# Fractional indices in supersymmetric theories

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The regularized index introduced by Witten for supersymmetric theories and discussed by Callias for elliptic operators is studied. We relate this index to Levinson's theorem for potential scattering and clarify its meaning even when it takes on fractional values. The index can be (and usually is) half-integer whenever the continuum in the spectrum extends down to zero energy.

One of the outstanding problems in supersymmetric theories is to understand the circumstances under which supersymmetry is broken, and to identify an order parameter associated to this breaking. Since the spectrum of supersymmetric theories is positive semidefinite, supersymmetry is broken if there are no states with zero energy. This has led Witten[1] to introduce an index that counts the number of bosonic zero-energy modes minus the number of fermionic zero-energy modes:

$$\Delta=\mathrm{Tr}(-1)^F=n_B(E=0)-n_F(E=0)\ . \tag{1}$$

If the Hilbert space of the theory is split into bosonic and fermionic subspaces, the supersymmetry charge can be written as[1]

$$Q=\begin{bmatrix}0 & L\\ L^\dagger & 0\end{bmatrix} \tag{2}$$

acting on the vector with a bosonic upper component and a fermionic lower component. Therefore $\Delta$ can be identified[2] with the index of the operator $L$:

$$\begin{aligned}\Delta&=\dim \mathrm{Ker}(L)-\dim\mathrm{Ker}(L^\dagger)\\ &=\dim\mathrm{Ker}(L^\dagger L)-\dim\mathrm{Ker}(LL^\dagger)\ .\end{aligned} \tag{3}$$

In most of the interesting cases $L$ is elliptic and one is led to the general study of the index of such operators.

In supersymmetric quantum mechanics

$$H=\tfrac{1}{2}\{Q,Q^\dagger\}=\begin{bmatrix}LL^\dagger & 0\\ 0 & L^\dagger L\end{bmatrix}, \tag{4}$$

and we shall concentrate on the behavior of the index for this problem. In general the formal expressions (1) and (3) are ill defined and need regularization. Witten has suggested a heat kernel regularization[1]

$$\Delta_\beta=\mathrm{Tr}(-1)^F e^{-\beta H}\ , \tag{5}$$

whereas Callias[2] introduced the form

$$\Delta(z)=\mathrm{Tr}(-1)^F\frac{z}{H+z}\ . \tag{6}$$

Actually (5) and (6) are directly related by a Laplace transform.

It has been recognized[3] that despite the fact that the nonzero-energy states of fermions and bosons are paired, the expressions (5) and (6) are actually regularization dependent if the theory has a continuum of states. Thus, the index of interest is defined as the limits

$$\Delta=\lim_{\beta\to\infty}\Delta_\beta \ \text{ and } \Delta=\lim_{z\to 0^+}\Delta(z)\ . \tag{7}$$

Moreover, in recent papers[3,4] it has been found that the index is fractional in certain circumstances. It has been suggested that such fractional values may arise when the continuum extends to $E=0$. However, no explanation has been provided for the particular fraction $\frac{1}{2}$ found in examples nor why no other values have been found. The quantity $\Delta(z)$ of Eq. (6) will be studied in the case of supersymmetric quantum mechanics and in the process we will clarify its physical meaning, interpretation, and possible values.

The action of the Dirac-type operator (2) onto the boson and fermion states can be cast as

$$Q\psi=\begin{bmatrix}0 & L\\ L^\dagger & 0\end{bmatrix}\begin{bmatrix}u\\ d\end{bmatrix}=\sqrt{E}\begin{bmatrix}u\\ d\end{bmatrix}\ . \tag{8}$$

In our examples we write

$$L=i\left[\frac{d}{dx}+\phi(x)\right],\quad L^\dagger=i\left[\frac{d}{dx}-\phi(x)\right], \tag{9a}$$

and then $H$ is diagonal with elements

$$\begin{aligned}H_{11}&\equiv H^+=LL^\dagger=-\frac{\partial^2}{\partial x^2}+\phi^2+\phi'\ ,\\ H_{22}&\equiv H^-=L^\dagger L=-\frac{\partial^2}{\partial x^2}+\phi^2-\phi'\ .\end{aligned} \tag{9b}$$

Therefore, we may write

$$\Delta(z)=\mathrm{Tr}\left[\frac{z}{H^-+z}-\frac{z}{H^++z}\right]=z\frac{d}{dz}\ln J_R(z)\ , \tag{10}$$

where

$$J_R(z)=\frac{J^-(z)}{J^+(z)}=\frac{\det\left[\dfrac{H^-+z}{H_\nu+z}\right]}{\det\left[\dfrac{H^++z}{H_\nu+z}\right]}\ . \tag{11}$$

 

The quantities $J^{\pm}(z)$ are the Jost functions[5,6] (Fredholm determinants) of an associated Schrödinger equation with energy $E=-z$. The comparison Hamiltonian $H_v$ will be chosen to ensure the existence of $J^-$ and $J^+$. If the spectrum of $H^{\pm}$ is *discrete* [$\phi(x)$ unbounded as $x\to\pm\infty$] the pairing properties of the nonzero-energy states assures that $\Delta(z)$ is $z$ independent and equal to the integer index. However, we are interested in the case when there is a continuum in the spectrum, and will therefore study the case when the background fields $\phi(x)$ in (9) are bounded as $x\to\pm\infty$. The general situation can then be considered by performing appropriate limits.

In Eq. (11), $H_v$ is any suitable and solvable Hamiltonian with potential $V(x)$. We will make the simplest possible choice for $V(x)$ such that it has *no* bound states and achieves the same asymptotic values as $\phi(x)$ when $x\to\pm\infty$. In the specific example $\phi(x=-\infty)=\phi_-$ and $\phi(x=+\infty)=\phi_+$, the potential $V(x)$ can be chosen to be $V(x)=\phi_-^2\theta(-x)+\phi_+^2\theta(x)$. This choice of $H_v$ guarantees the existence of the Jost function. Indeed the Jost functions can be written as

$$J^{\pm}=1-\frac{1}{\omega}\int_{-\infty}^{\infty}\phi_1(x)U^{\pm}(x)f^{\pm}(x)dx\ , \tag{12}$$

where $U^{\pm}=\phi^2(x)-V(x)\pm\phi'(x)$. The $f^{\pm}(x)$ are the solutions to

$$f^{\pm}=\phi_0(x)+\frac{1}{\omega}\left[-\phi_0(x)\int_x^{\infty}\phi_1(y)U^{\pm}(y)f^{\pm}(y)dy + \phi_1(x)\int_x^{\infty}\phi_0(y)U^{\pm}(y)f^{\pm}(y)dy\right]\ , \tag{13}$$

where $\phi_0,\phi_1$ are two linearly independent solutions of $H_v$ with Wronskian $\omega$. The normalization of $\phi_0(x)$ is fixed to be a unit incoming wave for $x<0$ and a pure transmitted wave for $x>0$. It can be easily seen that

$$J^{\pm}=\frac{T_v}{T^{\pm}}\ , \tag{14}$$

where $T_v,T^{\pm}$ are the transmission coefficients for $H_v,H^{\pm}$. The Jost functions have zeros at the bound states of $H^{\pm}$, and are complex above threshold with a phase equal to the negative of the scattering phase shift. Although an evaluation of $J^{\pm}(E)$ requires a nontrivial computation involving knowledge of the detailed behavior of $\phi(x)$, their ratio can be computed easily in terms of asymptotic quantities only.

The scattering states of $H^+$ have the boundary conditions

$$\psi^+(x)=\begin{cases} e^{ik_-x}+R^+e^{-ik_-x}, & x\to-\infty \\ T^+e^{ik_+x}, & x\to+\infty, \end{cases} \tag{15}$$

where $E=k_-^2+\phi_-^2=k_+^2+\phi_+^2$. Since the eigenstates of $H^-$ and $H^+$ with nonzero energy are related by the operator of $Q$ in Eq. (2), one then has

$$\sqrt{E}\ \psi^-=\begin{cases} i(ik_--\phi_-)e^{ik_-x}+R^+i(-ik_--\phi_-)e^{-ik_-x}, & x\to-\infty \\ i(ik_+-\phi_+)T^+e^{ik_+x}, & x\to+\infty\ . \end{cases} \tag{16}$$

Therefore the Jost ratio $J_R$ is itself a topological invariant given by

$$J_R(E)=\frac{T^+}{T^-}=\left[\frac{ik_--\phi_-}{ik_+-\phi_+}\right] = \frac{(\phi_-^2-E)^{1/2}+\phi_-}{(\phi_+^2-E)^{1/2}+\phi_+}\ . \tag{17}$$

It is remarkable that $J_R$ only depends on the asymptotic values of $\phi(x)$. Indeed the dependence of $J^+$ and $J^-$ on the local details of the background fields $\phi(x)$ cancels in the ratio $J_R$. To obtain $\Delta(z)$ in (10) we continue to negative $E=-z$,

$$(E-\phi^2)^{1/2}=i(z+\phi^2)^{1/2} \tag{18}$$

and finally achieve

$$\Delta(z)=z\frac{d}{dz}\ln J_R(z) = \frac{1}{2}\left[\frac{\phi_+}{(z+\phi_+^2)^{1/2}}-\frac{\phi_-}{(z+\phi_-^2)^{1/2}}\right]\ . \tag{19}$$

This is the expression given by Callias.[2] However, we have obtained it from the knowledge of the Jost function ratio and the behavior of the scattering states.

The functions $(d/dE)\ln J^{\pm}(E)$ have the following structure in the complex $E$ plane: isolated poles (zeros of $J^{\pm}$) at the bound-state energies and a cut for the continuum states starting at threshold $E_T=\min(\phi_-,\phi_+)$. However, the poles for $E\neq0$ cancel between $J^+$ and $J^-$ in $(d/dE)\ln J_R(E)$ due to the property of pairing of the energy levels [see Eqs. (2) and (6)]. The only possible remaining pole is at $E=0$ (a zero of $J^-$ or $J^+$). Now $\Delta(z)$ is analytic in the complex $z$ plane with a cut along the negative real axis from $(-\infty)$ to $z=-E_T$. Any possible pole at $z=0$ is canceled by the factor $z$ in the numerator in (19).

From the expression (19) the index is obtained as the $\lim_{z\to 0^+}\Delta(z)$. However, if we want to study the limiting case $\phi_+=0$ or $\phi_-=0$, the correct prescription is to keep $z$ finite, let $\phi_+$ or $\phi_-$ go to zero, and then let $z\to 0^+$. Since $\Delta(z)$ is analytic in the cut $z$ plane and vanishes for large $z$, it satisfies an unsubtracted dispersion relation of the form

$$\Delta(z)=\frac{1}{2\pi i}\oint_c \frac{\Delta(z')dz'}{z'-z}\ , \tag{20}$$

where the contour $c$ runs above the cut on the negative real axis from $z=-\infty$ to $z=-E_T$, around the edge of the cut, and then below the cut from $z=-E_T$ to $z=-\infty$. For $E_T\neq 0$ ($\phi_+\neq 0$ and $\phi_-\neq 0$) $J_R$ does not have a zero (or a pole) at the edge of the cut because of the pairing property. Therefore, the circle at the edge of the cut does not contribute to (18). Furthermore, even if $E_T=0$ ($\phi_+=0$ or $\phi_-=0$) and $J_R$ either vanishes or has a pole there, it can be seen that the circle around the edge of the cut still does not contribute to (20) for finite $z>0$.

Now above the cut, one has

$$\ln J_R(z)=\ln|J_R(z)|-i\delta_R(-z)\ , \tag{21}$$

where $\delta_R=\delta^- -\delta^+$ is the relative phase shift between the scattering states of $H^-$ and $H^+$. Therefore, for finite $z$, $\Delta(z)$ becomes

$$\Delta(z)=\frac{-1}{\pi}\int_{E_T}^{\infty}dE'\frac{E'}{E'+z}\frac{d}{dE'}\delta_R(E')\ , \tag{22}$$

and the index is

$$\Delta(0^+)=\frac{1}{\pi}[\delta_R(k=0)-\delta_R(k=\infty)]\ . \tag{23}$$

To relate (23) to the number of bound states, we use Levinson's theorem.[5,6] The standard proof proceeds by simply observing that the quantity $(d/dz)\ln J_R(z)$ has poles at the (zero-energy) bound states of $H^-$ and $H^+$. If $E_T\neq 0$, these poles are *within* the contour $c$ of Eq. (20) and we obtain

$$\oint_c \frac{d}{dz'}\ln J_R(z')dz'=\frac{1}{\pi}[\delta_R(k=0)-\delta_R(k=\infty)]$$
$$=n_B^-(E=0)-n_B^+(E=0)\ , \tag{24}$$

where $n_B^{\pm}(E=0)$ is the number of zero-energy bound states of $H^+$ and $H^-$.

However, *when* $E_T=0$ the situation is different. The contour $c$ now extends to $z'=0$ and $J_R$ has structure at $z'=0$. Indeed if $\phi_-=0$ then $J_R(z')$ vanishes as $\sqrt{z'}$ or if $\phi_+=0$ $J_R{}^{-1}(z')$ vanishes as $\sqrt{z'}$. Since there are no bound states within $c$, the contour integral in (24) has a contribution both from the continuum cut *and* from the small circle around the origin that yields a factor $\pm\frac{1}{2}$ (arising from the $\sqrt{z'}$ singularity) and we find

$$\oint_c \frac{d}{dz'}\ln J_R(z')dz'=\frac{1}{\pi}[\delta_R(k=0)-\delta_R(k=\infty)]\mp\tfrac{1}{2}$$
$$=0\ , \tag{25}$$

where the $(-)$ sign corresponds to $\phi_-=0$ and the $(+)$ sign to $\phi_+=0$. Levinson's theorem allows us to relate the behavior of the relative phase to the number of bound states.

Therefore, the index becomes

$$\Delta(0^+)=\begin{cases} n_B^-(E=0)-n_B^+(E=0), & \text{if } \phi_+\neq 0,\ \phi_-\neq 0\\ \frac{1}{2}, & \text{if } \phi_-=0\\ -\frac{1}{2}, & \text{if } \phi_+=0\ . \end{cases} \tag{26}$$

When $\phi_-=0$ the zero-energy (threshold) state that contributes the factor $+\frac{1}{2}$ to (25) is "half bound" in the sense that its wave function decays exponentially as $x\to+\infty$ and remains constant as $x\to-\infty$ (vice versa for $\phi_+=0$). This then translates into a $\sqrt{z}$ singularity in $J_R$. It is easy to see that this will always be the case when the continuum cut starts at the origin.

In a recent paper Akhoury and Comtet[3] have found that their computation of the index (with heat kernel regularization) was ambiguous in a particular model. The direct calculation of the index did not agree with the evaluation using the standard relation between the phase shifts and the density of state. However, we will show that there is no ambiguity—one must take care when the continuum can contain a threshold "bound state." The density of states is obtained in the following way. Since

$$2\pi i\rho(E)=\operatorname*{Tr}_{\eta\to 0^+}\left[\frac{1}{H-E-i\eta}-\frac{1}{H-E+i\eta}\right]\ , \tag{27}$$

from the definition of $\Delta(z)$ one has

$$[\rho^-(E)-\rho^+(E)]$$
$$=\frac{1}{2\pi i}\left[\left.\frac{\Delta(z)}{z}\right|_{z=-E-i\eta}-\left.\frac{\Delta(z)}{z}\right|_{z=-E+i\eta}\right]\ , \tag{28}$$

where $\rho^{\pm}$ are the density of states of $H^{\pm}$. Thus,

$$[\rho^-(E)-\rho^+(E)]=\delta(E)\Delta(0^+)+\frac{1}{\pi E}\mathrm{Im}\Delta(-E+i\eta)\ . \tag{29}$$

Using the result (19) we find

$$[\rho^-(E)-\rho^+(E)]=\tfrac{1}{2}[\epsilon(\phi_+)-\epsilon(\phi_-)]\delta(E)+\frac{1}{2\pi E}\left[\frac{\phi_-}{(E-\phi_-{}^2)^{1/2}}\theta(E-\phi_-{}^2)-\frac{\phi_+}{(E-\phi_+{}^2)^{1/2}}\theta(E-\phi_+{}^2)\right]\ , \tag{30}$$

where

$$\epsilon(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

and

$$\theta(x)=\begin{cases}1, & x>0\\ 0, & x<0\,.\end{cases}$$

From Eq. (30) we can easily understand why $\Delta(z)$ is generally not independent of $z$ when the spectrum contains a continuous part despite the pairing-of-states argument. We see that the density of states at finite $k$ are different for $H^-$ and $H^+$ as one would expect since their potentials differ.

It is instructive to examine Eq. (30) in a specific case. If $\phi_+$ is positive and $\phi_-$ is negative but smaller in magnitude, then the density of states *below* $E=\phi_+{}^2$ is

$$(\rho^- -\rho^+)=\delta(E)-\frac{1}{2\pi E}\frac{|\phi_-|}{(E-\phi_-{}^2)^{1/2}}\theta(E-\phi_-{}^2)\,,$$

whereas if $\phi_-\equiv 0$, then

$$(\rho^- -\rho^+)=\tfrac{1}{2}\delta(E)\,.$$

Now for finite $|\phi_-|$ the total difference in the number of states between $E=0$ and $\phi_+{}^2$ is

$$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left[\frac{|\phi_-|}{(\phi_+{}^2-\phi_-{}^2)^{1/2}}\right].$$

This is seen to smoothly approach $\frac{1}{2}$ as $|\phi_-|$ goes to zero, and 1 as $|\phi_-|\to\phi_+$. Thus, even though the density of states is quite discontinuous, the relative number of states is smooth. As $\phi_-\to 0$ and $\phi_+\to\infty$ we see from Eq. (30) that there is a "half state" at $E=0$ and a deficit of a half state at $E=\infty$. The former yields a contribution of $\frac{1}{2}$ whereas the latter gives vanishing contribution to the regularized index. This resolves the ambiguity and agrees with our general result (26).

We are now in a position to study the index using the heat kernel regularization. To this purpose we write (following Goldberger[7])

$$\Delta_\beta=\frac{1}{2\pi i}\oint_d e^{\beta z}\mathrm{Tr}\left[\frac{1}{H^-+z}-\frac{1}{H^++z}\right]dz\,, \qquad (31)$$

where the contour $d$ extends above the negative real axis from $z=0$ to $z=-\infty$ and returns below the axis closing at $z=0^+$. Then it is easy to see that Eq. (31) can be rewritten using Eq. (28) as

$$\Delta_\beta=\int_0^\infty e^{-\beta E}[\rho^-(E)-\rho^+(E)]\,, \qquad (32)$$

where $[\rho^-(E)-\rho^+(E)]$ is given by Eq. (30). A simple calculation yields

$$\Delta_\beta=\tfrac{1}{2}[\epsilon(\phi_+)-\epsilon(\phi_-)]+\tfrac{1}{2}\epsilon(\phi_-)\mathrm{erfc}((\beta\phi_-{}^2)^{1/2}) -\tfrac{1}{2}\epsilon(\phi_+)\mathrm{erfc}((\beta\phi_+{}^2)^{1/2})\,. \qquad (33)$$

The index is obtained as the limit $\beta\to\infty$. From (33) it is seen to coincide with $\Delta(0^+)$ in (19) and expressions (26). The dependence of $\Delta_\beta$ on $\beta$ is again due to the difference in the density of states of $H^+$ and $H^-$.

The limit of $\beta\to 0$ in (31) should give the formal expression $\mathrm{Tr}(-1)^F$ [same as $z\to\infty$ of $\Delta(z)$]. However, it is exactly zero because the total deficit of states (including zero modes) between $H^-$ and $H^+$ is zero, i.e., there are the same number of states in both. This is dictated by Levinson's theorem.

It is amusing that from Eq. (31), the Witten index can be interpreted as the difference of the second virial coefficients for particles whose interactions are given by $H^-$ and $H^+$.[7,8]

Although we have concentrated on one-dimensional examples, our results can be easily extended to three dimensions. The only difference is that if $J_R$ in (21) vanishes or is singular at $z'=0$, the extra $\frac{1}{2}$ only arises from the $S$-wave phase shifts[5] (in which case the state is half bound as in the one-dimensional case). For $l\geq 1$, it is a *true* normalizable bound state with a contribution of $\pm 1$.[5]

Therefore, we conclude that the index is an integer (or zero) whenever there is a mass gap in the theory. It is half-integer if the continuum extends to zero, indicating that there is a zero-energy (bound or resonant) state. Even though the index does not count states properly, only Levinson's theorem does this, it still contains the information about the existence of zero-energy states.

*Note added in proof.* The following references should also be mentioned: M. Hirayama, Prog. Theor. Phys. **70**, 1444 (1983); S. Cecotti and L. Girardello, Nucl. Phys. **B239**, 573 (1984); C. Imbimbo and S. Mukhi, Max-Planck Institute Report No. MPI-PAE 88/83 (unpublished); and M. Stone, Ann. Phys. (N.Y.) **155**, 56 (1984).

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