Large-N baryonic soliton and quarks

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Starting from Witten's large-N power counting we derive an equation identical to the so-called bootstrap condition of strong-coupling theory. The large-N baryons are therefore characterized by representations of the strong-coupling group (SCG). It is pointed out that the bootstrap relation is quite general and valid when the semiclassical expansion about soliton solutions is at work. The collective coordinates of the soliton correspond to the coordinates of induced representations of the SCG. One of the interesting representations of the SCG is the quark representation and this makes a bridge between the Skyrme solitons and the nonrelativistic quark model. We explicitly show that the induced representation is derived from N static quarks with $N\rightarrow\infty$. We further emphasize the generality and power of the algebraic method. For this purpose we present a modified chiral bag model which exhibits the algebraic relations in large N and approaches the Skyrme-soliton picture in the zero-bag-radius limit.

I. INTRODUCTION

In a recent paper[1] we derived exact equations for QCD to leading order in the number N of colors. Our starting point was Witten's analysis of baryon dynamics in the large-N limit[2] which shows that baryons are similar to solitons if one replaces the parameter of the semiclassical expansion by $1/N$.

In effect, the QCD baryons can be recovered as solitons of an effective chiral Lagrangian for which, the soliton effective action being of order N, one can indeed regard the semiclassical expansion parameter to be $1/N$.

Witten's identification[3] of Skyrme solitons[4] with QCD baryons is very beautiful but such a topological miracle calls, in our opinion, for a more physical understanding. In particular, since Skyrme solitons are similar to 't Hooft-Polyakov monopoles, they are somewhat remote from the naive quark bound-state picture. This situation raises a general problem which is quite interesting by itself. We have a theory (here QCD) with a parameter β (here $\beta=1/\sqrt{N}$) such that for $\beta\rightarrow 0$ the spectrum involves particles (here the baryons) which behave exactly as solitons. However, β is not the semiclassical expansion parameter of the original Lagrangian (QCD), rather it would be that of the effective Lagrangian, which we do not know *a priori*. The Skyrme-soliton model of Witten and his collaborators[5] is a specific model in which the effective Lagrangian is given by the chiral Lagrangian. Another possible effective Lagrangian model would be the chiral bag model[6] and probably some others. We believe, however, there should be some general features applied to all of these possible models.

The relevant features of the $\beta\rightarrow 0$ limit is as follows. There are heavy particles of mass $O(\beta^{-2})$ (the would-be solitons) and light particles of mass $O(\beta^0)$, strongly interacting with a coupling of order β^{-1}. This is reminiscent of the old static strong-coupling theories.[7] These were studied by algebraic methods[8,9] which as we pointed out in Ref. 1 can be applied, to leading order in β^{-1}, to any theory where the above small-β behavior is realized without making use of any specific solution of classical field equations. Hence one can directly study the baryons in QCD to leading order in N.

In Ref. 1 we only gave a brief account of our ideas. One aim in the present paper is to provide more pedagogical details. As we pointed out in Ref. 1, one solution of the large-N algebraic equations can be identified with the static quark model. Hence this is one step toward bridging the above-mentioned gap between large-N Skyrme solitons and the static quark model.[10] In general the above solution gives a quark picture of the would-be solitons which we further develop in the rest of the paper. The algebraic equations are of the same nature as the equations first derived in the static theories.[9] They are Lie algebra commutation relations associated with noncompact groups $K\times T$, semidirect products of the spin $\times$ flavor group K with a commutative group T. The representations of such algebras are naturally described by the induced-representation method,[11] where T is diagonalized. This corresponds to the usual picture of solitons. The collective coordinates of solitons[12] play the role of coordinates of the induced representation. Starting from the above-mentioned quarklike solution we develop an alternative method where the induced representation is derived from N static quarks with $N\rightarrow\infty$. This establishes a link between collective-coordinate methods and quark models.

II. SEMICLASSICAL APPROACH REVISITED

In order to facilitate the forthcoming discussion it is useful to review the essential features of the semiclassical approach by looking at simple field-theory models. First consider a two-dimensional theory with a spin-zero field $\phi(x,t)$ and a Lagrangian density

$$\mathscr{L}=\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2-\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2-\frac{1}{\beta^2}V[\beta\phi(x,t)] \ . \quad (2.1)$$

The coupling constant β has been defined (V is a function of one variable) in such a way that the semiclassical expansion parameter is $\hbar \sim \beta^2$. The static soliton solution, if it exists, has the form $\phi_s = (1/\beta)\phi_0$ with ϕ_0 independent of β. Its energy,

$$M_s = \frac{1}{\beta^2} \int dx \left[\frac{1}{2} \left[\frac{d\phi_0}{dx} \right]^2 + V[\phi_0] \right] , \tag{2.2}$$

is of order β^{-2}. It is equal to the mass of the quantized soliton to leading order in β^{-1}. For small β, the quantum theory describes a very massive particle, the soliton, coupled to the particle with mass of order 1 (say a meson of mass m) associated with ϕ. To leading order in β^{-1} one has[13,14]

$$\begin{aligned} \langle p' \,|\, \phi(0) \,|\, p \rangle &\sim \frac{1}{\beta} \int dx \, e^{-i(p-p')x} \phi_s(x) \\ &\equiv \frac{1}{\beta} \Gamma((p-p')^2) , \end{aligned} \tag{2.3}$$

where $|p\rangle$ and $|p'\rangle$ are quantum soliton states. The soliton-meson coupling constant

$$\frac{1}{\beta} \mathscr{A} \sim \lim_{q^2 \to -m^2} (q^2 + m^2) \Gamma(q^2) \frac{1}{\beta} \tag{2.4}$$

is of order β^{-1}. One can indeed verify that, quite generally,[13] the Fourier transform Γ of ϕ_s has a pole at $q^2 = -m^2$ so that $\mathscr{A}$ is not trivially zero. For small β the meson-soliton coupling is strong. On the other hand, the soliton is very massive and its recoil is small. Hence the meson-soliton interaction can be described by the nonrelativistic kinematics. In particular it has been shown[13] that to leading order in $1/\beta$ the meson-soliton elastic scattering amplitude $\mathscr{T}$ has the Born terms

$$\mathscr{T}^{(B)} = \frac{\mathscr{A}^2}{\beta} \left[\frac{1}{\omega - M_s + E_{p'}} - \frac{1}{\omega + M_s - E_{p'}} \right] \tag{2.5}$$

which correspond to the two possible one-soliton intermediate states. p, k and p', k' are the initial and final soliton and meson momenta. Moreover

$$\omega = (k'^2 + m^2)^{1/2}, \quad E_{p'} = (p'^2 + M_s^2)^{1/2} . \tag{2.6}$$

More generally one can show[13,14] that the reaction n mesons+soliton$\to m$ mesons+soliton has a connected scattering amplitude of order β^{m+n-1} at most. Hence (2.5) is the dominant term for the two-body elastic amplitude.

The above features are quite general and hold in any soliton model. The only particular point of the above model is that the soliton has no internal degrees of freedom since the theory has no internal symmetry. Since the soliton is very heavy and since its coupling with mesons is strong, the situation is reminiscent of the static strong-coupling models. These models have indeed rather simple nontrivial classical solutions which one may regard as the simplest examples of a soliton with internal degrees of freedom.

We now turn to one such model which is the so-called static charge-symmetric scalar meson theory described by the Lagrangian

$$\begin{aligned} L = &- \int d\vec{x} \left[\tfrac{1}{2} (\partial_\mu \phi^\alpha)^2 + \frac{m^2}{2} (\phi^\alpha)^2 \right] \\ &+ g \left[\sum_\alpha \left[\int d_3x \, \rho(x) \phi^\alpha \right]^2 \right]^{1/2} . \end{aligned} \tag{2.7}$$

ϕ^α is an isovector and ρ is a given c-number source function. We take it to depend on $\vec{x}^2$ only. For large g the power counting is the same as in the above typical soliton model if we let $g \sim 1/\beta$, and the theory can indeed be solved by expansion around the classical solution

$$\begin{aligned} &\phi_s^\alpha(x) = g v^\alpha u(x), \quad \sum_\alpha (v^\alpha)^2 = 1 , \\ &(m^2 - \nabla^2) u(x) = \rho(x) . \end{aligned} \tag{2.8}$$

v^α is an arbitrary unit vector. The arbitrariness is to be handled by collective coordinates as is well known by now. In this connection, one may recall that when we introduced collective coordinates in relativistic field theory[12] we were inspired by previous treatments[15,16] of static strong-coupling models. In order to illustrate our forthcoming discussion, it is useful to recall briefly the highlights of the collective-coordinate method on the typical example (2.7).

One performs a change of dynamical variables by letting at the quantum level

$$\phi = \phi_s^{[\Omega]} + \tilde{\phi}^{[\Omega]} , \tag{2.9}$$

where the new field $\tilde{\phi}$ satisfies

$$\sum_\alpha \int d_3x \, \tilde{\phi}^\alpha(x,t) \left[\frac{\partial \phi_s^{\alpha[\Omega]}}{\partial \Omega^\alpha} \right]_{\Omega=0} = 0 . \tag{2.10}$$

$\phi^{[\Omega]}$ denotes the results of applying a rotation in isospace on ϕ with parameters Ω. The parameters are dynamical variables and hence time dependent. This is consistent since (2.10), which serves to eliminate the zero modes, shows that $\tilde{\phi}$ describes fewer degrees of freedom than ϕ. To leading order in g one can drop $\tilde{\phi}$ and simply substitute $\phi_s^{[\Omega]}$, $\Omega \neq 0$, in (2.7). One obtains

$$L = -M_s + g^2 \tfrac{1}{2} \sum_\alpha (\dot{y}^\alpha)^2, \quad y = v^{[\Omega]} . \tag{2.11}$$

M_s is the classical energy of the solution (2.7), it is of the order of g^2. Apart from this term, (2.11) describes the free point particle of mass g^2 on the unit sphere. The corresponding Hamiltonian is

$$H = M_s + \frac{1}{2g^2} \vec{L}^2 , \tag{2.12}$$

where $\vec{L}$ is the usual isospin angular momentum operator. The Hamiltonian (2.12) acts on the space of square-integrable functions $\psi[\Omega]$ on the unit sphere. The spectrum is obviously given by

$$H\,|\,l,m\,\rangle=\left[M_s+\frac{l(l+1)}{2g^2}\right]|\,l,m\,\rangle\ ,\quad 0\le l<\infty\ . \tag{2.13}$$

There is an infinite number of very massive states (the so-called isobars) which become degenerate in the $g\to\infty$ limit.

The isobar-meson coupling is obtained in a way similar to the previously discussed relativistic case, namely,

$$\langle p',l'm'\,|\,\hat{\phi}^{\alpha}(0)\,|\,p,lm\,\rangle\cong\frac{1}{\beta}\int d_3x\,e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{x}}\langle l'm'\,|\,[\phi_s^{[\hat{\Omega}]}(x)]^{\alpha}\,|\,lm\,\rangle\equiv\frac{1}{\beta}\langle l'm'\,|\,\Gamma^{\alpha}\,|\,lm\,\rangle\ , \tag{2.14}$$

where $\hat{\Omega}$ are the quantum collective-coordinate operators. By using the standard formulas

$$\langle\Omega\,|\,lm\,\rangle=Y_l^m(\Omega),\quad \hat{\Omega}\,|\,\Omega\,\rangle=\Omega\,|\,\Omega\,\rangle\ ,$$

one obtains

$$\langle\,l'm'\,|\,\Gamma^{\alpha}\,|\,lm\,\rangle=\frac{\tilde{\rho}(q^2)}{m_\pi{}^2+\vec{q}\,^2}\int d\Omega(v^{[\Omega]})^{\alpha}Y_{l'}^{m'*}(\Omega)Y_l^m(\Omega)\ , \tag{2.15}$$

and the meson-isobar coupling constants read

$$\langle\,l'm'\,|\,\mathscr{A}^{\alpha}\,|\,lm\,\rangle\sim\tilde{\rho}(-m_\pi{}^2)\langle\,l'm'\,|\,(v^{[\hat{\Omega}]})^{\alpha}\,|\,lm\,\rangle\ . \tag{2.16}$$

From this formula it is clear that $\mathscr{A}^{\alpha}$ is diagonal in the basis where the operator $\hat{\Omega}$ is diagonal and one can write

$$\mathscr{A}^{\alpha}\,|\,\Omega\,\rangle=(\mathscr{A}_0^{[\Omega]})^{\alpha}\,|\,\Omega\,\rangle\ , \tag{2.17}$$

where $\mathscr{A}_0$ is proportional to v. Formula (2.17) shows that, considered as operators on the isobar internal space, the coupling operators $\mathscr{A}^{\alpha}$ commute, namely,

$$[\mathscr{A}^{\alpha},\mathscr{A}^{\beta}]=0\ . \tag{2.18}$$

This example is typical of the general case of solitons in theories with internal symmetries. One obtains infinite towers of very heavy particles which are degenerate to leading order. The internal space of solitons is described by square-integrable functions of the collective coordinates, associated with the internal symmetries of the theory which are broken by the classical solution. Formulas (2.17) and (2.18) hold in the corresponding Hilbert spaces.

III. THE ALGEBRAIC METHOD

In the previous section we have summarized the essential properties of a quantum theory where semiclassical expansion around soliton solutions is at work.

In the strong-coupling theory of Goebel[8] and its algebraic formulation[9] one reverses the argument so that one does not explicitly assume the existence of soliton solutions. Although this formalism has already been fully developed for the strong-coupling theory, we go through the derivation again to make this paper reasonably self-contained and to stress the relationship with semiclassical techniques.

The basic idea of the algebraic method is simple and applicable in any theory where the semiclassical power counting holds. Consider a theory with a typical parameter β such that for small β there exists a family of heavy particles of mass $M\sim 1/\beta^2$ and a family of "light" particles of mass $m\simeq\beta^0$. To leading order the elastic scattering of a light particle by a heavy particle occurs mostly in the lowest possible partial wave $\mathscr{T}_{JI}^{\beta\alpha}$, where I,J and α,β characterize the initial and final states of the heavy and light particles, respectively.

Assume further that the static coupling between two heavy and one light particle is of order β^{-1} and can thus be written as $(1/\beta)\mathscr{A}_{JI}^{\alpha}$ with $\mathscr{A}_{JI}^{\alpha}$ independent of β to leading order. The Born term to leading order is similar to (2.5):

$$\mathscr{T}_{JI}^{(B)\beta\alpha}\cong\frac{1}{\beta^2}\sum_L\left[\frac{\mathscr{A}_{JL}^{\beta}\mathscr{A}_{LI}^{\alpha}}{\omega+M_I-M_L}-\frac{\mathscr{A}_{JL}^{\alpha}\mathscr{A}_{LI}^{\beta}}{\omega-M_J+M_L}\right]\ , \tag{3.1}$$

where M_I is the mass of the heavy particle with quantum numbers I. Assume further that the reaction "1 heavy particle $+n$ light particles $\rightarrow$ 1 heavy particle $+m$ light particles" has a connected amplitude of order β^{m+n-1}. It then follows that (3.1) gives the dominant contribution which is of order β^{-2}. Such a term cannot be present, however, since from unitarity $\mathscr{T}$ is bounded for arbitrary β. Hence the two terms of (3.1) must cancel identically. Therefore the mass differences between heavy particles must vanish as $\beta\rightarrow 0$. Moreover,

$$\sum_L(\mathscr{A}_{JL}^{\beta}\mathscr{A}_{LI}^{\alpha}-\mathscr{A}_{JL}^{\alpha}\mathscr{A}_{LI}^{\beta})=0\ . \tag{3.2}$$

Hence the β power counting and unitarity alone ensure that the coupling constants $\mathscr{A}$ considered as matrices in the heavy-particle internal space commute as was already found by explicit computation in the standard soliton case. A typical situation where the β power counting holds in the absence of a true classical solution is when the heavy particles can be associated with minima of effective actions with typical parameter β^{-2}. We now briefly describe an example of this situation. Consider a model of an isovector $\vec{\phi}$ field interacting with N static isospinor quarks $\psi^{(K)}$. The Hamiltonian reads

$$H=\tfrac{1}{2}\int d_3x[\vec{\pi}^2+(\nabla\vec{\phi})^2+\mu^2\vec{\phi}^2]+\lambda\sum_K(\psi^{(K)\dagger}\vec{\tau}\cdot\vec{M}[\phi]\psi^{(K)})\ , \tag{3.3}$$

$$\vec{M}[\phi]=\int d_3x\,\rho(x)\vec{\phi}(x)\ . \tag{3.4}$$

We study it in the Schrödinger representation where $\vec{\phi}(\vec{x},0)\equiv\vec{\phi}(\vec{x})$ is a c number. For a given such function we introduce the two-component eigenvectors $u_{\pm}[\phi]$ of $\vec{\sigma}\cdot\vec{M}$,

$$\vec{\sigma}\cdot\vec{M}u^{(\pm)}=\pm(\vec{M}^2)^{1/2}u^{(\pm)}\ . \tag{3.5}$$

After expanding $\psi^{(K)}$ in terms of u,

$$\psi_i^{(K)} = \sum_{\alpha=\pm} q_\alpha^{(K)} u_i^{(\alpha)} ,$$

the interaction Hamiltonian becomes

$$\begin{aligned} H_I &= \lambda \sum_k \psi^{(K)\dagger} \vec{\tau} \cdot \vec{\mathrm{M}} \psi^{(K)} \\ &= \lambda (\vec{\mathrm{M}}^2)^{1/2} \sum_k (q_+^{(K)\dagger} q_+^{(K)} - q_-^{(K)\dagger} q_-^{(K)}) . \end{aligned} \tag{3.6}$$

The lowest-energy eigenstates have a quark part of the form $\prod_K (q_-^{(K)\dagger}) \,|\, 0 \rangle$ and, applied on this state, H_I takes the form

$$H_I \prod_K (q_-^{(K)\dagger}) \,|\, 0 \rangle = -N\lambda \left[\left[\int d_3 x \, \rho \vec{\phi} \right]^2 \right]^{1/2} \prod_K (q_-^{(K)\dagger}) \,|\, 0 \rangle . \tag{3.7}$$

We recover the model (2.7) with $g = N\lambda$. For $N \to \infty$ this effective theory has true semiclassical solitons as we recalled in Sec. II.

Going back to the general discussion we introduce the internal symmetry group K of $\mathscr{A}$. Let $\mathscr{I}^a$ and $D(a)^\alpha_\beta$ denote the infinitesimal generators of the representations spanned by the heavy- and light-particle multiplets. The content of the algebraic method is finally summarized by the equations

$$\begin{aligned} &[\mathscr{I}^a, \mathscr{I}^b] = i f^{abc} \mathscr{I}^c , \\ &[\mathscr{A}^\alpha, \mathscr{I}^a] = i D(a)^\alpha_\beta \mathscr{A}^\beta , \\ &[\mathscr{A}^\alpha, \mathscr{A}^\beta] = 0 . \end{aligned} \tag{3.8}$$

The power of these equations lies in their Lie algebra structure. Indeed they show that $\mathscr{I}^a$ and $\mathscr{A}^\alpha$ generate the algebra of the noncompact group $K \times T$, the semidirect product of K by the Abelian group T generated by it. As the above argument shows, (3.8) is a consequence of unitarity and small-β behavior alone. As is clear from Sec. II, Eq. (3.8) holds in the case of solitons. In QCD, Witten's analysis of the large-N limit shows that we can apply the above discussion if we replace β by $1/\sqrt{N}$.

The algebra (3.8) is similar to the Poincaré algebra, the coupling constant $\mathscr{A}^\alpha$ playing the role of the translations. One can thus apply the theory of induced representations[11,17] where one diagonalizes the Abelian part $\mathscr{A}^\alpha$. For the soliton case, this corresponds to the basis where the collective-coordinate operators are diagonal [see (2.17)], a semiclassical expansion around c-number classical solutions. There, however, the internal-symmetry-group properties of the heavy particle are not explicit. These become transparent only if we expand the diagonal state in terms of irreducible representations of K. For the static model of Sec. II this was done by expanding in terms of spherical harmonics [see (2.15)]. In general there appears an infinite tower of irreducible representations.

This problem can also be solved in all practical cases by the method of group contraction[9,18] which we now recall. First we note that in Eq. (3.8) we have to specify the representation generated by $D(a)$ in order to set up the problem. Physically this means that we have to specify the internal quantum numbers of the light particle. We shall discuss only the most common case where K is the group $SU_p \otimes SU_q$ and $D(a)$ is in the direct product of adjoint representation of SU_p and that of SU_q. This covers most of the practical cases. With this choice, the number of $\mathscr{A}^\alpha$ operators is $(p^2-1)(q^2-1)$ and the algebra (3.8) involves altogether

$$(p^2-1)(q^2-1) + (p^2-1) + (q^2-1) = (pq)^2 - 1 \text{ operators} .$$

This is the same number as for the SU_{pq} algebra and solution of (3.8) can be obtained by the method of group contraction of SU_{pq}. The general idea is as follows. Consider a representation of SU_{pq} where the generators of the $SU_p \otimes SU_q$ subgroup are $\mathscr{I}^a$. Denoting by $\mathscr{B}^\alpha$ the other generators, we can write

$$\begin{aligned} &[\mathscr{I}^a, \mathscr{I}^b] = i f^{abc} \mathscr{I}^c , \\ &[\mathscr{B}^\alpha, \mathscr{I}^a] = i D(a)^\alpha_\beta \mathscr{B}^\beta , \\ &[\mathscr{B}^\alpha, \mathscr{B}^\beta] = i (C^{\alpha\beta}_\gamma \mathscr{B}^\gamma + \tilde{C}^{\alpha\beta}_a \mathscr{I}^a) , \end{aligned} \tag{3.9}$$

where $C^{\alpha\beta}_\lambda$, $\tilde{C}^{\alpha\beta}_a$ are structure constants. The representations of $(SU_p \otimes SU_q) \times T$ are derived from representations of SU_{pq} as follows. Consider a family of representations of SU_{pq} depending on a parameter ϵ and such that for $\epsilon \to 0$ the matrix elements of $\mathscr{B}^\alpha$ are of order ϵ^{-1} at most. If we write

$$\mathscr{B} = \frac{\mathscr{A}}{\epsilon} + O(1) ,$$

$\mathscr{I}$ and $\mathscr{A}$ generate the algebra (3.8) of $(SU_p \otimes SU_q) \times T$ and have finite matrix elements for $\epsilon \to 0$. There are extensive discussions of the above procedure in the literature.[18] For the physical problem at hand, one particular representation which we call the quark representation is especially appealing. Indeed, consider the case of QCD where we can take the light particle to be the pseudoscalar mesons. The elastic scattering amplitude $\mathscr{I}^{\beta\alpha}_{JI}$ is then a P wave and the meson-hadron coupling is derivative. Hence (3.2) is to be replaced by

$$\sum_L [(k'^j \mathscr{A}^{\beta j}_{JL})(k^i \mathscr{A}^{\alpha i}_{LI}) - (k^i \mathscr{A}^{\alpha i}_{JL})(k'^j \mathscr{A}^{\beta j}_{LI})] = 0 ,$$

where the sum over L also includes the sum over the hadron spin. Since the meson momenta $\vec{k}, \vec{k}'$ are arbitrary, we get again (3.8) where $\mathscr{A}^\alpha$ is replaced by $\mathscr{A}^{\alpha i}$ and where

$$K = (SU_2)_{\text{spin}} \otimes (SU_n)_{\text{flavor}} . \tag{3.10}$$

Taking $n=3$ we see that (3.8) is obtained from the contraction of SU_6. The representations of SU_6 are characterized by five integers $\lambda_1, \ldots, \lambda_5$. Consider the completely symmetric representation $(\lambda,0,0,0,0)$. For fixed λ it is made up with all the symmetric states of λ quarks, and if we identify λ with the number N of color we recover the nonrelativistic SU_6 quark model. For $N \to \infty$ this representation becomes a representation of $(SU_2 \otimes SU_3) \times T$. This will be shown in detail in the next section.

IV. QUARK REPRESENTATION

The idea of induced representations of strong-coupling groups was described in the previous section and for the readers who are not familiar with this subject we shall review the content of Ref. 11 in the Appendix. In this section we assume that the reader has some familiarity with induced representations and proceed with the discussions on the quark representation introduced at the end of the previous section.

According to the induced-representation theory,[17] each irreducible unitary representation of $G=K\times T$, the semidirect product of a compact group K by a translation group T, is characterized by an orbit of the vectors $\mathscr{A}_i$, eigenvalues of T, and an irreducible unitary representation of the little group associated with the orbit.

In general the orbits are specified by the values of invariants. For example, in the case of $K=\mathrm{SU}_2$ and $T=T_3$ (i.e., $G=\mathrm{E}_3$, the Euclidean group in three dimensions) the invariant is given by A_i^2. For more complex cases there are several invariants and the most general orbits are described by a set of *uncorrelated* values of these invariants. The little group associated with these general orbits is in general uniquely determined. However, for a certain special orbit which is specified by a set of specifically *correlated* values of invariants the corresponding little group is larger than the general case.

Let us take an example of $K=\mathrm{SU}_2\otimes\mathrm{SU}_2$ and $T=T_9$, which are generated by $A_{i\alpha}$ ($i=1,2,3$ and $\alpha=1,2,3$). The invariants are given by

$$\mathscr{P}_2=A_{i\alpha}A_{i\alpha},\quad \mathscr{P}_3=\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}A_{i\alpha}A_{j\beta}A_{k\gamma}\ ,$$
$$\mathscr{P}_4=A_{i\alpha}A_{i\beta}A_{j\beta}A_{j\alpha}\ . \tag{4.1}$$

The general orbits are therefore specified by three uncorrelated real numbers: $\mathscr{P}_2$, $\mathscr{P}_3$, and $\mathscr{P}_4$. The little group in this case is trivial, i.e., identity. One of the special orbits is the orbit given by

$$\mathscr{P}_2=\mathscr{P}_4=1,\quad \mathscr{P}_3=0\ . \tag{4.2}$$

This orbit contains a point $\mathring{A}_{i\alpha}=\delta_{i1}\delta_{\alpha 1}$ and the little group associated with it is $\mathrm{U}(1)\otimes\mathrm{U}(1)$. Another interesting orbit is given by

$$\mathscr{P}_2=\mathscr{P}_4=3,\quad \mathscr{P}_3=6\ , \tag{4.3}$$

which contains a point $\mathring{A}_{i\alpha}=\delta_{i\alpha}$. The little group in this case is SU(2), which is generated by $J_i+\mathring{A}_{i\alpha}I_\alpha$.

Another interesting way to specify these special orbits is to express $\mathscr{A}^\alpha$ in terms of other commuting group operators of fewer degrees of freedom. For the case of $\mathrm{SU}_2\otimes\mathrm{SU}_2\times T_9$ we set

$$A_{i\alpha}=\xi^+\sigma_i\tau_\alpha\xi\ , \tag{4.4}$$

where ξ transforms as a spinor by SU(2)⊗SU(2) and $\xi^\dagger$ is a conjugate of ξ. We assume ξ and $\xi^\dagger$ are commuting operators. Equation (4.4) is a construction of $\mathrm{SU}_2\otimes\mathrm{SU}_2\times T_9$ algebra from an algebra generated by $\mathrm{SU}_2\otimes\mathrm{SU}_2$ generators and commuting ξ and $\xi^\dagger$. Since the conditions

$$\xi^\dagger\sigma_i\xi=0,\quad \xi^\dagger\tau_\alpha\xi=0\ ,\quad i=1,2,3,\quad \alpha=1,2,3\ , \tag{4.5}$$

are invariant by $\mathrm{SU}_2\otimes\mathrm{SU}_2$, they define an orbit. It is not difficult to prove that (4.4) and (4.5) together with the normalization $\xi^\dagger\xi=1$ lead to (4.3).

For the orbit (4.3) the little group is SU(2). Therefore, if we specify a representation for SU(2) we obtain an induced representation of the strong-coupling group $\mathrm{SU}(2)\otimes\mathrm{SU}(2)\times T_9$. If we choose the identity representation for the little group, we obtain the $J=I$ series of isobars. Since this representation is obtained from a symmetric quark representation of SU(4) by contraction, we call it the quark representation.

Let us denote by $q_A^{\dagger(K)}$ a creation operator of a static quark. We use K for color index and A for spin and isospin. The symmetric color-singlet state of N quarks is given by

$$|A_1A_2\cdots A_N\rangle = \frac{1}{N!}\epsilon_{k_1K_2\cdots K_N}q_{A_1}^{(K_1)\dagger}q_{A_2}^{(K_2)\dagger}\cdots q_{A_N}^{(K_N)\dagger}|0\rangle\ . \tag{4.6}$$

This is a symmetric state of N indices and it contains $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, $\ldots$, $(N/2,N/2)$ series of SU(2)⊗SU(2) states for odd N.

Let us multiply $\xi_{A_1}\xi_{A_2}\cdots\xi_{A_N}$ to (4.6) and define

$$|\xi\rangle=C_N(q^{\dagger(1)}\xi)(q^{\dagger(2)}\xi)\cdots(q^{\dagger(N)}\xi)|0\rangle\ , \tag{4.7}$$

where C_N is a normalization constant to be fixed later. The matrix elements of SU(4) generators defined by

$$J_i=\sum_K q^{\dagger(K)}\sigma_i q^{(K)},\quad I_\alpha=\sum_K q^{\dagger(K)}\tau_\alpha q^{(K)}\ ,$$
$$B_{i\alpha}=\sum_K q^{\dagger(K)}\sigma_i\tau_\alpha q^{(K)} \tag{4.8}$$

are then given by

$$\langle\xi'|J_i|\xi\rangle=NC_N^2(\xi'^\dagger\sigma_i\xi)(\xi'^\dagger\xi)^{N-1}\ ,$$
$$\langle\xi'|I_\alpha|\xi\rangle=NC_N^2(\xi'^\dagger\tau_\alpha\xi)(\xi'^\dagger\xi)^{N-1}\ ,$$
$$\langle\xi'|B_{i\alpha}|\xi\rangle=NC_N^2(\xi'^\dagger\sigma_i\tau_\alpha\xi)(\xi'^\dagger\xi)^{N-1} \tag{4.9}$$

for normalized ξ and ξ', i.e.,

$$\xi^\dagger\xi=\xi'^\dagger\xi'=1\ , \tag{4.10}$$

$$|\xi'^\dagger\xi|\le 1\ . \tag{4.11}$$

Therefore, in the large-N limit the matrix elements (4.9) are nonzero only when ξ' is a neighbor of ξ.

At this point we choose ξ which satisfies (4.5). Then, ξ is specified by the coset parameters, which we simply denote by Ω. So from now on we write ξ and ξ' as $\xi(\Omega)$ and $\xi(\Omega')$, and further $|\xi\rangle$ and $|\xi'\rangle$ by $|\Omega\rangle$ and $|\Omega'\rangle$, respectively.

We shall prove later that the normalization constant can be chosen such that

$$\lim_{N\to\infty}[\xi^\dagger(\Omega')\xi(\Omega)]^N C_N^2=\delta(\Omega',\Omega)\ , \tag{4.12}$$

where $\delta(\Omega',\Omega)$ is a δ function in the coset space with Haar measure (see Appendix). A simple proof will be given by considering the integral

$$\int d\Omega' F(\Omega')[\xi^\dagger(\Omega')\xi(\Omega)]^N \tag{4.13}$$

and evaluating it for large N by the standard saddle-point approximation. We simply mention that at this point the conditions (4.5) are crucial for the proof.

Using (4.12) we obtain

$$\langle \Omega' | \Omega \rangle \cong \delta(\Omega',\Omega) , \tag{4.14}$$

$$\langle \Omega' | J_i | \Omega \rangle \cong i\Delta_i \delta(\Omega',\Omega) , \tag{4.15a}$$

$$\langle \Omega' | I_\alpha | \Omega \rangle \cong i\Delta_\alpha \delta(\Omega',\Omega) , \tag{4.15b}$$

$$\langle \Omega' | B_{i\alpha} | \Omega \rangle \cong N[\xi^\dagger(\Omega)\sigma_i\tau_\alpha\xi(\Omega)]\delta(\Omega',\Omega) , \tag{4.15c}$$

where Δ_i and Δ_α are Lie derivatives acting on Ω. These expressions nicely exhibit the contraction and the induced representation for the quark representation. Indeed, because of the well-known properties of Lie derivatives (4.15a) and (4.15b) show that J_i, I_α are represented by differential operators acting on Ω which satisfy the $SU_2 \otimes SU_2$ algebra and B transforms as a vector under each SU_2. Moreover it follows from (4.15c) that if we define

$$A_{i\alpha} = \frac{1}{N} B_{i\alpha} , \tag{4.16}$$

we have indeed for $N \to \infty$

$$A_{i\alpha} | \Omega \rangle = [\xi^\dagger(\Omega)\sigma_i\tau_\alpha\xi(\Omega)] | \Omega \rangle . \tag{4.17}$$

This completes the derivation of the induced representation for the $(SU_2 \otimes SU_2) \times T_9$ group.

We finally comment about the general case of $K = SU_p \otimes SU_q$ $(p \le q)$. The construction proceeds along the same path, taking ξ in the fundamental representations of each group, and (4.9) holds if we replace σ_α and τ_α by the matrix λ_α and $\tilde{\lambda}_\alpha$ of SU_p and SU_q. Ω now denotes the parameters of $(SU_p \otimes SU_q)/H$ with H the little group of ξ (i.e., SU_p if $p \le q$). To complete the argument we now give the derivation of Eq. (4.12) suitably generalized. From unitarity we can write

$$[\xi^\dagger(\Omega')\xi(\Omega)] = [\xi^\dagger d(\Omega'')\xi] ,$$

where d is the matrix of the fundamental representation of $SU_p \otimes SU_q$ and

$$d(\Omega'') = d^\dagger(\Omega')d(\Omega) .$$

The formula to be proven reduces to

$$\lim_{N\to\infty} C_N{}^2 \int d\Omega [\xi^\dagger d(\Omega)\xi]^N F(\Omega) = F(0) , \tag{4.18}$$

where F is an arbitrary function on the coset space $SU_p \otimes SU_q / SU_p$. One uses saddle-point methods around $\Omega = 0$ where $| [\xi^\dagger d(\Omega)\xi] |$ is maximum. We recall that d being unitary $| [\xi^\dagger d(\Omega)\xi] | \le 1$. From the very definition of the infinitesimal generators it follows that

$$\frac{\partial}{\partial\Omega^a} \{ \ln[\xi^\dagger d(\Omega)\xi] \} \Big|_{\Omega=0} = i(\xi^\dagger \Sigma_a \xi) ,$$

where Σ is the relevant linear combination of λ and $\tilde{\lambda}$. Hence, in general even though $| \xi^\dagger d(\Omega)\xi |$ is maximum at $\Omega = 0$, its phase is not stationary. For stationarity, we impose that $(\xi^\dagger \Sigma_a \xi) = 0$ for all infinitesimal directions in the group space corresponding to $(SU_p \otimes SU_q)/H$. This also holds trivially in H, however, since H leaves ξ invariant. Hence we have the general condition

$$(\xi^\dagger \Sigma_a \xi) = 0 \tag{4.19}$$

for all generators of K.

Next, the second derivatives

$$\frac{\partial^2}{\partial\Omega_a \partial\Omega_b} \ln[\xi^\dagger d(\Omega)\xi] \Big|_{\Omega=0} = -(\xi^\dagger \Sigma_a \Sigma_b \xi)$$

are negative definite so that the saddle-point method applies and (4.18) holds. Condition (4.19) is sufficient for any group K which is such that the group-contraction method applies.

Consider, for illustration, the case of $K = U_1$, $G = U_1 \times T_2$ which is obtained by group contraction of SU_2. One has, in this case,

$$| \Omega \rangle = \prod_K (e^{i\Omega/2}\xi_+ q_+^{(K)\dagger} + e^{-i\Omega/2}\xi_- q_-^{(K)\dagger}) | 0 \rangle ,$$

$$\langle \Omega' | \Omega \rangle = (\alpha e^{i(\Omega-\Omega')/2} + \beta e^{-i(\Omega-\Omega')/2})^N , \tag{4.20}$$

$$\alpha = | \xi_+ |^2, \quad \beta = | \xi_- |^2 .$$

By standard asymptotic evaluations of binomial coefficients one obtains

$$\langle \Omega' | \Omega \rangle \underset{N\to\infty}{\cong} e^{(i/2)(\Omega-\Omega')N(\alpha-\beta)} e^{-(\Omega-\Omega')^2(N/2)\alpha\beta} ,$$

and, unless $\alpha = \beta$, the stationary point is complex. This last condition is an immediate consequence of (4.19).

One interesting point of our construction is that for finite N the states $| \Omega \rangle$ for different Ω are not all linearly independent since they are vectors in a finite-dimensional space. For $N \to \infty$, however, the states $| \Omega \rangle$ do become linearly independent and span the infinite-dimensional Hilbert space of the induced representation.

Going back to the beginning of the section we see that (4.19) specifies in general the orbit of the induced representation. Let us look at practical examples. For $SU_2 \otimes SU_2$ it is convenient to rewrite (4.5) as

$$\mathrm{tr}[\xi^\dagger \tau_i \xi] = \mathrm{tr}[\xi^\dagger \xi \tau_\alpha^T] = 0 , \tag{4.21}$$

where now the two SU_2 groups act on ξ from the left and from the right, respectively. From this condition one obtains

$$\xi^\dagger \xi = \xi \xi^\dagger = \mathbb{1} . \tag{4.22}$$

Therefore ξ is a unitary matrix and the set of ξ is isomorphic to the sphere S_3. This is indeed the coset space $(SU_2 \otimes SU_2)/SU_2$. The orbit specified by (4.21) can be best viewed as follows. As expected, this condition is $SU_2 \otimes SU_2$ invariant. By a suitable $SU_2 \otimes SU_2$ transformation we can always replace ξ by the unit matrix. In the physical problem where the two SU_2 groups correspond to spin and isospin, respectively, this orbit thus gives back the well-known nontrivial mixing between rotations in space and internal space which is the key to nontrivial classical solutions. More generally, that is for $SU_2 \otimes SU_n$, one satisfies (4.19) by similarly linking the SU_2 group indices with those of an SU_2 subgroup of SU_n.

The reader should be pretty much convinced by now that the Lie algebra relations of $K \times T$ are a powerful tool. Once a particular representation is chosen, the spectrum and topological properties of the heavy particle (the would-be soliton) are completely specified to leading order. We now present a criterion to specify the choice of representation. As we explained at the beginning, there are several classes of orbits. For the general orbits, the values of the invariants are arbitrary and hence can vary continuously. For the special orbits, on the contrary, these invariants only take special values [see, e.g., (4.2) and (4.3)]. According to Michel and Radicati,[19] the minima of the potential occur at the exceptional orbits if the potential is solely a function of $\mathscr{A}^\alpha$. Soliton solutions are minima of an effective potential and although the potential is a function of other variables as well as $\mathscr{A}^\alpha$, it is likely the minima occur at the exceptional orbits. Therefore, it is reasonable to propose as a likely practice that one would keep only the representations of $K\times T$ which are characterized by exceptional orbits. This is the algebraic version of the topological stability arguments for classical solutions. As the above discussion shows, the quark representation does correspond to an exceptional orbit and hence satisfies the general stability criterion we just discussed.

The quark language used in the present study of the algebraic equations should be of special value in theories where the collective object is made up of fermion constituents. As an example we consider a static model of a complex ϕ field interacting with N very massive static isospinor quarks. The Hamiltonian reads

$$\begin{aligned} H&=H_0+H_I\ ,\\ H_0&=\int d_3x(\,|\pi|^2+|\nabla\phi|^2+\mu^2|\phi|^2)\ ,\\ H_I&=\lambda\sum_K\left[\int d_3x\,\phi(x)\psi^{(K)\dagger}\sigma_+\psi^{(K)}+\text{H.c.}\right]\ . \end{aligned} \tag{4.23}$$

Here the internal-symmetry group K is U_1 and there are two mesons. Hence we expect the noncompact algebra to be $U_1\times T_2$ which is a contraction of SU_2. Looking at (4.20), we see that we can simply identify $\psi_i^{(K)}$ with $q_\pm^{(K)}$. If we apply H_I to the state $|\Omega\rangle$ given by (4.20), it will become diagonal to leading order in N since it is proportional to that part of the SU_2 algebra which reduces to translations in the contraction. Hence we have

$$H_I|\Omega\rangle \underset{N\to\infty}{\cong} \lambda N\left[\int d_3x\,\phi(x)(\xi_+^*\sigma_+\xi_-)e^{-i\Omega} + \text{H.c.}\right]|\Omega\rangle\ . \tag{4.24}$$

The crucial point is that the vectors $|\Omega\rangle$ diagonalize the interaction term to leading order in N. This is a general feature which we shall further develop in the next section. In the present model one sees on (4.24) that one has again reduced the problem to a strong-coupling model of the type shown in Sec. II. However, a word of caution is needed here. We are not saying that all eigenstates of H are given by the quark representation at $N\to\infty$. The algebraic method certainly is applicable here but the representation may be reducible, one irreducible component is the quark representation.

V. ARE SOLITONLIKE SOLUTIONS THE GENERAL ANSWER?

As we recall in Sec. II, in the case of semiclassical solitons the form factor is the Fourier transform of the solution of the classical field equations. Clearly the small-β behavior will also be realized if the heavy particle is associated with a nontrivial minimum of an effective action which for $\beta\to 0$ is of order β^{-2}. In this case the form factor is the Fourier transform of the corresponding classical solution. This is the situation with the Skyrme-soliton picture of hadrons.[4]

On the other hand, we believe that hadrons are made out of constituent quarks. Our algebraic discussion shows that the spectrum of large-N baryons is indeed in agreement with the quark model. However in the Skyrme-soliton picture of hadrons, the quarks seem to remain only as "algebraic spirits" not physical constituents. At this point one wonders whether our discussion of large N is strictly equivalent to Skyrme solitons. More generally the question is the following: If the basic hypothesis of the small-β behavior is satisfied, does there always exist a unique action, effective or not, and an associated c-number solution? We now show that the answer to this question is negative. The sum of all the diagrams of Fig. 1 can be written compactly as a functional of the field ϕ taken as an external field. One gets

$$\widetilde{S}(q)=\int d\vec{y}\,e^{-i\vec{y}\cdot\vec{q}}S(y)\ , \tag{5.1}$$

$$S(y)=\exp\left[i\frac{\lambda}{\beta}\int dt\,\phi(y,t)\right]\ , \tag{5.2}$$

$$q=p-p',\quad \lambda=\widetilde{\lambda}\pi/M\beta^2\ . \tag{5.3}$$

From this one can easily compute the ϕ expectation value between heavy-particle states, to which one associates a fundamental field ψ. The following discussion was inspired by an earlier work of Manton.[20]

Consider first the case of two-dimensional solitons. Replace the action (2.1) by

$$\mathscr{L}=\tfrac{1}{2}[-(\partial_\mu\phi)^2-(\partial_\mu\psi)^2-M^2\psi^2] +\frac{1}{\beta^2}\left[V(\beta\phi)+\frac{\widetilde{\lambda}}{\beta}\psi^2\phi\right]\ .$$

There is now a fundamental field ψ with mass $M\sim\beta^{-2}$ and strongly coupled to ϕ. Note that we set this coupling to be $\widetilde{\lambda}/\beta^3$ because it is the relativistic coupling. The nonrelativistic coupling will indeed be of order β^{-1}. For small β, the ψ particle is very heavy and one can treat the emission of the ϕ particle by an eikonal approximation. The result is

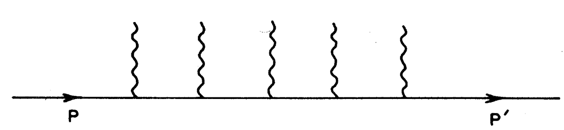

FIG. 1. Soft meson emission.

$$\langle p' | \phi(0) | p \rangle \equiv \tilde{\Gamma}[q] \propto \int dy\, e^{iqy} \int \mathscr{D}\phi\, \phi(0,0) \exp\left[i \left[S[\phi] + \frac{\lambda}{\beta} \int dt\, \phi(y,t) \right] \right] .$$

Using translation invariance we can rewrite the formula as

$$\tilde{\Gamma}[q] = \int dy\, e^{-iqy} \Gamma(y) \ , \quad \Gamma(y) = \int \mathscr{D}\phi\, \phi(y,0) \exp\left[i \left[S[\phi] + \frac{\lambda}{\beta} \int dt\, \phi(0,t) \right] \right] . \tag{5.4}$$

For small β, (5.4) is dominated by the minima of the complete action including the source term and we get

$$\Gamma(y) \simeq \phi_c(y) \ , \tag{5.5}$$

where ϕ_c satisfies

$$\partial_\mu{}^2 \phi_c + \frac{1}{\beta^2} \frac{\partial V}{\partial \phi_c} + \frac{\lambda}{\beta} \delta(x) = 0 \ . \tag{5.6}$$

We find the classical equations with a source term.

Away from $x=0$ we have the homogeneous equations with two general soliton solutions $\phi_s^{\pm}(x-X)$ where X is an arbitrary constant and where $\phi_s^{\pm}$ can be defined so that $\phi_s^{-}(-x) = -\phi_s^{+}(x)$. We can satisfy (5.6) by suitably choosing two different solutions of the homogeneous equation for $x>0$ and $x<0$. Since the heavy particle has its own fundamental field we must take the soliton number of the solution to be zero. Therefore we select the same vacuum at $x=-\infty$ and $x=+\infty$. Hence ϕ_c is typically of the form

$$\phi_c = \theta(x) \phi_s^{+}(x-X) + \theta(-x) \phi_s^{-}(x+X) \ . \tag{5.7}$$

On the other hand, it is well known that if we choose the minima of V to be at $V=0$, we have

$$\beta \phi_s^{\pm\prime} = \mp (2V[\phi_s^{\pm}])^{1/2} \ . \tag{5.8}$$

Hence ϕ_c is a solution of (5.6) if we let

$$\lambda = \{2V[\phi_s^{+}(-X)]\}^{1/2} + \{2V[\phi_s^{-}(X)]\}^{1/2} \ . \tag{5.9}$$

This equation has solutions only if λ lies in the finite range of values taken by the right-hand side. Going back to the form factor Γ_R one sees that in general its residue on the meson mass shell is not equal to λ. The coupling constant has been renormalized. On the other hand, the basic hypotheses of the algebraic method certainly hold. They are thus more general than the soliton picture described in Sec. III. We now point out this latter is the limit of bare coupling constant $\lambda=0$. In this case ψ decouples. Equation (5.8) gives $X=+\infty$. The soliton emerges by pulling apart a soliton-antisoliton pair.

Finally we exhibit a similar phenomenon on a chiral-invariant large-N theory which is conceived as a phenomenological model of large-N QCD. The model is a slight modification of the chiral bag model.[6] The dynamics are specified by the action

$$S = \int dt \left[\int_{x^2<R^2} d_3x \sum_K \tfrac{1}{2} (\bar{\psi}^{(K)} \gamma_\mu \overleftrightarrow{\partial}_\mu \psi^{(K)}) + \int_{x^2=R^2} d\sigma\, \tfrac{1}{2} \sum_K \bar{\psi}^{(K)} (\phi^0 + i\gamma_5 \vec{\tau}\cdot\vec{\phi}) \psi^{(K)} \right] + \frac{f_\pi{}^2}{16} \int d_4x\, \mathrm{tr}(\partial_\mu u \partial_\mu u^\dagger) \ , \tag{5.10}$$

where $\phi^0, \vec{\phi}$ is the usual chiral field

$$u = \phi^0 + i\vec{\phi}\cdot\vec{\tau}, \quad (\phi^0)^2 + \vec{\phi}^2 = 1 \ , \tag{5.11}$$

and $\psi^{(K)}$, $K=1,\dots,N$ are the quark fields. Since $f_\pi \propto \sqrt{N}$ (see Ref. 2) the chiral action is of order N. Contrary to the standard bag[6] theory in this model we treat the spherical bag radius R as a parameter. We investigate the system first by setting a trial state for quarks and then by using the large-N semiclassical method. The trial state we use is

$$| \Psi_{\text{trial}} \rangle = \prod_{K=1}^{N} \left[\int d_3x\, \psi^{\dagger(K)}(x) \xi(x) \right] | \Psi_0 \rangle \ , \qquad \left[\int d_3x\, \psi^{\dagger(K)} \xi(x) \right]^\dagger | \Psi_0 \rangle = 0 \ , \tag{5.12}$$

which is similar to (4.7). We treat $\xi(x)$ as variational parameters. In this paper, we further restrict $\xi(x)$ to be nonzero only for the upper two components in a γ_0 diagonal representation in order to keep the correct baryon number (i.e., 1) for the trial state.[21] It is easy to count the baryon number of (5.12) by using the very massive quarks. The baryon number is 1, since

$$(1-\gamma_0)\xi = 0, \quad \int \xi^\dagger(x) \xi(x) d_3x = 1 \ . \tag{5.13}$$

If we evaluate the effective Lagrangian using this trial state we obtain an effective Lagrangian identical to the chiral Lagrangian with an additional source term

$$\frac{N}{2} \int_{x^2=R^2} d\sigma\, \phi^0 \ . \tag{5.14}$$

We now look for radial-like solutions with the standard ansatz

$$u = e^{iF\hat{x}\cdot\vec{\tau}} \ , \tag{5.15}$$

and the action becomes

$$S = \int dt \int_0^\infty dr \left\{ \frac{f_\pi{}^2}{8} \left[\left[r \frac{d}{dr} F \right]^2 + 2\sin^2 F \right] + \delta(r-R) \frac{N}{2} \cos F \right\} . \tag{5.16}$$

The classical equation is now

$$-\frac{d}{dr} \left[r^2 \frac{dF}{dr} \right] + \sin(2F) - \delta(r-R) \lambda \sin F = 0 \ , \qquad \lambda = \frac{2N}{f_\pi{}^2} \ . \tag{5.17}$$

At this point the discussion proceeds in a way similar to the previous two-dimensional soliton case. For $r \neq R$ we have the homogeneous equation

$$r^2F''+2rF'=\sin(2F)\ , \tag{5.18}$$

which is invariant by scale transformation $r\to\mu r$. Hence we can solve (5.17) by combining two different solutions with different scales. Denote by $F^{\pm}$ the two solutions of (5.18) such that

$$F^{+}(0)=0,\ \ F^{+}(\infty)=\pi\ , \tag{5.19}$$

$$F^{-}(0)=\pi,\ \ F^{-}(\infty)=0\ . \tag{5.20}$$

$F^{\mp}$ are the two Skyrme solitons which describe the baryons and antibaryons, respectively, in the soliton picture. Indeed the baryonic number[3–5] of soliton is

$$B_{\text{soliton}}=-\frac{1}{\pi}\int_0^\infty dr\frac{d}{dr}(F-\sin F\cos F)\ . \tag{5.21}$$

In $F^{\pm}$ we pick up a particular scale. The relevant solution of (5.16) is of the form

$$F=\theta(r-R)F^{-}(\mu_1 r)+\theta(R-r)F^{+}(\mu_2 r)\ . \tag{5.22}$$

We can adjust μ_1, μ_2 such that F is continuous at R and such that its derivative has a discontinuity equal to $-\lambda\sin F$. Since (5.22) is continuous at $r=R$ and since $F(0)=F(\infty)=0$, the total baryon number of solitons (5.22) is zero. This is consistent since the constituent quarks carry the baryon number 1. On the other hand, the total baryon number (i.e., quarks+soliton) inside $r<R$ is

$$B_{\text{ins}}=1-\frac{1}{\pi}\{F^{+}(\mu_2R)-\sin[F^{+}(\mu_2R)]\cos[F^{+}(\mu_2R)]\}\ . \tag{5.23}$$

It is a function of R. This is a picture analogous to the standard chiral-bag situation.[22]

We may recover the purely soliton picture of baryons by considering the limit where $\mu_2R\to 0$. Then $F^{+}(\mu_2R)\simeq\pi$ and the source term $\sin F\delta(r-R)$ disappears. Moreover, from (5.23) $B_{\text{ins}}\to 0$ and the constituent-quarks baryon number canceled with the soliton baryon number inside the bag. Hence the Skyrme-soliton picture of hadrons seems to be a limiting case.[23] The algebraic equations are thus more general. Only a more detailed study of the physical properties will decide which particular dynamical realization underlies the algebraic structure of QCD we have just put forward.

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APPENDIX

In this appendix we review the induced representations of the strong-coupling group, whose Lie algebra is defined by (3.8). This appendix is an improved version of Ref. 11.

We first diagonalize all the commuting generators $\mathscr{A}^\alpha$ of T. Let us denote the eigenvalues by p^α:

$$\mathscr{A}^\alpha\,|\,p\,\rangle=p^\alpha\,|\,p\,\rangle\ . \tag{A1}$$

The symmetry group K acts on $\mathscr{A}^\alpha$ according to the algebra (3.3), namely,

$$u^\dagger(g)\mathscr{A}^\alpha u(g)=O_{\alpha\beta}(g)\mathscr{A}^\beta\ , \tag{A2}$$

where $g\in K$ and we assume $O(g)$ is an orthogonal matrix. Accordingly, K acts as a transformation group in the space of p:

$$g:p\to gp,\ \ (gp)^\alpha=O_{\alpha\beta}(g)p^\beta\ . \tag{A3}$$

One can draw orbits of K in the space of p. The orbits are classified by a set of invariants constructed by p^α. Consider a point on a given orbit and denote it by $\mathring{p}^\alpha$. Then every point on the orbit is generated by $g\mathring{p}$. Elements of K which make $\mathring{p}$ invariant form a subgroup, which is called the "little group" and denoted by H:

$$g_0\in H,\ \ g_0\mathring{p}=\mathring{p}\ . \tag{A4}$$

Let us denote an element of the left coset K/H by Ω. Every element in K is then uniquely specified by

$$g=\Omega g_0,\ \ g\in K,\ \ \Omega\in K/H,\ \ g_0\in H\ . \tag{A5}$$

The p space in a given orbit is then specified by Ω, so that (A1) can be expressed as

$$\mathscr{A}^\alpha\,|\,\Omega,\mathring{p}\,\rangle=(\Omega\mathring{p})^\alpha\,|\,\Omega,\mathring{p}\,\rangle\ . \tag{A6}$$

One defines the Haar measure of K/H as a quotient of Haar measure of K by that of H:

$$dg=d\Omega\,dg_0\ . \tag{A7}$$

One then defines the δ function in K/H by

$$f(\Omega)=\int d\Omega' f(\Omega')\delta(\Omega,\Omega')\ . \tag{A8}$$

We normalize the vector $|\,\Omega,\mathring{p}\,\rangle$ in (4.6) as

$$\langle\,\Omega',\mathring{p}\,|\,\Omega,\mathring{p}\,\rangle=\delta(\Omega',\Omega)\ . \tag{A9}$$

Let us denote a unitary irreducible representation of the little group H by

$$u(g_0)\,|\,LM\,\rangle=\sum_{M'}\mathscr{D}^L_{M'M}(g_0)\,|\,LM'\,\rangle\ . \tag{A10}$$

Since H is a compact group the representation space V is finite dimensional. The basis of V which we specified by $|\,LM\,\rangle$ is normalized as

$$\langle\,LM\,|\,L'M'\,\rangle=\delta_{LL'}\delta_{MM'}\ . \tag{A11}$$

Although we used the same notation as the angular momentum of the rotation group, H will not necessarily be O(3), of course.

The induced representations of $G=K\times T$ are defined in the space of $K/H\otimes V$ as in the following:

$$\begin{aligned}&\mathscr{A}^\alpha\,|\,\Omega,M;\mathring{p}L\,\rangle=(\Omega\mathring{p})^\alpha\,|\,\Omega,M;\mathring{p}L\,\rangle\ ,\\ &u(g)\,|\,\Omega',M;\mathring{p}L\,\rangle=\sum_{M'}\mathscr{D}^L_{M'M}(g_0)\,|\,\Omega,M';\mathring{p}L\,\rangle\ ,\end{aligned} \tag{A12}$$

where Ω' and g_0 are given by

$$g\Omega'=\Omega g_0\ . \tag{A13}$$

The induced representation constructed in (A12) is known to be irreducible.

In order to see the internal-symmetry properties of isobars it is necessary to decompose the space of induced representation into irreducible components of symmetry group K. For this purpose we first set $\Omega'=0$ in (A12),

$$u(g)\,|\,0,M;\mathring{p}L\,\rangle=\sum_{M'}\mathscr{D}^{L}_{M'M}(g_0)\,|\,\Omega,M';\mathring{p}L\,\rangle\ . \qquad \text{(A14)}$$

Let us use μ for the irreducible representation, ν for its basis, and $D^{\mu}_{\nu\nu'}(g)$ for the unitary representation. An irreducible tensor operator is then defined by

$$\int D^{\mu}_{\nu\sigma}(g)^{*}u(g)dg\equiv X(\mu\nu\,|\,\sigma)\ , \qquad \text{(A15)}$$

which has the property

$$u(g)X(\mu\nu\,|\,\sigma)=\sum_{\nu'}D^{\mu}_{\nu'\nu}(g)X(\mu\nu'\,|\,\sigma)\ . \qquad \text{(A16)}$$

Thus, one defines

$$\begin{aligned}|\,\mu\nu;\mathring{p}L\sigma\,\rangle&\equiv X(\mu\nu\,|\,\sigma)\,|\,0,M;\mathring{p}L\,\rangle\\&=\int D^{\mu *}_{\nu\sigma}(g)u(g)dg\,|\,0,M;\mathring{p}L\,\rangle\\&=\int D^{\mu *}_{\nu\sigma}(g)\sum_{M'}\mathscr{D}^{L}_{M'M}(g_0)\,|\,\Omega,M';\mathring{p}L\,\rangle dg\\&=\sum_{\sigma'}\int D^{\mu *}_{\nu\sigma'}(\Omega)\,|\,\Omega,M';\mathring{p}L\,\rangle d\Omega\\&\qquad\times\int D^{\mu *}_{\sigma'\sigma}(g_0)\mathscr{D}^{L}_{M'M}(g_0)dg_0\ . \qquad \text{(A17)}\end{aligned}$$

In order to evaluate the integral we must investigate how H is contained in K. The components of an irreducible representation μ of K has been specified by ν (or σ). Since H is a subgroup of K the index ν (or σ) may be written (ξLM) where ξ specifies irreducible representations of a chain of subgroups of K; $K\supset K_1\supset\cdots\supset H$. Then the matrix $D^{\mu}_{\sigma'\sigma}(g_0)$ can be written

$$D^{\mu}_{(\xi'L'M')(\xi LM)}(g_0)=\delta_{\xi\xi'}\delta_{LL'}\mathscr{D}^{L}_{M'M}(g_0)\ . \qquad \text{(A18)}$$

Thus, setting $\sigma\equiv\xi\widetilde{L}M$ we obtain the following expression for (A17):

$$|\,\mu\nu;\mathring{p}L(\xi\widetilde{L}M)\,\rangle=\delta_{L'L}\sum_{M'}\int d\Omega\,D^{\mu}_{\nu(\xi LM')}(\Omega)\,|\,\Omega,M';\mathring{p}L\,\rangle\ .$$

The right-hand side is obviously independent of M, and furthermore $L=\widetilde{L}$ so that the above expression can be written as

$$|\,\mu\nu;\mathring{p}\xi L\,\rangle=\sum_{M'}\int d\Omega\,D^{\mu *}_{\nu(\xi LM')}(\Omega)\,|\,\Omega,M';\mathring{p}L\,\rangle\ . \qquad \text{(A19)}$$

From this expression one sees that the multiplicity of the representation μ is given by the number of different values that ξ may take on the same L.

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