasonably good, with errors of 10% and 15% in the ground-state wave functions for $m_q=1.310$ and 3.276, and errors of less than 3% for the $n=2$ and $n=3$ states. The accuracy of the results for the excited states is especially striking in view of the very large conversion factor in Eq. (7). We are therefore confident that proper spin-dependent fits to the Hostler-Repko spectra would lead to results as accurate as those in Table II.

IV. RELATIVISTIC EXTENSION OF THE COULOMB CASE

The relation in Eq. (10) between $|\psi^{\rm nonrel}_{nS}(0)|^2$ and the inverse density of states for potentials with Coulomb singularities was derived by Bell and Pasupathy[9] using a modified JWKB argument and by Fröman and Fröman[10] using a phase-integral method. We will use the Bell-Pasupathy method to derive a relativistic extension of Eq. (10), but note that our assumptions could be weakened somewhat in the phase-integral approach.[10,11]

The Bell-Pasupathy technique is based on the assumption that the exact wave function in the potential

$$V(r)=-\frac{4\alpha_s}{3r}+V_c(r),\quad V_c(0)\text{ finite},\tag{21}$$

can be approximated by a pure Coulomb wave function for the shifted energy $E-V_c(0)$ for r small, and by the JWKB wave function for the full potential for r large. By matching the Coulomb and JWKB wave functions at intermediate r where V_c is relatively unimportant, they construct a phase-shifted JWKB function which is a valid solution of the Schrödinger equation from the matching point out. This corrected JWKB function is then normalized. The normalization depends primarily on the behavior of the wave function in the outer region where the full potential acts, and is insensitive to the form of the wave functions very close to the origin. The result in Eq. (10) is then obtained by continuing the normalized wave function to the origin using the exact Coulomb function.

The essential feature of Eq. (10) is that the effects of the long-range confining interaction $V_c(r)$ appear only through the inverse density of states. The extra multiplicative factor relative to Eq. (3) is calculable using only the color-Coulomb interaction. This relation has been tested numerically for the Coulomb-plus-linear potential by Bell and Pasupathy, Table I in Ref. 9. We give a similar test in Table IV using a different fit to the spectrum E_n.[15] Except for the ground state, which is not expected to be well described in the JWKB approximation, Eq. (10) gives excellent values for $|\psi^{\rm nonrel}_{nS}(0)|^2$. An analytic test of Eq. (10) for the exactly solvable Hulthén potential was given some time ago in a different context by one of the authors[16] and has been repeated in more detail by Fröman and Fröman,[10] again with excellent results.

Our extension of the Bell-Pasupathy analysis to relativistic $q\bar{q}$ systems involves some subtleties which are not present in the nonrelativistic problem, but the principle of

TABLE IV. Numerical test of the nonrelativistic JWKB relation for $|\psi^{\rm nonrel}_{nS}(0)|^2$ in Eq. (10) for the singular potential $V(r)=-a/r+br$ with $a=0.25$, $b=0.18$ GeV^{-2}, and $m_q=1.45$ GeV. dE_n/dn was calculated from cubic-polynomial fits to $\ln E_n$ as a function of $\ln n$ (see Ref. 15).

n	E_n (GeV)	dE_n/dn (GeV)	$\lvert\psi^{\rm nonrel}_{nS}(0)\rvert^2$ Predicted (GeV^3)	$\lvert\psi^{\rm nonrel}_{nS}(0)\rvert^2$ Exact (GeV^3)	Error (%)
1	0.5161	0.6273	0.035 85	0.038 69	−7.4
2	1.0556	0.4667	0.032 44	0.032 37	+0.2
3	1.4779	0.3890	0.030 10	0.030 20	−0.3
4	1.8451	0.3472	0.028 96	0.028 95	+0.02
5	2.1768	0.3185	0.028 16	0.028 29	−0.5

the analysis is the same. Our objective is the calculation of $|\chi_{nS}(\vec{0},0)|^2$, the square of the Bethe-Salpeter wave function at the space-time origin, for use in the exact expression for $\Gamma_n(e^+e^-)$, Eq. (1). We carry out the calculation in two steps. We first consider the region in which the quarks have a spacelike separation $r > m_q^{-1}$. We make the standard assumptions that the $q\bar{q}$ Fock state gives the dominant contribution to χ in this region, and that the $q\bar{q}$ interaction is adequately described by the instantaneous Coulomb-gauge interaction in Eq. (21), and ignore possible long-range spin-dependent effects. The dominant S-state part of the Bethe-Salpeter wave function for $t=0$, $r \gg m_q^{-1}$ is then given by a solution of the instantaneous Salpeter equation for the $q\bar{q}$ system, $\chi_{nS}(\vec{r},0)=\Psi_{nS}^{\rm rel}(r)$, where

$$[2(p^2+m_q^2)^{1/2}-M_n+V(r)]\Psi_{nS}^{\rm rel}(r)=0 , \tag{22}$$

with m_q the on-shell quark mass. We solve this reduced problem using relativistic JWKB methods, and determine the normalization of the Salpeter wave function using a generalization of the Bell-Pasupathy technique.

In the inner region $r \lesssim m_q^{-1}$, relativistic retardation effects in the $q\bar{q}$ interaction and radiative effects involving transverse gluons are important, and we need a full relativistic treatment of the problem. However, the variation of the long-range confining interaction can be ignored in this region. The quarks act essentially as "free" particles with the usual short-distance QCD interactions and a total energy $M_n - V_c(0)$, and the variation of χ for $r \lesssim m_q^{-1}$ can be determined using perturbation theory. We will return to this point later.

We will begin by considering the outer region and determining the normalization of the JWKB solution of Eq. (22) constructed by Cea *et al.*[17] The radial wave function $u_n(r)$ found by those authors has the form

$$u_n(r) \simeq N^{1/2}\{[M_n - V(r)]/[M_n - V_c(0)]\}^{1/2} w_n(r) , \tag{23}$$

where $V(r)$ is the potential in Eq. (21) and $w_n(r)$ is a JWKB-type function,

$$w_n(r)=[p/p(r)]^{1/2}\sin\Delta(r) . \tag{24}$$

Here $p(r)$ is the local momentum,

$$p(r)=\left[\tfrac{1}{4}[M_n-V(r)]^2-m_q^2-\frac{(l+\frac{1}{2})^2}{r^2}\right]^{1/2} = \left[m_q(E_n-V_{\rm eff})-\frac{(l+\frac{1}{2})^2}{r^2}\right]^{1/2} , \tag{25}$$

$$V_{\rm eff}(E_n,r)=V-\frac{1}{4m_q}(E_n-V)^2 , \tag{26}$$

and p is the relativistic momentum of a free quark with total energy

$$\tfrac{1}{2}[M_n-V_c(0)]=m_q+\tfrac{1}{2}[E_n-V_c(0)] .$$

$\Delta(r)$ is the usual JWKB phase shifted by a constant $\phi(E)$ so that $N^{-1/2}u_n(r)$ matches smoothly for $r \simeq r_{\rm min}$ to the exact solution[18] of Eq. (22) for the shifted Coulomb potential $-4\alpha_s/3r + V_c(0)$, normalized to unit asymptotic amplitude,

$$\Delta(r)=\int_{r_{\rm min}}^{r} p(r)dr+\phi(E) . \tag{27}$$

The matching conditions are discussed by Castorina *et al.*[17] We note that $w_n(r)$ is the JWKB solution to the Klein-Gordon equation

$$\frac{d^2}{dr^2}w_n(r)+m_q[E_n-V_{\rm eff}(E_n,r)]w_n(r)=0 . \tag{28}$$

To determine the normalization constant N in Eq. (23), we add a small δ-function perturbation to $V_{\rm eff}$ in Eq. (28),

$$V_{\rm eff} \to V_{\rm eff}+\lambda\delta(r-R) , \tag{29}$$

and calculate its effect on the energy eigenvalues in two different ways. We first observe from Eq. (28) that the perturbation induces a discontinuity of height $m_q\lambda$ in the logarithmic derivative of $w_n(r)$ at $r=R$, and therefore changes the asymptotic phase of $w_n(r)$ by an amount ϵ,

$$\epsilon=-m_q\lambda\frac{\sin^2\Delta(R)}{p(R)}=-\frac{m_q\lambda}{p}w_n^2(R) . \tag{30}$$

The extra phase changes the JWKB quantization condition from

$$\pi n(E_n)\equiv\int_{r_{\rm min}}^{r_{\rm max}} p(r)dr+\phi(E_n)+\frac{\pi}{4}=n\pi \tag{31}$$

to

$$\pi n(E_n)+\epsilon=n\pi . \tag{32}$$

Since n is fixed for a given state, the energy eigenvalues E_n must change as the result of the perturbation by an amount δE_n determined by the condition

$$\pi\frac{dn(E_n)}{dE_n}\delta E_n+\epsilon=0 . \tag{33}$$

Thus, from Eqs. (30) and (33),

$$\delta E_n=\frac{m_q\lambda}{\pi p}\frac{dE_n}{dn}w_n^2(R) . \tag{34}$$

We can obtain a second relation for δE_n using first-order perturbation theory. Because the effective potential in Eq. (28) is energy dependent, the calculation is somewhat different from usual. We find that

$$\lambda w_n^2(R)=\delta E_n\int_0^\infty w_n^2(r)\left[1-\frac{dV_{\rm eff}}{dE_n}\right]dr =\delta E_n\int_0^\infty w_n^2(r)\left[1+\frac{1}{2m_q}(E_n-V)\right]dr . \tag{35}$$

The extra factor in the integrand is just what is necessary to relate w_n to u_n. Thus, using Eq. (23) and the normalization condition for u_n,

$$\lambda w_n^2(R)=\delta E_n\frac{M_n'}{2m_q}\frac{1}{N}\int_0^\infty u_n^2(r)\,dr =\frac{M_n'}{2m_q}\frac{1}{N}\delta E_n, \quad M_n'=M_n-V_c(0) . \tag{36}$$

Combining Eqs. (34) and (36), we find that the normalization constant N is given by

$$N=\frac{M_n'}{2\pi p}\frac{dE_n}{dn},\quad p=(\tfrac{1}{4}M_n'^2-m_q^2)^{1/2}. \tag{37}$$

We next observe that, by construction, the JWKB wave function connects smoothly with the exact solution of Eq. (22) for a pure color-Coulomb interaction,

$$\begin{aligned} &u_n(r)\to N^{1/2}u_s^{\mathrm{Coul}}(r)\\ &\quad=(4\pi N)^{1/2}r\Psi_S^{\mathrm{Coul}}(r),\\ &V_c(r)-V_c(0)<4\alpha_s/3r,\quad r\geq m_q^{-1}. \end{aligned} \tag{38}$$

Here Ψ_S^{Coul} is the Salpeter Coulomb wave function for energy $E_n-V_c(0)$, with the usual plane wave normalization, i.e., $u_s^{\mathrm{Coul}}(r)$ goes asymptotically to a sine wave of unit amplitude for $r\to\infty$. This function (which we recently constructed analytically[18]) gives a solution of the instantaneous form of the free Bethe-Salpeter problem (the problem in which the confining interaction is neglected) for $r\gtrsim m_q^{-1}$. With a proper phase shift, it connects smoothly with the $q\bar{q}$ component $\chi_S^{\mathrm{free}}(r,0)$ of the complete free solution valid in the inner region $r\lesssim m_q$.[19]

We conclude that the full Bethe-Salpeter wave function $\chi_{nS}(r,0)$ can be approximated smoothly for different (overlapping) ranges of r by

$$\chi_{nS}(r,0)\sim\frac{1}{(4\pi)^{1/2}r}u_n^{\mathrm{JWKB}}(r),\quad r>r_0, \tag{39a}$$

$$\simeq N^{1/2}\Psi_S^{\mathrm{Coul}}(r),\quad m_q^{-1}<r<r_0, \tag{39b}$$

$$\simeq N^{1/2}\chi_S^{\mathrm{free}}(r,0),\quad 0<r<m_q^{-1}, \tag{39c}$$

where $u_n^{\mathrm{JWKB}}(r)$ is defined in Eq. (23), and r_0 is a JWKB-Coulomb matching point. We emphasize that the effectiveness of this method of constructing χ_{nS} (as checked by extensive numerical calculations in the nonrelativistic and instantaneous relativistic problems) depends on the smoothness of the wave functions and confining interaction, and is insensitive to the particular choice of matching points for the wave functions constructed in the various regions.[10]

If we evaluate $\chi_{nS}(0,0)$ using Eqs. (39c) and (37), we find that

$$\begin{aligned} |\chi_{nS}(0,0)|^2&\simeq N|\chi_S^{\mathrm{free}}(0,0)|^2\\ &\simeq\frac{M_n'}{2\pi p'}|\chi_S^{\mathrm{free}}(0,0)|^2\frac{dM_n}{dn}. \end{aligned} \tag{40}$$

To complete our derivation of the relativistic generalization of Eq. (10), we need to determine $|\chi_S^{\mathrm{free}}(0,0)|^2$. We observe first that the square of the exact, properly normalized Salpeter Coulomb wave function for $r\sim m_q^{-1}$ involves an overall factor[18]

$$\begin{aligned} \frac{p^2}{4\pi}F(v^{\mathrm{rel}})&=\frac{p^2}{4\pi}\frac{4\pi\alpha_s}{3v^{\mathrm{rel}}}[1-\exp(-4\pi\alpha_s/3v^{\mathrm{rel}})]^{-1}\\ &=\frac{p^2}{4\pi}\left[1+\frac{2\pi\alpha_s}{3v^{\mathrm{rel}}}+\tfrac{1}{3}\left[\frac{2\pi\alpha_s}{3v^{\mathrm{rel}}}\right]^2+\cdots\right]. \end{aligned} \tag{41}$$

This factor, which sets the scale of $\chi_S^{\mathrm{free}}(r,0)$ at $r\sim m_q^{-1}$ is just the square of an ordinary Coulomb wave function at $r=0$. The characteristic dependence of $F(v^{\mathrm{rel}})$ on $\alpha_s/v^{\mathrm{rel}}$ is associated with the longer-range (infrared) part of the Coulomb interaction, and cannot be modified by short-range effects as $r\to 0$. The function $|\chi_S^{\mathrm{free}}(0,0)|^2$ must therefore contain $F(v^{\mathrm{rel}})$ as an overall factor, and must be of the form

$$|\chi_S^{\mathrm{free}}(0,0)|^2=\frac{p^2}{4\pi}F(v^{\mathrm{rel}})[1+O(\alpha_s)]. \tag{42}$$

In order to determine the corrections to Eq. (42), we must deal with the retardation and spin-dependent corrections to our instantaneous approximation, and include the effects of interactions involving transverse gluons. The standard approach to this problem involves a perturbative calculation of the corrections to χ_{nS} beginning with the solution of Eq. (22) as input. This is essentially equivalent to the calculation of the "radiative" corrections to the widths $\Gamma_n(l^+l^-)$ for the leptonic decay of the 1^- states carried out by a number of authors.[20] One expands the Bethe-Salpeter equation relative to the instantaneous approximation in Eq. (22), includes the retardation and spin-dependent effects and the contributions of transverse gluons as perturbations, and calculates the corrections to χ_{nS} directly, hence, by Eq. (1), the corrections to $\Gamma_n(l^+l^-)$. [See, for example, Bergström *et al.*[20] for a calculation of this type which starts with the Schrödinger approximation to Eq. (22).] Rather than repeat this calculation, we will identify the radiative corrections using an approach based on duality,[21] that is, the equality of appropriate energy averages of the physical cross section for $e^+e^-\to$ resonances, and the total cross section for e^+e^- annihilation into channels containing a heavy-quark pair calculated in perturbative QCD.[22,23] (We will ignore the small mixing with channels in which the annihilation photon initially produces a light-quark pair.) This approach has been discussed in more detail elsewhere.[2]

To implement the duality argument, we express the resonance cross section in terms of $|\chi_S^{\mathrm{free}}|^2$ using Eqs. (1) and (40),

$$\begin{aligned} W^2\sigma(e^+e^-\to\text{hadrons})&=\sum_n 6\pi^2\Gamma_n(e^+e^-)\delta(W-M_n)+\text{continuum contributions}\\ &=\sum_n\frac{48\pi^2\alpha^2e_q^2}{pW}|\chi_S^{\mathrm{free}}(0,0)|^2(1-\Delta)\frac{dM_n}{dn}\delta(W-M_n)+\cdots, \end{aligned} \tag{43}$$

average over a range of energies in the resonance region, and equate the result to the average of the QCD cross section.[22,23] Upon replacing the sum on n by an integral, we find that

$$\left\langle \frac{1}{pW} |\chi_S^{\rm free}(0,0)|^2(1-\Delta)\right\rangle = \left\langle \frac{v^{\rm rel}}{8\pi}(1-\tfrac{1}{3}v^{\rm rel2})\left[1+\frac{2\pi\alpha_s}{3v^{\rm rel}}-\frac{16\alpha_s}{3\pi}g(v^{\rm rel})+O(\alpha_s{}^2)\right]\right\rangle , \tag{44}$$

where Δ is the correction defined in Eq. (1) and[1,2]

$$g(v)\simeq 1+0.046v-v(1-v)^2 \tag{45}$$

is an accurate approximation to Schwinger's exact expression.[22,24] The factor $(1-\frac{1}{3}v^{\rm rel2})$ in Eq. (44) may be identified with $(1-\Delta)$ as will be discussed in Sec. IV. The first two terms in the remaining expression are just the leading terms in the expansion of the expected color-Coulomb factor, Eq. (41), which can be extracted as an overall factor on the right-hand side of Eq. (44). The remaining term is the leading radiative correction.

We observe finally that the radiative corrections are short-range effects in r, or equivalently, involve mass scales on the order of m_q, and are therefore insensitive to energy averaging on the scale set by the resonance spacings. Restricting the energy average to the region of a single resonance, we conclude that

$$|\chi_S^{\rm free}(0,0)|^2\simeq\frac{p^2}{4\pi}F(v^{\rm rel})\left[1-\frac{16\alpha_s}{3\pi}g(v^{\rm rel})+O(\alpha_s{}^2)\right], \quad M_n>2m_q . \tag{46}$$

The correction term in this equation evaluated for $v=0$ ($g=1$) is just the radiative correction to $|\chi_{nS}(0,0)|^2$ calculated by a number of authors[20] neglecting the motion of the quarks,

$$\Gamma_n=\Gamma_n^{(0)}\left[1-\frac{16\alpha_s}{3\pi}+O(\alpha_s{}^2)\right] . \tag{47}$$

Finally, combining Eqs. (40) and (46), we obtain our relativistic generalization of Eq. (10) for 1^- states:

$$|\chi_{nS}(0,0)|^2=F(v^{\rm rel})\frac{M_n{}^2}{16\pi^2}v_n^{\rm rel}\frac{dM_n}{dn}\times\left[1-\frac{16\alpha_s}{3\pi}g(v^{\rm rel})+O(\alpha_s{}^2)\right], \quad M_n>2m_q . \tag{48}$$

If $V_c(0)\neq 0$, M_n should be replaced in this expression by $M_n-V_c(0)$, and the relativistic velocity $v^{\rm rel}$ modified accordingly. We conjecture but have not proved that Eq. (48) should be modified for $M_n<2m_q$ to

$$|\chi_{nS}(0,0)|^2=\frac{\alpha_s M_n{}^2}{12\pi}\frac{dM_n}{dn}\left[1-\frac{16\alpha_s}{3\pi}+O(\alpha_s{}^2)\right] . \tag{49}$$

The two expressions connect smoothly for $v^{\rm rel}\to 0$ and agree with Eqs. (10) and (12) in the nonrelativistic limit.

These results can be generalized to different spin-parity states by using the Coulomb factor for nonzero angular momentum, and changing the relativistic and radiative correction factors to those appropriate for the process of interest.

In Table V we present a numerical check of our results. We have calculated the solutions to the Salpeter equation, Eq. (22), for the potential

$$V(r)=-\frac{a}{r}+br \tag{50}$$

for $a=0.25$, $b=0.18$ GeV^2, and $m_q=1.45$ GeV (values in the range needed for charmonium) and divided the results by the solution of the Coulomb Salpeter equation[18] at small r. The ratio of the two wave functions for $r\to 0$ should equal the spectrum-dependent factor in Eq. (48),

$$\lim_{r\to 0}\frac{p^2}{4\pi}|\Psi_{nS}(r)/\Psi_S^{\rm Coul}(r)|^2\simeq\frac{M_n{}^2}{16\pi^2}v_n^{\rm rel}\frac{dM_n}{dn} . \tag{51}$$

We see from Table V that the agreement of the numerical and theoretical results is excellent. The remaining arguments needed to justify the transition from Eq. (40) to Eq. (48) depend only on the separation of short-range and long-range effects, and well-established results in perturbation theory.

We note finally that we can use Eqs. (10) and (48) to relate nonrelativistic and relativistic systems which have the wave functions for the same spectrum. Eliminating $dM_n/dn=dE_n/dn$ between the two equations, we find that

TABLE V. Numerical test of the relativistic JWKB relation for $\lim_{r\to 0}(p^2/4\pi)|\Psi_{nS}(r)/\Psi_S^{\rm Coul}(r)|^2$, Eq. (51), for the singular potential $V(r)=-a/r+br$ with $a=0.25$, $b=0.18$ GeV^2, and $m_q=1.45$ GeV. dE_n/dn was calculated from a polynomial fit to the lowest six energies.

n	E_n (GeV)	$v_n^{\rm rel}$	$\frac{p^2}{4\pi}\lvert\Psi_{nS}(0)/\Psi_S^{\rm Coul}(0)\rvert^2$ (GeV^3)	$\frac{M_n{}^2}{16\pi^2}v^{\rm rel}\frac{dM}{dn}$ (GeV^3)	Error (%)
1	0.4924	0.519	0.0242	0.0229	−5.4
2	1.0022	0.669	0.0281	0.0280	−0.3
3	1.3925	0.737	0.0309	0.0306	−1.0
4	1.7260	0.779	0.0334	0.0332	−0.6
5	2.0236	0.808	0.0355	0.0350	−1.3

$$|\chi_{nS}(0,0)|^2 \sim \frac{F(v^{\rm rel})}{F(v^{\rm nonrel})}\frac{M_n^2}{4m_q^2}\frac{v^{\rm rel}}{v^{\rm nonrel}}|\psi_{nS}^{\rm nonrel}(0)|^2 \times\left[1-\frac{16\alpha_s}{3\pi}g(v^{\rm rel})+O(\alpha_s^2)\right], \quad M_n\geq 2m_q . \tag{52}$$

The combination of factors multiplying $|\psi_{nS}^{\rm nonrel}(0)|^2$ from the left is always greater than unity [this follows from Eq. (8), the observation that $F(v)$ is a monotonically decreasing function of v, and the fact that $v^{\rm nonrel}\geq v^{\rm rel}$]. As a result, in the absence of the radiative corrections, $|\chi_{nS}(0,0)|^2$ would always be larger than $|\psi_{nS}^{\rm nonrel}(0)|^2$ in agreement with the "tighter confinement" argument given following Eq. (8). (This assumes, of course, that it is possible to fit the relativistic spectrum using a Schrödinger model and the given α_s.)

For $M<2m_q$, Eq. (12) and our conjectured formula for $|\chi_{nS}(0,0)|^2$ in Eq. (49) give the alternative expression

$$|\chi_{nS}(0,0)|^2=\frac{M_n^2}{4m_q^2}|\psi_{nS}^{\rm nonrel}(0)|^2\left[1-\frac{16\alpha_s}{3\pi}+O(\alpha_s^2)\right], \quad M_n\leq 2m_q . \tag{53}$$

IV. THE CORRECTIONS Δ AND Δ'

We now return to a brief discussion of the radiative and relativistic corrections Δ,Δ' to the leptonic widths of $q\bar{q}$ bound states defined in Eqs. (1) and (2). In Refs. 1 and 2, we used duality and the QCD expansion of $\sigma(e^+e^-\to q\bar{q})$ to estimate these quantities to order α_s^2. Equation (48) in the present paper provides a simple derivation of the duality relation, and justifies our earlier extraction of the Coulomb-related terms (powers of α_s/v) from the QCD perturbation series. These terms are part of $|\chi_{nS}(0,0)|^2$.

The correction Δ_n was derived in Ref. 1. In the present notation it is given to order v^2/c^2 by

$$\Delta_n=\frac{4m_q}{3\pi}[\chi_{nS}(0,0)]^{-1}\times\int_0^\infty dr\,K_0(m_qr)\left[\chi_{nS}(0,0)-\frac{d}{dr}[r\chi_{nS}(r,0)]+\sqrt{2}\chi_{nD}(0,0)-\sqrt{2}\frac{d}{dr}[r\chi_{nD}(r,0)]\right], \tag{54}$$

where χ_{nD} is the large-large D-state component of the Bethe-Salpeter wave function and $K_0(m_qr)$ is the exponentially decreasing hyperbolic Bessel function. We will suppose that the dominant S-state wave function can be approximated for r small by a series solution to a relativistic Schrödinger equation[25] with the proper normalization at $r=0$,

$$\chi_{nS}(r,0)\simeq\chi_{nS}(0,0)[1-\tfrac{2}{3}\alpha_s m_q r-\tfrac{1}{6}k^2r^2+O(\alpha_s^2)], \tag{55}$$

where $k^2\simeq m_q[E-V_c(0)]$. We will neglect the D-state contribution. The integrals over r in Eq. (54) are easily performed, and we find that

$$\Delta_n\simeq\frac{16\alpha_s}{9\pi}+\frac{E_n-V_c(0)}{3m_q}+O(\alpha_s^2) . \tag{56}$$

In Refs. 1 and 2, we identified Δ_n with $[E_n-V_c(0)]/3m_q\sim\frac{1}{3}v_n^2(0)$, the result appropriate for nonsingular interactions as shown in Eq. (61) of Ref. 1. In the presence of a color-Coulomb singularity, the approximation to Δ_n in Eq. (56) contains an extra piece $16\alpha_s/9\pi$ which contributes one third of the "radiative correction" in Eqs. (43) or (48). It is *not* an extra contribution to the radiative correction: in the absence of a confining interaction, one must get the same *total* value for $|\chi_S(0,0)|^2$ or $\sigma(e^+e^-\to q\bar{q})$ whether the calculation is done using perturbation theory or the Bethe-Salpeter equation. The effect of including a confining interaction is simply to renormalize $\chi_{nS}(0,0)$ as in Eq. (40). The short-range corrections defined by *ratios* of wave functions are unchanged to the accuracy to which we are working. By *choosing* $\Delta_n=\frac{1}{3}v_n^2$ in Eq. (44) and Refs. 1 and 2, we have in fact *redefined* Δ_n and must delete the term $16\alpha_s/9\pi$ from Eq. (56).

We remark also that the division of the various α_s-dependent terms into corrections Δ_n and Δ_n' [the "radiative" correction relative to a particular choice of wave function in Eq. (2)] is gauge dependent, though the total result is not. Our treatment of the Bethe-Salpeter equation presupposes the use of the Coulomb gauge, whereas the free QCD calculations are usually done in the Feynman gauge.

We will henceforth *define* Δ_n to be $\frac{1}{3}v_n^2$, thus retaining only the last term in Eq. (56). However, we note that $(1-\frac{1}{3}v_n^2)$ is a natural factor to isolate in Eq. (1) for two reasons: First, it is the analog of the overall factor $(1-\frac{1}{3}v^2)$ which appears in the free cross section for $e^+e^-\to q\bar{q}$ [a fact which we used in Refs. 1 and 2 and the discussion following Eq. (42)] and second, it depends only on v_n, while all other terms depend on α_s. With this definition the function $|\chi^{\rm free}(0,0)|^2$ needed in Sec. III is given by Eq. (46).

If we wish finally, to separate $|\chi_{nS}(0,0)|^2$ in Eq. (1) into a leading term and a "radiative correction" Δ_n' as in Eq. (2), we must choose an appropriate initial approximation for the Bethe-Salpeter wave function. A usual choice in phenomenological studies of quarkonium is to equate the function $\Psi_{nS}^{\rm rel}(0)$ in Eq. (2) with the nonrelativistic Schrödinger wave function for the potential in Eq. (21), but this choice neglects essentially kinematic relativistic effects which can be quite large for light-quark systems. We prefer to circumvent this problem in a way useful for numerical studies and consistent with our earlier duality arguments[2] (but perhaps awkward for perturbative calculations) by calculating $\Psi_{nS}^{\rm rel}(r)$ for $r\geq m_q^{-1}$ using the Salpeter equation and matching the result to a solution of the relativistic Schrödinger equation[25] for $r<m_q^{-1}$ as in Sec. II. In this case, the argument in Ref. 2 gives

$$\Gamma_n \simeq \Gamma^{(0)}(1-\tfrac{1}{3}v_n^2)\left[1-\frac{16\alpha_s}{3\pi}g(v^n)+C_2'\frac{\alpha_s^2}{\pi^2}\right], \quad (57)$$

where $g(v)$ is defined in Eq. (45) and

$$C_2' \simeq 24.26-0.115N_f . \quad (58)$$

The first-order correction $16\alpha_s/3\pi$ in this expression is independent of (any reasonable) choice of $\Psi_{nS}^{\rm rel}(0)$ for the reasons sketched by Celmaster[20] and by Kummer and Wirthumer.[26] The value of C_2' depends on this choice (and on the choice of renormalization scheme). Our result is equal for the free cross section $\sigma(e^+e^-\to q\bar{q})$ to that obtained by simply extracting the known color-Coulomb factor $F(v^{\rm rel})$ from the QCD perturbation series in the modified minimal-subtraction ($\overline{\rm MS}$) scheme.[2]

VI. SUMMARY

In this paper, we have derived a JWKB relation between the square of the Bethe-Salpeter two-fermion wave function at the origin and the inverse density of states of the system. Our derivation holds for the realistic situation in which the two-fermion (quark-antiquark) interaction includes a color-Coulomb component at short distance and a long-range confining interaction. Our principal results are as follows.

For 1^- states, we find that $|\chi_{nS}(0,0)|^2$ is given for $M_n > 2m_q$ by

$$|\chi_{nS}(0,0)|^2 \simeq F(v^{\rm rel})\frac{M_n^2}{16\pi^2}v^{\rm rel}\frac{dM_n}{dn} \times\left[1-\frac{16\alpha_s}{3\pi}g(v^{\rm rel})+O(\alpha_s^2)\right], \quad (59)$$

where $F(v)$ is the Coulomb factor defined in Eq. (11), M_n is the mass of the nth state, and $v^{\rm rel}$ is the relativistic velocity of a free quark with mass m_q and total energy $M_n/2$. This expression includes the effects of the short-range gluonic radiative corrections; the function $g(v)$ is defined in Eq. (44). For $M_n < 2m_q$, we believe (but have not proved) that

$$|\chi_{nS}(0,0)|^2 = \frac{\alpha_s M_n^2}{12\pi}\frac{dM_n}{dn}\left[1-\frac{16\alpha_s}{3\pi}+O(\alpha_s^2)\right], \quad (60)$$

a result which connects smoothly with Eq. (59) for $v^{\rm rel}\to 0$ and reduces to the (proven) nonrelativistic expression in Eq. (12) for

$$|M_n-2m_q|/2m_q \ll 1 .$$

Using these results, we find that the leptonic width for the decay of 1^- states is given in terms of the inverse density of states by

$$\Gamma_n(e^+e^-) \simeq \frac{\alpha^2 e_q^2}{\pi}v_n^{\rm rel}F(v_n^{\rm rel})\frac{dM_n}{dn} \times\left[1-\frac{16\alpha_s}{3\pi}g(v_n^{\rm rel})+O(\alpha_s^2)\right](1-\tfrac{1}{3}v_n^{\rm rel2}), \quad M_n > 2m_q . \quad (61)$$

or by

$$\Gamma_n(e^+e^-) \simeq \tfrac{4}{3}\alpha^2 e_q^2\alpha_s\frac{dM_n}{dn}\left[1-\frac{16\alpha_s}{3\pi}+O(\alpha_s^2,v_n^2)\right], \quad M_n < 2m_q . \quad (62)$$

We can also use the results in Eqs. (59) and (60) and the corresponding results for a Schrödinger potential model fitted to the same spectrum to relate the relativistic and nonrelativistic wave functions. If we eliminate $dM_n/dn\,(=dE_n/dn)$ between Eqs. (10) and (59), or (12) and (60), we find that for a fixed α_s

$$|\chi_{nS}(0,0)|^2 = \frac{F(v^{\rm rel})}{F(v^{\rm nonrel})}\frac{M_n^2}{4m_q^2}\frac{v^{\rm rel}}{v^{\rm nonrel}}|\psi_{nS}^{\rm nonrel}(0)|^2 \times\left[1-\frac{16\alpha_s}{3\pi}g(v^{\rm rel})+O(\alpha_s^2)\right], \quad M_n > 2m_q , \quad (63)$$

$$|\chi_{nS}(0,0)|^2 = \frac{M_n^2}{4m_q^2}|\psi_{nS}^{\rm nonrel}(0)|^2\left[1-\frac{16\alpha_s}{3\pi}+O(\alpha_s^2)\right], \quad M_n < 2m_q . \quad (64)$$

These relations allow us to correct the wave functions or leptonic widths calculated in the phenomenological Schrödinger theory to obtain reliable relativistic predictions. The corrections are quite large for light-quark systems.

The present results can be generalized to different spin-parity states by using the Coulomb factor for nonzero angular momentum (see Refs. 5 and 9), and changing the relativistic and radiative corrections to those appropriate to the process of interest.

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[12]Our procedure is to match the JWKB wave function for a pure Coulomb potential in the region $E<0$, $-\alpha/Er\gg 1$,

$$u^{\mathrm{JWKB}}(r)=\left[m_q\left[\frac{\alpha}{r}+E\right]\right]^{-1/4}\times\sin\left[m_q^{1/2}\int_0^r\left[\frac{\alpha}{r}+E\right]^{1/2}dr+\phi\right]$$

$$\simeq(r/m_q\alpha)^{1/4}\sin[2(m_q\alpha r)^{1/2}+\phi]$$

to the exact solution of the Schrödinger equation,

$$u^{\mathrm{Coul}}(r)=2(-m_qE)^{1/2}re^{-(-m_qE)^{1/2}r}\times{}_1F_1\left[1-\frac{\alpha}{2}\left[\frac{m_q}{-E}\right]^{1/2},2,2(-m_qE)^{1/2}r\right]$$

$$\simeq\left[-\frac{4E}{\alpha\pi}\right]^{1/2}\left[\frac{r}{m_q\alpha}\right]^{1/4}\times\sin\left[2(m_q\alpha r)^{1/2}-\frac{\pi}{4}\right].$$

This determines the phase ϕ of the JWKB wave function and the relative normalization of the two functions. The normalization of the JWKB wave function with a nonsingular confining potential present is then determined exactly as in Ref. 9, and the normalized wave function is continued to $r=0$ using the matched Coulomb wave function. The result is given in Eq. (12) for a color-Coulomb potential, $\alpha\to 4\alpha_s/3$. The JWKB quantization condition for the phase-shifted wave function is

$$n\pi=m_q^{1/2}\int_0^{r_0}\left[\frac{\alpha}{r}-V_c+E\right]^{1/2}dr,\quad n=1,2,\ldots.$$

The quantization condition is exact for a pure Coulomb potential.
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[15]Our procedure of fitting $\ln E_n$ to a polynomial in $\ln n$ was suggested by two considerations, first, that the JWKB quantization condition in the presence of a Coulomb interaction depends on n rather than $(n-\frac{1}{4})$ (see Ref. 9), and second, the observation of A. Martin, Phys. Lett. **93B**, 338 (1980), that charmonium potentials (of which the Coulomb-plus-linear potential is an example) are well approximated for the most relevant values of r by simple power laws, $V(r)\sim Kr^{\nu}$. The power-law approximation gives a spectrum with $E\propto n^{2\nu/(2+\nu)}$. Simple fits to the spectrum using polynomials in n do not change the results in Table IV significantly.
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tion

$$\left[\frac{1}{m_q}\nabla^2+\frac{1}{4m_q}(M_n{}^2-4m_q{}^2)-\frac{M_n}{2m_q}V(r)\right]\psi_S(r)=0$$

with a Coulomb potential $V=-4\alpha_s/3r$ has correct relativistic kinematics, gives the exact S-state eigenvalues for the Salpeter-Coulomb wave equation derived in Ref. 18,

$$M_n=2m_q/\left[1+\frac{4\alpha_s{}^2}{9n^2}\right]^{1/2},\quad n=1,2,\ldots,$$

agrees in phase with the exact solution to $O(\alpha_s v^2)$ for r large, and extends smoothly to $r=0$ where it has exactly the value in Eq. (42),

$$|\psi_S(0)|^2=\frac{p^2}{4\pi}F(v^{\rm rel})\ .$$

We expect, therefore, that it gives a quite reasonable approximation to the r dependence of $\chi_{nS}(r,0)$. The normalization at the origin is absorbed in the overall factor $\chi_{nS}(0,0)$ in Eq. (55).

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