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Comment on Einstein-massless-scalar field equations and conformally flat solutions

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An (infinite) family of conformally flat solutions of the Einstein-massless-scalar field equations is derived when the space-time dependence of the conformal factor and of the scalar field is of the type $u=k\cdot x$, k being a constant and uniform *lightlike* four-vector. This includes the Penney solution and completes Gürses's discussion on the subject. Parallel developments are discussed when the space-time dependence is given by the variable $v=x^2$: they lead to the unique Gürses solution.

I. INTRODUCTION

Conformally flat solutions of the coupled massless-scalar and gravitational field Einstein equations have been given by Penney[1] and Gürses.[2] In particular, Gürses has asserted that there can be only two distinct solutions of these equations, characterized by a dependence only on $u=k\cdot x=k_\lambda x^\lambda$ or on $v=x^2=\eta_{\mu\nu}x^\mu x^\nu$, $k=\{k_\lambda, \lambda=0,1,2,3\}$ being a constant and uniform four-vector.

We derive here, in the case of a u dependence with a lightlike k, an explicit particular infinite family of solutions parametrized by a nonzero real number m. For $m=\frac{1}{2}$, we recover Penney's solution.[1] We also show that all the solutions characterized by different values of the parameter m are nonequivalent: they cannot be related through coordinate transformations as is clear from a detailed study based on the use of the program CLASSI from SHEEP.[3] These results are thus at a variance with Gürses's general conclusions when $k^2=0$.

The program CLASSI also shows that our conformally flat solutions (with $k^2=0$) can be interpreted in terms of a null electromagnetic field. These solutions describe special plane-fronted gravitational waves with a constant null vector k, of the type considered by McLenaghan, Tariq, and Tupper[4] (Cf. Ref. 5, theorem 32.17).

Section II contains the discussion of scalar fields whose u dependence corresponds to lightlike (IIa) and to nonlightlike (IIb) four-vectors k, leading to the above results and to the Penney solution in particular. The discussion of a v dependence is presented in Sec. III in a complete parallel way leading to the unique Gürses solution.

The signature $(+,-,-,-)$ is chosen for the metric tensor and the Ricci tensor is defined by $R_{\mu\nu}=R^\alpha{}_{\mu\nu\alpha}$.

II. ON SCALAR FIELDS $\phi(k\cdot x)$ AND $\psi(k\cdot x)$

Let us consider a conformally flat space-time with metric tensor

$$g_{\mu\nu}(x)=\phi^2(u)\eta_{\mu\nu}, \quad u=k\cdot x \ , \tag{2.1}$$

where k is an arbitrary constant and uniform four-vector. The coupled Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=-\kappa T_{\mu\nu} \ , \tag{2.2}$$

$$T_{\mu\nu}=\partial_\mu\psi\partial_\nu\psi-\tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi \ , \tag{2.3}$$

where ψ is a scalar field whose space-time dependence is also given in terms of the variable u, are equivalent to the system

$$2\frac{k_\mu k_\nu}{\phi^2}[\phi\phi''-2(\phi')^2]+\frac{k^2\eta_{\mu\nu}}{\phi^2}[(\phi')^2-2\phi\phi'']=-\kappa T_{\mu\nu} \tag{2.4}$$

and

$$T_{\mu\nu}=(\psi')^2(k_\mu k_\nu-\tfrac{1}{2}k^2\eta_{\mu\nu}) \ . \tag{2.5}$$

The derivatives are taken with respect to the variable u and $k^2=\eta^{\mu\nu}k_\mu k_\nu$.

From Eqs. (2.4) and (2.5), the Einstein equations are equivalent to

$$k_\mu k_\nu A+\eta_{\mu\nu}k^2B=0, \quad \mu,\nu=0,1,2,3 \ , \tag{2.6}$$

with

$$A=\frac{2}{\phi^2}[\phi\phi''-2(\phi')^2]+\kappa(\psi')^2 \tag{2.7}$$

and

$$B=\frac{1}{\phi^2}[(\phi')^2-2\phi\phi'']-\tfrac{1}{2}\kappa(\psi')^2 \ . \tag{2.8}$$

Let us distinguish different cases corresponding to the lightlike ($k^2=0$) or nonlightlike ($k^2\neq 0$) character of k.

(a) If $k^2=0$, then we immediately get from Eq. (2.6)

$$A=0 \ . \tag{2.9}$$

This relation implies from Eq. (2.7) that

$$\psi=\sqrt{2/\kappa}\int^{u}[\phi(\phi^{-1})'']^{1/2}du' \ . \tag{2.10}$$

In particular, if we choose for the scalar field ϕ the explicit form (suggested by Penney's work)

$$\phi(k\cdot x)=(k\cdot x+d)^m \ , \tag{2.11}$$

where d and m are arbitrary constants, we get

$$\psi=\left[\frac{2m(1+m)}{\kappa}\right]^{1/2}\ln(k\cdot x+d)+b \tag{2.12}$$

leading to a family

$$g_{\mu\nu}(x)=(k\cdot x+d)^{2m}\eta_{\mu\nu}, \quad \psi(k\cdot x)\equiv(2.12) \tag{2.13}$$

of solutions satisfying the Einstein-massless-scalar field equations.

The program CLASSI from SHEEP,[3] developed in order to check the eventual equivalence between different metrics of four-dimensional Riemannian manifolds, solutions of Einstein's field equations, has been used to show explicitly that the various solutions (2.13) corresponding to different values of the parameter m are really intrinsically different. This is at variance with Gürses's[2] contention that there exist only two distinct solutions when the source of Einstein's equations is a massless scalar field. If $m=\frac{1}{2}$ (and $b=0$), we recover the Penney solution while, if $m=-1$, we get the Minkowski metric.

As shown by the use of the program CLASSI, the Ricci tensor associated with solutions (2.13) is of the algebraic type $A3[(11,2)]$ in Segré notation (see, for example, Ref. 5), implying that the related energy-momentum tensor can be interpreted as that of a pure radiation field which can itself be interpreted in terms of a null electromagnetic field.

This result is in agreement with the well-known fact that the line element associated with the metric tensor (2.1) when $k^2=0$ can be brought into the following Brinkmann form (cf. Ref. 5, Sec. 21.5)

$$ds^2=-dx^2-dy^2+\kappa\Phi(u)(x^2+y^2)\frac{d^2u}{2}+2du\,dv \ . \tag{2.14}$$

This metric tensor is, in fact, the only existing conformally flat solution with a null electromagnetic field.[4]

Let us also notice, in connection with Gürses's developments,[2] that our ϕ solution (2.11) satisfies

$$\Omega\equiv\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}=0 \quad\text{and}\quad \Box\phi\equiv\eta^{\mu\nu}\partial_\mu\partial_\nu\phi=0 \ , \tag{2.15}$$

so that we immediately recover the Gürses-Penney solution in this case.

(b) If $k^2\neq 0$ (k is timelike or spacelike), we obtain from Eq. (2.6)

$$A=0 \text{ and } B=0 \ , \tag{2.16}$$

so that we have to solve the only condition:

$$\phi\phi''+(\phi')^2=0 \ . \tag{2.17}$$

We are led to the unique solution (up to arbitrary constants C_1 and C_2)

$$\phi(k\cdot x)=[C_1(k\cdot x)+C_2]^{1/2} \tag{2.18}$$

corresponding to the Penney solution when $k^2\neq 0$ and $C_1=1$, $C_2=d$.

III. ON SCALAR FIELDS $\phi(x^2)$ AND $\psi(x^2)$

Let us assume now that the space-time dependence of the scalar fields ϕ and ψ is given by the variable $v=x^2$, namely,

$$g_{\mu\nu}=\phi^2(v)\eta_{\mu\nu}, \quad \psi=\psi(v) \ . \tag{3.1}$$

Equations (2.2) and (2.3) imply that

$$x_\mu x_\nu A'+\eta_{\mu\nu}B'=0 \tag{3.2}$$

with

$$A'=\frac{4}{\phi^2}[\phi\phi''-2(\phi')^2]+2\kappa(\psi')^2 \tag{3.3}$$

and

$$B'=\frac{2}{\phi^2}\{v[(\phi')^2-2\phi\phi'']-3\phi\phi'\}-\kappa v(\psi')^2 \ , \tag{3.4}$$

where the derivatives are taken with respect to the variable v.

Equation (3.2) is satisfied if and only if

$$A'=0 \tag{3.5a}$$

and

$$B'=0 \ . \tag{3.5b}$$

By combining Eqs. (3.4) and (3.5), it follows that

$$v[\phi\phi''+(\phi')^2]+3\phi\phi'=0 \ . \tag{3.6}$$

This equation is easily integrated by putting

$$aS=\phi\phi' \ , \tag{3.7}$$

where a is an arbitrary constant. Indeed, Eq. (3.6) is then reduced to the form

$$vS'+3S=0 \ , \tag{3.8}$$

which implies that

$$S=\frac{b}{v^3}, \quad b=\text{const} \ . \tag{3.9}$$

From Eqs. (3.7) and (3.9) we immediately get

$$\phi=(C_1+C_2/x^4)^{1/2} \ , \tag{3.10}$$

where C_1 is an arbitrary constant and $C_2=-ab$.

Using Eqs. (3.3), (3.5a), and (3.10), we recover finally Gürses's solution

$$\begin{aligned} &g_{\mu\nu}=(C_1+C_2/x^4)\eta_{\mu\nu}, \\ &\psi=-\sqrt{6/\kappa}\tanh^{-1}(x^2\sqrt{C_1/ab}\,) \ . \end{aligned} \tag{3.11}$$

This means that, if the space-time dependence of the scalar fields ϕ and ψ is given by the Lorentz invariant $v=x^2$, then Eqs. (2.2) and (2.3) admit Gürses's solution as the *unique* solution.

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