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# Ultraviolet divergences in $1/N$ expansions of asymptotically free theories

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We calculate the ultraviolet divergences of the nonlinear $\sigma$ and Gross-Neveu models in the $1/N$ expansion near two dimensions ($d=2-\epsilon$). Beyond the leading order the theories develop logarithmic as well as pole singularities in $\epsilon$. Identical divergences occur when the renormalization constants calculated in perturbation theory are expanded in powers of $1/N$ rather than in powers of the coupling constants. The necessary infinite renormalizations of the $1/N$ expansions are completely determined by the two-loop $\beta$ functions and one-loop anomalous dimensions of perturbation theory. These results can be extended to non-Abelian gauge theories in four dimensions which admit $1/N$ expansions.

## I. INTRODUCTION

We have considered in some detail the nature of the ultraviolet singularities and the behavior of the renormalization constants of quantum field theories with $1/N$ expansions.[1] In particular, we deal here with the nonlinear $\sigma$ model (NLSM) and the Gross-Neveu model (GNM).[2] Of course, a great deal is known already about these models in two-dimensional space-time. They are exactly integrable, renormalizable theories; their $S$ matrices and spectra have been calculated exactly.[3,4] Since we are ultimately interested in applying our results and techniques to four-dimensional theories, these properties are not what directly concern us here. Rather, we concentrate on the details of the renormalization procedure when the Green's functions for the models are expanded in powers of $1/N$ ($N$ is the number of components of the fields) and the theory is regularized by dimensional continuation[5] about $d=2$.

Both of the models are asymptotically free in two dimensions.[2,6] For $d>2$ they have a two-phase structure. For small coupling constant, $g$, there is a perturbative phase in which the quanta are massless and a strong-coupling phase in which a mass gap appears and no massless states remain. In the NLSM, mass generation is associated with the restoration of the O($N$) symmetry apparently violated by enforcing the constraint equation of the model, and in the GNM dynamical mass generation is accompanied by violation of a chiral symmetry of the Lagrangian.

As $d\to 2$ the critical coupling constant which separates the two phases approaches zero, and for $d<2$ the theories exist only in the massive phases. If one replaces $d=2$ by $d=4$ and "massive phase" by "confining phase," then there is a close parallel between the behavior of these models and the expected behavior of QCD or any unbroken non-Abelian gauge theory in four dimensions.

The massive phases of the two-dimensional models are readily realized if the coupling constants of the respective models are replaced by $g/\sqrt{N}$ and the Green's functions are expanded in powers of $1/N$. In the case of non-Abelian gauge theories in four dimensions the $1/N$ expansion thus defined is consistent with confinement, but it has not been proved that there exists a two-phase structure for which $d=4$ is the critical dimension.[7]

Though general discussions of the renormalizability of the $1/N$ expansions have appeared before,[8,9] we are not aware of any thorough discussions of the renormalization constants which go beyond the leading order in $1/N$. In carrying out the next-to-leading-order calculations using the method of dimensional regularization, we have discovered some new results. Beyond the leading order the dimensionally continued Green's functions develop logarithmic as well as pole singularities in $\epsilon=2-d$. (Note that $\epsilon>0$ for $d<2$.) We have found also that these singularities do not contradict the Laurent expansion in $\epsilon$ of the renormalization constants found in perturbation theory but are, in fact, predicted by expanding these renormalization constants in powers of $1/N$ rather than in powers of the coupling constant. This reconciliation of the $1/N$ and perturbative results leads to further predictions about the ultraviolet singularities of the $1/N$ expansions for these models. They are effectively super-renormalizable. All the divergences of the $1/N$ expansion are determined by the one-loop anomalous dimension and the two-loop Gell-Mann—Low—Callan—Symanzik $\beta$ function of the perturbative expansion. Some renormalization constants have divergences to all orders in $1/N$ while others have divergences only to a finite order of $1/N$ or are in fact finite. In all cases the results are predictable from the low-order perturbation theory calculations.

In Sec. II we carry out explicit calculations for the NLSM and GNM to demonstrate the emergence of the $\ln\epsilon$ singularities. The consistency of our results with the behavior of the renormalization constants calculated in perturbation theory is demonstrated in Sec. III. Here we also determine the general structure of the ultraviolet singularities in the $1/N$ expansion and demonstrate the super-renormalizable properties alluded to above. In the concluding Sec. IV we discuss our results and, in particular, their generalizations to $1/N$ expansion of four-dimensional gauge theories. Some details of the proofs of renormalizability of the NLSM and the GNM in the $1/N$ expansion are given in the two appendices.

## II. CALCULATIONS

### A. Nonlinear Sigma Model

The Lagrangian density for the O($N$) NLSM is

$$\mathscr{L}=\tfrac{1}{2}\eta_\mu\cdot\eta_\mu \tag{1}$$

with the constraint

$$\eta^2=\eta\cdot\eta=N/g^2 \,. \tag{2}$$

We use the summation convention for repeated space-time indices and a dot to indicate summation over internal spin indices.

The generating functions for Green's functions can be written in functional-integral form[10] as

$$Z(J)=\int [d\eta]\prod_x \delta(\eta(x)\cdot\eta(x)-N/g^2) \times\exp\left[i\int(\mathscr{L}+J\cdot\eta)\right] . \tag{3}$$

It is convenient to introduce an auxiliary field, $\lambda$, and write a Fourier representation of the $\delta$-function constraints, viz.

$$\prod_x \delta(\eta^2(x)-N/g^2) =\int [d\lambda]\exp\left[-i\int dx\frac{\lambda}{2}(\eta^2-N/g^2)\right] . \tag{4}$$

Making this substitution in Eq. (3) renders the exponent quadratic in the $\eta$ field, and the corresponding Gaussian functional integral can be evaluated. There results an effective nonlocal action for the auxiliary field, $\lambda$, and a loop expansion of this effective action generates the $1/N$ expansion. The result can be achieved directly by shifting $\lambda\to\lambda+m^2$ (the value of $m^2$ will be determined later) and writing the action as

$$S=S_0+S_I$$

with

$$S_0=\int \tfrac{1}{2}(\eta_\mu\cdot\eta_\mu-m^2\eta^2)+\frac{N}{2}\int\int \lambda(x)E^{-1}(x,y)\lambda(y) \,,$$
$$S_I=\int \frac{\lambda}{2}(N/g^2-\eta^2)-\frac{N}{2}\int\int \lambda(x)E^{-1}(x,y)\lambda(y) \,, \tag{5}$$

where $E^{-1}(x,y)$, the effective wave operator for the $\lambda$ field, corresponds to the bubble diagram of Fig. 1 and is given by

$$E^{-1}(x,y)=\frac{-i}{2}\langle x\,|\,(-\partial^2-m^2)^{-1}\,|\,y\rangle \times\langle y\,|\,(-\partial^2-m^2)^{-1}\,|\,x\rangle \,. \tag{6}$$

In momentum space the inverse propagator can be written as a hypergeometric function

$$E^{-1}(p^2)=-\frac{i}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{[(k+p)^2-m^2][k^2-m^2]} =\frac{\Gamma(1+\epsilon/2)}{8\pi m^2}\left[\frac{4\pi}{m^2}\right]^{\epsilon/2} \times F(1,1+\epsilon/2;\tfrac{3}{2};-p^2/4m^2) \,. \tag{7}$$

The presence of $E^{-1}(x,y)$ in the interaction term cancels the bubble diagrams (see Fig. 1) generated by the three-point interaction. Therefore, the prescription for calculation is to omit both $E^{-1}$ insertions and elementary one-loop bubbles. As shown in Appendix A, two counterterms are required to renormalize graphs with only external $\eta$ lines,

$$\mathscr{L}_{\rm CT}=\frac{a}{2}\eta_\mu\cdot\eta_\mu+\frac{b}{2}\frac{N}{g^2}\lambda \,. \tag{8}$$

Therefore, the Feynman rules are as given in Fig. 2.

The power in $1/N$ of a given graph can be found as follows. Remove all $\eta$ lines connected directly to external $\eta$ particles. One now has a graph with $E_\lambda$ external $\lambda$ lines connected via closed $\eta$ loops. Contract each $\eta$ loop to a point. Count the number of closed $\lambda$ loops, $L_\lambda$, of the resulting reduced graph. The power in $1/N$ of the graph is $E_\lambda+L_\lambda-1$. Thus, only a finite number of graphs contribute to each order in $1/N$. The renormalization constants, therefore, can also be expanded in powers of $1/N$, i.e.,

$$Z=1+\sum_{j=0} z^{(j)}/N^j \,. \tag{9}$$

Both of the counterterms can be determined from the self-energy corrections for the $\eta$ particles. To zeroth order in $1/N$ the self-energy is given by the diagrams of Fig. 3. There is no wave-function renormalization to this order. Calculation of the diagrams gives

$$\Pi^{(0)}=m^2\left[\frac{2}{\epsilon}-\frac{4\pi}{g^2}\frac{1}{\Gamma(1+\epsilon/2)} \times\left[\frac{m^2}{4\pi\mu^2}\right]^{\epsilon/2}[1+(z_g{}^{-1})^{(0)}]\right] . \tag{10}$$

The ultraviolet divergence at $d=2$ can be removed by the minimal-pole subtraction

$$(z_g{}^{-1})^{(0)}=g^2/2\pi\epsilon \,. \tag{11}$$

Then at $d=2$ the renormalized self-energy is

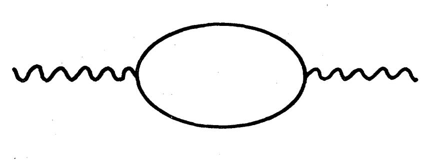

FIG. 1. Bubble diagram for $\lambda$ wave operator.

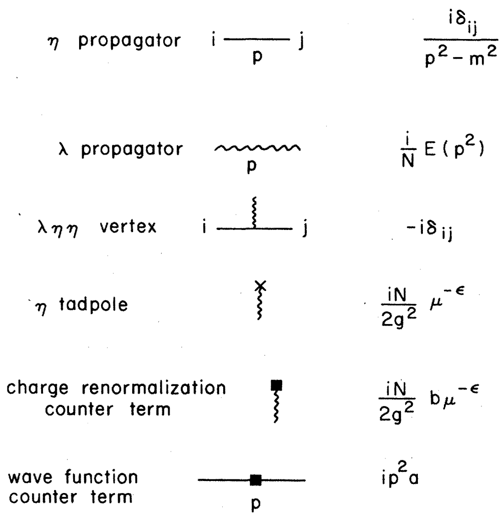


FIG. 2. Feynman rules for $1/N$ expansion of nonlinear $\sigma$ model.

$$\Pi^{(0)}=m^2[-4\pi/g^2+\ln(\hat{\mu}^2/m^2)]\ ,$$
$$\hat{\mu}^2=4\pi\mu^2e^{-\gamma},\quad \gamma=\text{Euler's constant}\ . \tag{12}$$

The parameter $m^2$ is fixed to be the physical mass of the $\eta$ particles. Thus, we require

$$\Pi(m^2)=0\ ,$$

which has the solutions

$$m^2=0$$

and

$$m^2=\hat{\mu}^2\exp[-(4\pi/g^2)]\ . \tag{13}$$

The zero-mass solution is known to be physically unacceptable for $d\leq 2$,[11] and one adopts the dynamical mass generation given by the second solution.

The diagrams which contribute to the self-energy to $O(1/N)$ are shown in Fig. 4. To calculate the ultraviolet divergences of these diagrams we need the asymptotic behavior of $E^{-1}$ for large Euclidean momentum. A standard result for hypergeometric functions[12] gives

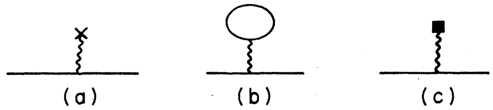


FIG. 3. $(1/N)^0$ radiative corrections to $\eta$ self-energy.

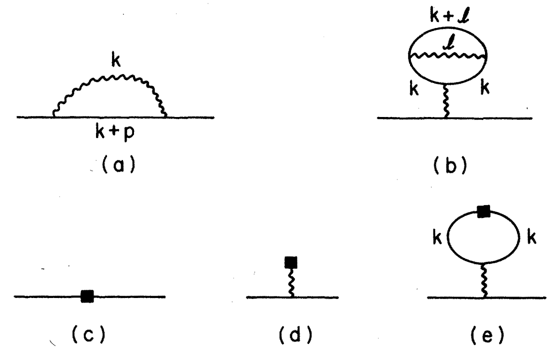


FIG. 4. $(1/N)$ radiation corrections to $\eta$ self-energy.

$$E^{-1}(-p_E^2)=\frac{1}{16\pi m^2}\left[\frac{4\pi}{m^2}\right]^{\epsilon/2}\Gamma(\epsilon/2)\left[\frac{4m^2}{p_E^2}\right]^{1+\epsilon/2}\times\left[\left[\frac{p_E^2}{4m^2}\right]^{\epsilon/2}-\frac{\Gamma(1-\epsilon/2)\Gamma(1/2)}{\Gamma(1/2-\epsilon/2)}\right]\times[1+O(4m^2/p_E^2)]\ . \tag{14}$$

Note also that

$$E^{-1}(0)=\frac{\Gamma(1+\epsilon/2)}{8\pi m^2}\left[\frac{4\pi}{m^2}\right]^{\epsilon/2}. \tag{15}$$

The contributions of Figs. 4(a) and 4(b) are

$$\Pi_a^{(1)}=i\int\frac{d^dk}{(2\pi)^d}\frac{E(k^2)}{(k+p)^2-m^2} \tag{16}$$

and

$$\Pi_b^{(1)}=-\frac{E(0)}{2}\int\int\frac{d^dk\,d^dl}{(2\pi)^{2d}}\frac{E(l^2)}{[(k+l)^2-m^2](k^2-m^2)^2}\ . \tag{17}$$

Individually these two terms are quadratically divergent, but their leading divergence cancel in agreement with the power-counting results of Appendix A. To establish this we use the following result to factor $\Pi_b^{(1)}$:

$$I(l^2)=i\int\frac{d^dk}{(2\pi)^d}\frac{1}{[(k+l)^2-m^2](k^2-m^2)^2}=\frac{\Gamma(3-d/2)}{2(4\pi)^{d/2}}\int_0^1 d\alpha[m^2-\alpha(1-\alpha)l^2]^{d/2-3}\ . \tag{18}$$

Integrating by parts, we can write $I(l^2)$ in terms of bubble diagrams,

$$I(l^2)=-\frac{\Gamma(1+\epsilon/2)}{(4\pi)^{d/2}}\frac{1}{(l^2-4m^2)}\left[(m^2)^{d/2-2}+(3-d)\int_0^1 d\alpha[m^2-\alpha(1-\alpha)l^2]^{d/2-2}\right]$$
$$=-\frac{2}{l^2-4m^2}[E^{-1}(0)+(3-d)E^{-1}(l^2)]\ . \qquad (18')$$

We will refer to this formula as the cutting rule because of its similarity to a result expressing a general one-loop graph in two dimensions as a sum of bubble diagrams.[13] Substituting this result in Eq. (17) gives

$$\Pi_b^{(1)}=-i\int\frac{d^dl}{(2\pi)^d}\frac{E(l^2)}{l^2-4m^2}$$
$$-i(3-d)E(0)\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2-4m^2}\ . \qquad (19)$$

Evaluating the second integral and combining $\Pi_a^{(1)}$ and $\Pi_b^{(1)}$ gives

$$\Pi_{a+b}^{(1)}=i\int\frac{d^dl}{(2\pi)^d}\left[\frac{1}{(l+p)^2-m^2}-\frac{1}{l^2-4m^2}\right]E(l^2)$$
$$-\frac{4m^2}{\epsilon}(1+\epsilon)(2)^{\epsilon}\ . \qquad (20)$$

We can write, therefore,

$$\Pi_{a+b}^{(1)}=[(4/d-1)p^2-3m^2]\ln\epsilon$$
$$-\frac{4m^2}{\epsilon}(1+\epsilon)2^{\epsilon}+F(p^2,m^2,\epsilon)\ , \qquad (24)$$

where $F$ is finite at $\epsilon=0$.

The counterterms of Figs. 4(c) and 4(d) are straightforward,

$$\Pi_c^{(1)}=-p^2z_\eta^{(1)}\ ,$$
$$\Pi_d^{(1)}=-\frac{4\pi m^2}{g^2\Gamma(1+\epsilon/2)}\left[\frac{4\pi\mu^2}{m^2}\right]^{\epsilon/2}b^{(1)} \qquad (25)$$

with

$$b^{(1)}=(z_g{}^{-1})^{(1)}-[1+(z_g{}^{-1})^{(0)}]z_\eta^{(1)}\ .$$

The wave-function renormalization insertion in the loop of Fig. 4(e) generates a mass counterterm:

$$-[(4/d-1)p^2-3m^2]\int\frac{d^dl}{(2\pi)^d}\frac{E(l^2)}{(l^2-4m^2)^2}$$
$$+\text{finite terms}\ . \qquad (21)$$

Using the asymptotic form for $E(l^2)$ from Eq. (14) the divergent piece of the first integral can be evaluated as

$$J=i\int\frac{d^dl}{(2\pi)^d}\frac{E(l^2)}{(l^2-4m^2)^2}\bigg|_{\text{uv}}$$
$$=-\frac{2^{-\epsilon}}{\Gamma(1-\epsilon/2)\Gamma(\epsilon/2)}\int_{x_0}^{\infty}\frac{dx}{x(x^{\epsilon/2}-C)}$$
$$=\frac{2^{-\epsilon}}{\Gamma(1-\epsilon/2)\Gamma(1+\epsilon/2)}\frac{1}{C}\ln\frac{x_0{}^{\epsilon/2}-C}{x_0{}^{\epsilon/2}}\ , \qquad (22)$$

where we have made a Wick rotation and set $x=l_\epsilon{}^2/4m^2$. $x_0$ is an infrared cutoff and

$$C=\Gamma(1/2)\Gamma(1-\epsilon/2)/\Gamma(1/2-\epsilon/2)$$
$$=-(\epsilon/2)[\Psi(1)-\Psi(2)]+O(\epsilon^2)$$

with (23)

$$\Psi(x)=\frac{d}{dx}\ln\Gamma(x)\ .$$

Thus, we obtain[14]

$$J=\ln\epsilon+\text{finite terms}\ . \qquad (23)$$

$$-\frac{4\pi m^2}{g^2\Gamma(1+\epsilon/2)}\left[\frac{4\pi\mu^2}{m^2}\right](z_g{}^{-1})^{(1)}-\Pi^{(0)}z_\eta^{(1)}\ . \qquad (27)$$

Since $\Pi^{(0)}=0$ to $O(1/N)$, we can drop the last term. Then the full contribution to $\Pi^{(1)}$ for $\epsilon\to 0$ is

$$\Pi^{(1)}=(p^2-3m^2)\ln\epsilon+F(p^2,m^2)-4m^2/\epsilon+(m^2-p^2)z_\eta^{(1)}$$
$$-\frac{4\pi m^2}{g^2}\left[1-\frac{\epsilon}{2}\ln(\hat{\mu}^2/m^2)\right](z_g{}^{-1})^{(1)}\ . \qquad (28)$$

The singular parts of the $O(1/N)$ renormalization constants can then be identified as

$$z_\eta^{(1)}\,|_{\text{sing}}=\ln\epsilon\ ,$$
$$(z_g{}^{-1})^{(1)}\,|_{\text{sing}}=-g^2/2\pi\epsilon-(g^2/2\pi)\ln\epsilon \qquad (29)$$

or

$$Z_g{}^{-1}=1+\frac{g^2}{2\pi}\left[\left[1-\frac{2}{N}\right]\frac{1}{\epsilon}-\frac{1}{N}\ln\epsilon\right]\ ,$$
$$Z_\eta=1+\frac{1}{N}\ln\epsilon\ . \qquad (30)$$

In Sec. III we show that these results are consistent with the $(1/\epsilon)$ expansion of perturbation theory and complete the renormalization of the self-energy.

### B. Gross-Neveu Model

The Gross-Neveu model contains an $N$-component fermion field with a chiral-invariant interaction

$$\mathscr{L}=\bar{\psi}i\partial\!\!\!/\psi+\frac{g^2}{2N}(\bar{\psi}\psi)^2 \, . \tag{31}$$

To derive the $1/N$ expansion for the model, it is convenient to introduce an auxiliary field, $\sigma$, and write

$$\mathscr{L}=\bar{\psi}i\partial\!\!\!/\psi-\frac{N}{2g^2}\sigma^2-\sigma\bar{\psi}\psi \, . \tag{32}$$

The $1/N$ expansion can be generated as in the NLSM by performing the Gaussian integral over the physical fields and making a loop expansion of the resultant effective nonlocal action for the auxiliary field, $\sigma$. Equivalently, one can shift the $\sigma$ field, $\sigma\to\sigma+m^2$, and write the action as

$$S=S_0+S_I$$

with

$$S_0=\int\bar{\psi}(i\partial\!\!\!/-m)\psi +\frac{N}{2}\int\int\sigma(x)\left[\frac{-1}{g^2}\delta(x-y)+\Delta^{-1}(x,y)\right]\sigma(y) \, , \tag{33}$$

$$S_I=\frac{N}{2g^2}m\sigma-\sigma\bar{\psi}\psi-\frac{N}{2}\int\int\sigma\Delta^{-1}\sigma \, .$$

$\Delta^{-1}(x,y)$ is given by a one-loop fermion bubble, viz.

$$\Delta^{-1}(x,y)=i\langle x\,|\,(i\partial\!\!\!/-m)^{-1}\,|\,y\rangle\langle y\,|\,(i\partial\!\!\!/-m)^{-1}\,|\,x\rangle \, . \tag{34}$$

In addition, there are the counterterms discussed in Appendix B. After translation of the $\epsilon$ field, they can be written as

$$\mathscr{L}_{\rm CT}=a\bar{\psi}i\partial\!\!\!/\psi-b(N\mu^{-\epsilon}/2g^2)(\sigma^2+2m\sigma) \, . \tag{35}$$

The fermion loop is divergent in two dimensions, but this ultraviolet singularity can be removed by a coupling constant counterterm. The effective wave operator for the $\sigma$ field in the action of Eq. (33) is

$$\begin{aligned} D_0{}^{-1}(p^2)&=-\mu^{-\epsilon}/g^2+\Delta^{-1}(p)\\ &=-\frac{\mu^{-\epsilon}}{g^2}+i\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\left[\frac{1}{(k\!\!\!/+p\!\!\!/-m)(k\!\!\!/-m)}\right]\\ &=-\frac{\mu^{-\epsilon}}{g^2}+\frac{2^{d/2}\Gamma(\epsilon/2)}{(4\pi)^{d/2}}(m^2)^{-\epsilon/2}\\ &\quad+2^{d/2}(p^2-4m^2)E^{-1}(p^2) \, . \end{aligned} \tag{36}$$

The singularity of the second term can be canceled by including the $b^{(0)}\sigma^2$ counterterm in the wave operator. The pole is canceled by setting

$$b^{(0)}=(z_g{}^{-1})^{(0)}=g^2/\pi\epsilon \, . \tag{37}$$

Then at $d=2$ the renormalized one-loop wave operator is

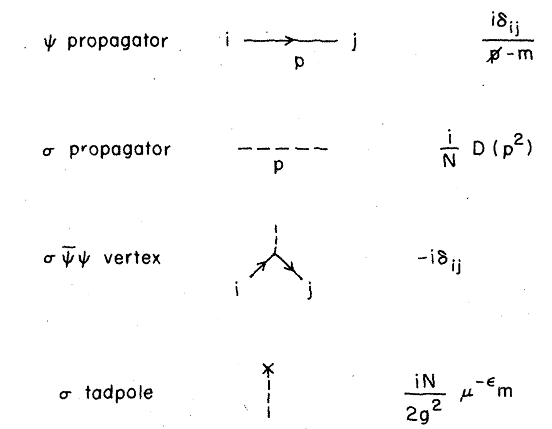


FIG. 5. Feynman rules for $1/N$ expansion of Gross-Neveu model.

$$D^{-1}(p^2)=-\frac{1}{g^2}-\frac{1}{2\pi}\ln(2m^2/\hat{\mu}^2) +2(p^2-4m^2)E^{-1}(p^2) \, . \tag{38}$$

The Feynman rules are given in Fig. 5. The power of $1/N$ of a given graph is determined in close analogy to the NLSM. Erase the fermionic lines connected directly to external fermions, and contract the internal fermion loops to point vertices. The resulting diagram has $E_\sigma$ external lines and $L_\sigma$ loops. Its power in $1/N$ is $E_\sigma+L_\sigma-1$.

To zeroth order in $1/N$ the fermion self-energy is given by the graphs of Fig. 6. Using the value of $b^{(0)}$ from above, their combined contribution at $\epsilon=0$ is

$$\Sigma^{(0)}=mD(0)\left[\frac{1}{g^2}+\frac{1}{2\pi}\ln(2m^2/\hat{\mu}^2)\right] . \tag{39}$$

The condition $\Sigma(p\!\!\!/=m)=0$ leads to dynamical mass generation via the relation

$$m^2=(\hat{\mu}^2/2)\exp(-2\pi/g^2) \, . \tag{40}$$

This dimensional transmutation relation, extended to $\epsilon\neq 0$, implies that

$$D^{-1}(p^2)=2^{d/2}(p^2-4m^2)E^{-1}(p^2)+O(1/N) \, . \tag{41}$$

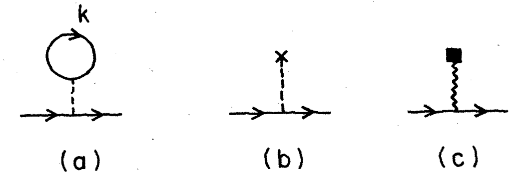


FIG. 6. $(1/N)^0$ radiative corrections to $\psi$ self-energy.

The $O(1/N)$ contributions to the fermion self-energy are given by the diagrams of Fig. 7 as follows:

$$\Sigma_a^{(1)} = D(0) \int \int \frac{d^d k\, d^d l}{(2\pi)^{2d}} \mathrm{tr} \left[ \frac{1}{(\not k + \not l - m)(\not k - m)^2} \right] D(l^2) \, . \tag{42}$$

Evaluating the trace and using the cutting rule of Eq. (18′) leads to

$$\Sigma_a^{(1)} = mD(0) 2(d-1) i \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - 4m^2}$$
$$= mD(0) \frac{1-\epsilon}{2\pi} \Gamma(\epsilon/2) \left[ \frac{m^2}{\pi} \right]^{-\epsilon/2} \tag{43}$$

$-m/\epsilon +$ finite terms,

$$\Sigma_b^{(1)} = i \int \frac{d^d k}{(2\pi)^d} \frac{D(k^2)}{(\not k + \not p - m)}$$
$$= \not p A + mB$$

with

$$A(p^2) = i \int \frac{d^d k}{(2\pi)^d} \frac{(1 + p \cdot k / p^2) D(k^2)}{(p+k)^2 - m^2} \, ,$$
$$B(p^2) = i \int \frac{d^d k}{(2\pi)^d} \frac{D(k^2)}{(p+k)^2 - m^2} \, . \tag{44}$$

Expanding the denominators of $A$ and $B$ in powers of $p^2$ gives

$$A = i \left[ 1 - \frac{2}{d} \right] \int \frac{d^d k}{(2\pi)^d} \frac{D(k^2)}{k^2 - 4m^2} + \text{finite terms} \, . \tag{45}$$

The most singular part of the integral can be evaluated using the method of Sec. II A to obtain

$$A_{\text{sing}} = -\frac{\epsilon}{2} \ln \epsilon \, . \tag{46}$$

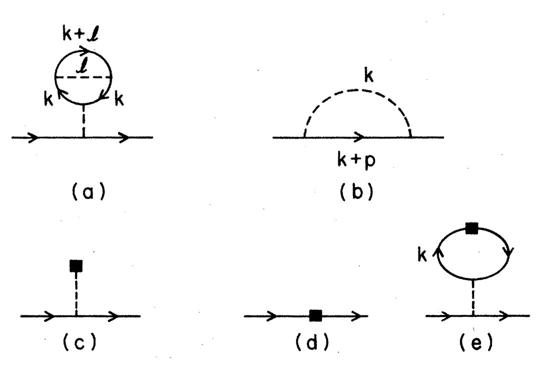


FIG. 7. $(1/N)$ radiative corrections to $\psi$ self-energy.

But this does not diverge at $\epsilon = 0$ and does not require a wave-function renormalization counterterm. For the mass counterterm we obtain by similar manipulation

$$B = \tfrac{1}{2} \ln \epsilon + \text{finite terms} \, . \tag{47}$$

The counterterm contributions are

$$\Sigma_c^{(1)} = mD(0) b^{(1)} \mu^{-\epsilon} / g^2 \, , \tag{48}$$

$$\Sigma_d^{(1)} = \not p a^{(1)} \, , \tag{49}$$

$$\Sigma_e^{(1)} = m a^{(1)} \left[ -\frac{1}{4} - \frac{2\Gamma(1-d/2)}{d(4\pi)^{d/2}} (m^2)^{d/2} \right] . \tag{50}$$

As noted above we can set $a^{(1)} = 0$ and, therefore, drop $\Sigma_d^{(1)}$ and $\Sigma_e^{(1)}$. For $d = 2, D(0) = -\pi_0$. Thus, the only necessary counterterm of $O(1/N)$ is

$$b^{(1)} = (z_b{}^{-1})^{(1)} = -\frac{g^2}{\epsilon \pi} + \frac{g^2}{2\pi} \ln \epsilon \tag{51}$$

and

$$Z_g{}^{-1} = 1 + \frac{g^2}{\pi} \left[ \left[ 1 - \frac{1}{N} \right] \frac{1}{\epsilon} + \frac{g^2}{2\pi N} \ln \epsilon \right] ,$$
$$Z_\psi = 1 \, . \tag{52}$$

## III. RENORMALIZATION-GROUP ANALYSIS

We will show that the singularities calculated in Sec. II are consistent with those found in perturbation theory and are, in fact, predicted when the renormalization constants calculated by minimal-pole subtraction are reexpanded in powers of $1/N$. Moreover, only the one- and two-loop contributions to the perturbative renormalization effects give rise to singularities in the $1/N$ expansion.

The renormalization-group $\beta$ function and the anomalous dimension have been calculated to three-loop order for the nonlinear $\sigma$ model.[15] The results expressed in terms of $\alpha = g^2/2\pi$ are

$$\beta(\alpha) = -\left[ 1 - \frac{2}{N} \right] \alpha^2 \left[ 1 + \frac{\alpha}{N} + O\left[ \frac{\alpha^2}{N} \right] \right] ,$$
$$\gamma_\eta(\alpha) = \frac{\alpha}{2N} + O\left[ \frac{\alpha^2}{N} \right] . \tag{53}$$

From the definitions of the $Z$ factors in Eq. (A21) differentiation with respect to $\mu$ gives the differential equations[5]

$$\beta(\alpha) + [\beta(\alpha) - \epsilon\alpha] \alpha \frac{\partial}{\partial \alpha} \ln Z_\alpha = 0 \, ,$$
$$\gamma_\eta(\alpha) = \tfrac{1}{2} [\beta(\alpha) - \epsilon\alpha] \frac{\partial}{\partial \alpha} \ln Z_\eta \, . \tag{54}$$

These can be solved for $Z_\alpha$ and $Z_\eta$ as

$$\ln Z_\alpha{}^{-1} = \int_0^\alpha \frac{\beta(x)/x^2}{[-\epsilon + \beta(x)/x]} dx \, ,$$
$$\ln Z_\eta = 2 \int_0^\alpha \frac{\gamma_\eta(x)}{[-\epsilon x + \beta(x)]} dx \, . \tag{55}$$

First, let us integrate to find $Z_\alpha$ using the two-loop $\beta$ function. Actually, it is convenient to use the modified form

$$\beta(\alpha)=-a\alpha^2/(1-b\alpha)\ ,$$
$$a=1-\frac{2}{N},\quad b=\frac{1}{N}\ , \tag{56}$$

which differs from the two-loop answer only in higher orders. We will see that the divergences of $Z_\alpha$ are insensitive to the higher-order contributions, and the approximation of Eq. (56) gives a simpler answer after integration.

$$\ln Z_\alpha{}^{-1}=\left[1-\frac{b\epsilon}{a}\right]^{-1}\ln\left[1+\frac{a-b\epsilon}{\epsilon}\alpha\right]\ . \tag{57}$$

Expanding in powers of $\alpha$ gives

$$\ln Z_\alpha{}^{-1}=\frac{a\alpha}{\epsilon}\sum_{n=0}\frac{(-1)^n}{n+1}\left[\frac{a-b\epsilon}{\epsilon}\alpha\right]^n \tag{58}$$

which has the expected structure for $\ln Z_\alpha$ as a sum of poles in $\epsilon$. However, an expansion in powers of $b\sim O(1/N)$ gives the result

$$\ln Z_\alpha{}^{-1}=\left[\sum_{n=0}\left[\frac{b\epsilon}{a}\right]^n\right]\left[\ln\left[1+\frac{a\alpha}{\epsilon}\right]-\sum_{j=0}\frac{1}{j}\left[\frac{b\alpha}{\epsilon+a\alpha}\right]^j\right]\ . \tag{59}$$

If one now takes the limit $\epsilon\to 0$, we get

$$Z_\alpha{}^{-1}=1+\frac{a\alpha}{\epsilon}-b\alpha\ln\epsilon+(\text{convergent terms})\ , \tag{60}$$

which agrees exactly with the calculations of Sec. II. Taking the one-loop result for the anomalous dimension in Eq. (55) and the $\beta$ function from Eq. (56), we calculate the wave-function renormalization constant as

$$\ln Z_\eta=2\int_0^\alpha\frac{c(1-bx)}{\epsilon+(a-b\epsilon)x}dx$$
$$=\frac{2c}{a-b\epsilon}\left[\frac{a}{a-b\epsilon}\ln\left[1+\frac{a-b\epsilon}{\epsilon}\alpha\right]-b\alpha\right]\ , \tag{61}$$
$$c=-\frac{1}{2N}\ .$$

Expanding in powers of $1/N$ and letting $\epsilon\to 0$ gives

$$\ln Z_\eta=-\frac{2c}{a}\ln\epsilon+\text{finite terms}$$
$$=\frac{1}{N-2}\ln\epsilon+\text{finite terms}\ , \tag{62}$$

which is also in agreement with the result of Sec. II. Furthermore, we can show that the higher-order perturbative contributions to $\beta$ and $\gamma$ give only terms which do not diverge as $\epsilon\to 0$. Hence, these extra terms can be scaled away by finite renormalizations and the forms given in Eq. (60) and (62) are sufficient to render the theory finite. To demonstrate this explicitly write

$$\beta(\alpha)=-a\alpha^2[1+b\alpha+\alpha^2h(\alpha)] \tag{63}$$

and

$$\gamma(\alpha)=c\alpha+\alpha^2g(\alpha)\ , \tag{64}$$

where $h(\alpha)$ and $g(\alpha)$ are $O(1/N)$ and finite at $\alpha=0$. Using the superscript (2) to denote quantities calculated with $h(\alpha)=g(\alpha)=0$, we find

$$\ln Z_\alpha^{(2)}-\ln Z_\alpha=\frac{\epsilon}{a}\int_0^\alpha dx\frac{h(x)}{(1+bx)[1+bx+x^2h(x)]}$$
$$+O(\epsilon^2\ln\epsilon)\ . \tag{65}$$

The difference vanishes as $\epsilon\to 0$ so that the full $Z_\alpha{}^{-1}$ and the result of Eq. (60) differ only by the addition of finite and vanishing terms. Therefore, we can perform finite multiplicative renormalizations and take the coupling-constant renormalization factor to be

$$Z_\alpha{}^{-1}=1+\left[1-\frac{2}{N}\right]\frac{\alpha}{\epsilon}-\frac{\alpha}{N}\ln\epsilon\ . \tag{66}$$

Similarly, we have for the wave-function renormalization constant

$$\ln Z_\eta^{(2)}-\ln Z_\eta=2\int_0^\alpha dx\frac{g(x)(1+bx)-xch(x)}{a(1+bx)[1+bx+x^2h(x)]}$$
$$+O(\epsilon\ln\epsilon)\ . \tag{67}$$

Therefore, a finite rescaling enables us to set

$$Z_\eta=\exp\left[\frac{1}{N-2}\ln\epsilon\right]\ . \tag{68}$$

The charge renormalization constant has divergent contributions only to $O(1/N)$. While the wave-function renormalization has divergences to all orders in $1/N$, the higher-order terms are all determined by exponentiation of the result given by low-order perturbation theory.

A similar analysis goes through for the Gross-Neveu model. When expressed in terms of $\alpha=g^2/\pi$, the perturbative renormalization-group functions calculated to two-loop orders are[16]

$$\beta(\alpha)=-\left[1-\frac{1}{N}\right]\alpha^2\left[1-\frac{\alpha}{2(N-1)}\right]\ ,$$
$$\gamma_\psi(\alpha)=\frac{\alpha^2}{8N}\ . \tag{69}$$

The charge renormalization constant with the minimally required divergent structure may be found by the same procedure used for the NLSM with the result

$$Z_\alpha{}^{-1}=1+\left[1-\frac{1}{N}\right]\frac{\alpha}{\epsilon}+\frac{\alpha}{2(N-1)}\ln\epsilon\ . \tag{70}$$

The terms up to $O(1/N)$ agree with the calculations in Sec. II B. In this case we predict $\ln\epsilon$ divergences in $Z_\alpha$ to all orders in $1/N$, but the higher-order terms are all predictable from the second-order perturbative results.

The anomalous dimension is second order in $\alpha$. Therefore, the wave-function renormalization constant should be finite in the $1/N$ expansion as indicated below,

$$\begin{aligned}\ln Z_\psi &= 2\int_0^\alpha \frac{\gamma_\psi(x)}{-\epsilon x+\beta(x)} \\ &= 2c\int_0^\alpha \frac{x(1+bx)}{\epsilon+(a+b\epsilon)x}dx \\ &= 2c\left[\frac{a\alpha}{(a+b\epsilon)^2}+\frac{1}{2}\frac{b\alpha^2}{(a+b\epsilon)} \right. \\ &\quad \left. -\frac{a\epsilon}{(a+b\epsilon)^3}\ln\left[1+\frac{a+b\epsilon}{\epsilon}\alpha\right]\right] \qquad (71)\end{aligned}$$

with $a=(1-1/N)$, $b=1/[2(N-1)]$, and $c=-1/8N$.

When expanded in powers of $\alpha$, $\ln Z_\psi$ has a pole structure in $\epsilon$, but when expanded in powers of $1/N$, it is finite as $\epsilon\to 0$. This is consistent with the results of Sec. II B where we found no divergent wave-function renormalization to $O(1/N)$. Therefore, no infinite wave-function renormalization should be required for the fermion fields in the $1/N$ expansion of the GNM.

Using Eqs. (66) or (70) we can use the first line of Eq. (54) to compute the $\beta$ functions for the models in the $1/N$ renormalization scheme. At $\epsilon=0$, we have the same form for both models,

$$\beta(\alpha)=-a\alpha^2 \qquad (72)$$

which agrees with the one-loop result in perturbation theory. This appears to contradict a general result that in any mass-independent renormalization scheme, the form of the $\beta$ function is the same up to two-loop order.[17] However, this result assumes that one can expand the coupling constant in one scheme as a power series in the coupling constant of another scheme to at least second order, i.e.,

$$\alpha'=\alpha+k\alpha^2+O(\alpha^3)\ . \qquad (73)$$

In the present case this expansion does not exist. Turning the argument around one can compute the relation between the coupling constants in two schemes by solving the differential equation

$$\beta'(\alpha')=\frac{\partial\alpha'}{\partial\alpha}\beta(\alpha)\ . \qquad (74)$$

For example, if we take

$$\beta'(\alpha')=-a(\alpha')^2 \qquad (75)$$

and

$$\beta(\alpha)=-a\alpha^2/(1-b\alpha)$$

as used in our calculations, then Eq. (74) gives

$$\alpha'=\alpha/[1+b\alpha\ln(\alpha/\alpha_0)] \qquad (76)$$

which does not admit a power series expansion about $\alpha=0$.[18]

The renormalized self-energy determines the dynamical mass generation. Using our results to $O(1/N)$ and the relation of Eq. (76) leads to

$$\begin{aligned}m^2 &= K\mu^2\exp(-2a/\alpha') \\ &= K\mu^2(\alpha/\alpha_0)^{-2ab}\exp(-2a/\alpha)\ . \qquad (77)\end{aligned}$$

$K$ is a finite calculable number. The last expression for $m^2$ agrees with the two-loop result of perturbation theory (Brezin and Zinn-Justin[6]).

In our $1/N$ renormalization scheme we found $Z_\eta$ to be coupling-constant independent and $Z_\psi=1$, which implies that there are no anomalous dimensions for the $\eta$ or $\psi$ fields. An argument similar to that given above for the $\beta$ function shows that this does not contradict the usual argument that the form of the one-loop anomalous dimension should be scheme independent.

## IV. DISCUSSION

The graphs of the $1/N$ expansions considered here can be related to infinite sums of Feynman graphs in perturbation theory. Our results show that many of the ultraviolet divergences of perturbation theory are damped by this resummation. The infinite renormalizations of the $1/N$ expansions are determined completely from the one-loop anomalous dimensions and the two-loop $\beta$ functions of the perturbative expansions.

The only natural expansion parameter for the $S$ matrices and the spectra of the NLSM and GNM is $1/N$. The same may be true as well for unbroken non-Abelian gauge fields in four dimensions. For QCD the most important point is not whether $\frac{1}{3}$ is a small expansion parameter but whether the $1/N$ expansion starts out in the correct (confining) phase.

We have seen that the ultraviolet singularities of the field theories are less severe in the $1/N$ expansion than in ordinary perturbation theory. The simplification of the renormalization constants has, however, the corollary that the $1/N$ method tells us less about the detailed behavior of the renormalized theory at short distances than does perturbation theory.

It is not yet clear whether it is possible to carry out an operator-product expansion in the $1/N$. Therefore, it may turn out that while the $1/N$ expansion is well suited to studying the on-shell behavior of asymptotically free theories, the short-distance and light-cone behavior probed by such processes as deep-inelastic lepton scattering and the production of high-mass muon pairs are better explained via perturbation theory.

There is a possible further restriction implied by the $1/N$ scheme. Because of the $\ln\epsilon$ singularities there are branch points of the amplitudes at $\epsilon=0$, the critical dimension. Therefore, it is not clear that we can continue the amplitudes calculated beyond the leading order in $1/N$ and renormalized by our method to higher dimensions. It is above the critical dimension that the two-phase structures manifest themselves. In QCD, however, one would certainly be willing to pay the price of losing this information in exchange for real understanding of the physical content of the theory at the critical dimension $d=4$.

Our conclusions about the nature of the ultraviolet singularities deduced from the behavior of the renormalization-group functions in Sec. III do not depend upon the detailed calculations of Sec. II. These explicit examples, however, are important to convince us that there are no additional singularities or other problems lurking in the summation of an infinite set of perturbative Feynman diagrams implicit in each order of the $1/N$ expansion (not to mention the fact that this is also the route by which we first discovered this behavior).

At this point, however, we can use the renormalization-group approach in theories for which explicit calculation in the $1/N$ approach is not tractable. We have in mind particularly non-Abelian gauge theories in four dimensions. For an SU($N$) theory with $N_f$ flavors of fermions in the fundamental representation and with the gauge constant written as $g^2=\alpha/N$ the two-loop $\beta$ function is[19]

$$\beta(\alpha)=-B_0\alpha^2+B_1\alpha^3\;,$$
$$B_0=\frac{1}{8\pi^2}\left[\frac{11}{3}-\frac{2N_f}{3N}\right],\tag{78}$$

and

$$B_1=\frac{1}{(16\pi^2)^2}\left[-\frac{34}{3}+\frac{10N_f}{3N}+\frac{(N^2-1)}{N^2}N_f{}^2\right].$$

This $\beta$ function can be written in the form of Eq. (56) with $a=B_0$ and $b=-B_1/B_0$. In this case $b=O(1)$, but the singular part of $Z_\alpha{}^{-1}$ is still given by Eq. (60) as

$$Z_\alpha{}^{-1}=1+\frac{B_0\alpha}{\epsilon}+\frac{B_1}{B_0}\alpha\ln\epsilon\;.\tag{79}$$

This technique can also be used to treat anomalous dimensions and their associated renormalization constants in gauge theories. The $\ln\epsilon$ singularity enters at the zeroth order of $1/N$ rather than at the first order as in the two-dimensional models. This is not surprising since the lowest-order contributions for the NLSM and GNM are sums of an infinite set of bubble graphs, which require only single-pole renormalizations, while the lowest-order terms for gauge theories consist of all planar graphs which have more complicated ultraviolet divergences.[20]

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## APPENDIX A

In this appendix we combine functional integration techniques with power counting to show that if the $1/N$ expansion of the nonlinear $\sigma$ model can be renormalized by local counterterms, then just two counterterms are required to renormalize graphs with external $\eta$ lines only.

First, consider a vertex function with $\eta_e$ external $\eta$ lines and $\lambda_e$ external $\lambda$ lines. Using the Feynman rules for the $1/N$ expansion derived in Sec. II, standard power counting gives the superficial degree of divergence of such a vertex as

$$D=2(1-\lambda_e)\;.\tag{A1}$$

Therefore, we initially require counterterms with zero or one $\lambda$ field and any number of $\eta$ fields. The general form of the counterterms can be written as

$$\begin{aligned}\mathcal{L}_{\rm CT}=&\tfrac{1}{2}(\eta_\mu\cdot\eta_\mu)\sum_{k=0}a_k(\eta^2)^k+\frac{\lambda}{2}\sum_{k=0}b_k(\eta^2)^k\\&+(\eta\cdot\eta_\mu)(\eta\cdot\eta_\mu)\sum_{k=0}c_k(\eta^2)^k\\&+m^2\sum_{k=1}d_k(\eta^2)^k\;.\end{aligned}\tag{A2}$$

If all these terms were really needed, the counterterm Lagrangian would be nonlocal. However, for graphs without external $\lambda$ lines only the $a_0$ and $b_0$ terms are required. Essentially, it is rescaling of $\lambda$ and the constraint equation

$$\eta^2=N/g^2$$

which allow us to eliminate all the other terms.

In the $1/N$ expansion the lowest-order graphs contributing divergent terms of these forms determine that

$$\begin{aligned}&a_k=O(1/N^{k+1}),\;\; b_k=O(1/N^k)\;,\\&c_k=O(1/N^{k+2}),\;\; d_k=O(1/N^k)\;.\end{aligned}\tag{A3}$$

Green's functions for $\eta$ particles are generated by adding to the Lagrangian a source term for the $\eta$ fields, viz.

$$\begin{aligned}Z(J)=&\int[d\eta\,d\lambda]\exp\left[i\int(J\cdot\eta+\mathcal{L}+\mathcal{L}_{\rm CT})\right]\\=&\int[d\eta\,d\lambda]\exp\left[i\int\left[J\cdot\eta+\tfrac{1}{2}(\eta_\mu\cdot\eta_\mu)\sum_1 a_k(\eta^2)^k+(\eta\cdot\eta_\mu)^2\sum_0 c_k(\eta^2)^k+m^2\sum_1 d_k(\eta^2)^k\right.\right.\\&\left.\left.+\frac{\lambda}{2}\sum_1 b_k(\eta^2)^k+\tfrac{1}{2}(1+a)(\eta_\mu\cdot\eta_\mu)-\tfrac{1}{2}m^2\eta^2-\frac{\lambda}{2}[\eta^2-(1+b)N/g^2]\right]\right]\;.\end{aligned}\tag{A4}$$

We have made obvious definitions of $a$ and $b$ in terms of $a_0$ and $b_0$, respectively. Since $\lambda$ is a dummy integration variable, we can rescale and define

$$\lambda'=\lambda(1-b_1)\;.\tag{A5}$$

The Jacobian of this transformation can be absorbed in the overall normalization of the functional integral. The $b_k$ counterterms have the same form as before when written in terms of $\lambda'$ and $b_k'$ with

$$b_k' = b_k/(1-b_1) . \tag{A6}$$

In the $1/N$ expansion we write

$$1/(1-b_1) = 1 + b_1 + (b_1)^2 + \cdots \tag{A7}$$

so that we still have

$$b_k' \sim O(1/N^k) ,$$

except that the $b_1$ counterterm has disappeared. We can now write (dropping primes)

$$Z(J) = \int [d\eta\, d\lambda] \exp\left[ i \int \left[ J\cdot\eta + \cdots + \frac{\lambda}{2} \sum_3 b_k(\eta^2)^k - \frac{\lambda}{2}\eta^2(1-b_2\eta^2) + \frac{N}{g^2}(1+b)\lambda \right]\right] \tag{A8}$$

with $b_2 \sim O(1/N^2)$.

Next we set

$$\lambda' = \lambda(1-b_2\eta^2) . \tag{A9}$$

The Jacobian of this transformation is

$$\prod_x [1-b_2\eta^2(x)] = \exp\left[ \sum_x \ln[1-b_2\eta^2(x)] \right] = \exp\left[ \delta^{(n)}(0) \int dx \ln[1-b_2\eta^2(x)] \right] . \tag{A10}$$

Since $\delta^{(n)}(0)=0$ in dimensional regularization, the Jacobian can be set equal to 1.

For the $b_k$ counterterms, substituting Eq. (A9) into Eq. (A8) and expanding $1/(1-b_2\eta^2)$ as a geometric series gives

$$\sum_3 b_k \lambda(\eta^2)^k = \sum_3 b_k' \lambda'(\eta^2)^k$$

with (A11)

$$b_k' \sim O(1/N^k), \quad k \geq 3 .$$

The term in the action involving the $\lambda$ field only transforms as

$$\frac{N}{g^2}(1+b)\lambda = \frac{N}{g^2}(1+b)[\lambda' + \lambda' b_2\eta^2 + \lambda'(b_2\eta^2)^2 + \cdots ] . \tag{A12}$$

Dropping primes, we have a new $\lambda\eta^2$ term on the right-hand side of Eq. (A12) which again can be removed by a $c$ number rescaling of $\lambda$. Since $b_2 \sim O(1/N^2)$, the coefficient of $\lambda(\eta^2)^2$ which was originally $O(1/N^2)$ has been reduced to $O(1/N^3)$ by the change of variable in Eq. (A9). By repeating this process we have an inductive proof that the $\lambda(\eta^2)^2$ counterterm can be removed to all orders of $1/N$.

Next we prove inductively that all $\lambda(\eta^2)^k$ counterterms can be removed. Assume that all such terms up to $k=m$ have been transformed away. We have

$$Z(J) = \int [d\lambda\, d\eta] \exp\left[ i \int \left[ J\cdot\eta + \cdots - \frac{\lambda}{2}\eta^2[1-b_{m+1}(\eta^2)^m] + \frac{N}{2g^2}(1+b)\lambda \right]\right] . \tag{A13}$$

Define

$$\lambda' = \lambda[1-b_{m+1}(\eta^2)^m] .$$

With dimensional regularization the Jacobian is unity as in the previous case. The term in $\lambda$ alone transforms as

$$\frac{N}{2g^2}(1+b)\lambda = \frac{N}{g^2}(1+b)\lambda'[1+b_{m+1}(\eta^2)^m + b_{m+1}{}^2(\eta^2)^{2m} + \cdots ] . \tag{A14}$$

By our inductive hypothesis the $\lambda(\eta^2)^m$ term is removable. For $m>1$, higher-order terms in the expansion correspond to counterterms with powers of $(\eta^2)$ greater than $m+1$. The redefinition of $\lambda$ has removed the $\lambda(\eta^2)^{m+1}$ term and redefined the $b_k$'s for $k>m+1$. By repetition of this procedure all the $\lambda(\eta^2)^k$ counterterms can be eliminated to any order in $1/N$.

Next we consider the $(\eta\cdot\eta_\mu)(\eta\cdot\eta_\mu)(\eta^2)^k$ counterterms. We can use the relation

$$(\eta\cdot\eta_\mu) = -\partial_\mu \frac{\delta}{\delta\lambda(x)} S , \tag{A15}$$

where $S$ is the action with only the $a$ and $b$ counterterms, to write

$$Z(J) = \int [dn\, d\lambda] \exp\left\{ i \int \left[ J\cdot\eta + \cdots + \sum_0 c_k(\eta^2)^k(\eta\cdot\eta_\mu) \left[ i\partial_\mu \frac{\delta}{\delta\lambda(x)} \right]\right]\right\} \exp(iS) . \tag{A16}$$

Since the only $\lambda$ dependence is now in $S$, ordinary and functional integrations by parts set the coefficient of the $c_k$'s to zero.

Finally, for the $a_k$ and $d_k$ terms we use

$$(\eta^2) = \left[ 2i\frac{\delta}{\delta\lambda(x)} + \frac{N}{g^2}(1+b) \right] S \tag{A17}$$

and functional integration by parts to obtain

$$Z(J)=\int [d\eta\, d\lambda]\exp\left[i\int\left\{J\cdot\eta+\tfrac{1}{2}(\eta_\mu\cdot\eta_\mu)\left[1+a+\sum_1 a_k\left[\frac{N}{g^2}(1+b)\right]^k\right]-\tfrac{1}{2}m^2\eta^2-\frac{\lambda}{2}[\eta^2-(1+b)N/g^2]\right.\right.$$
$$\left.\left.+\tfrac{1}{2}m^2\sum_1 d_k[(1+b)N/g^2]^k\right\}\right]. \tag{A18}$$

The $m^2$ term is an irrelevant phase factor which can be absorbed into the implicit normalization. With an obvious redefinition of the coefficient of the kinetic energy term, we have the desired result

$$Z(J)=\int [d\eta\, d\lambda]\exp\left[i\int\left\{\tfrac{1}{2}(1+a)(\eta_\mu\cdot\eta_\mu)-\tfrac{1}{2}m^2\eta^2-\frac{\lambda}{2}[\eta^2-(1+b)N/g^2]+J\cdot\eta\right\}\right]. \tag{A19}$$

Writing the Lagrangian in terms of unrenormalized quantities as

$$\mathscr{L}=\tfrac{1}{2}(\eta_\mu^0\cdot\eta_\mu^0)-\tfrac{1}{2}m_0{}^2\eta_0{}^2-\frac{\lambda_0}{2}[(\eta_0\cdot\eta_0)-N/g_0{}^2]\,, \tag{A20}$$

we see that renormalization is achieved by a multiplicative scaling

$$\begin{aligned}&\eta_0=\sqrt{Z_\eta}\,\eta,\quad Z_\eta=1+a\,,\\ &m_0{}^2=m^2/Z_\eta\,,\\ &\lambda_0=\lambda/Z_\eta\,,\\ &g_0{}^2=g^2\mu^\epsilon Z_g,\quad Z_g{}^{-1}Z_\eta{}^{-1}=1+b\,.\end{aligned} \tag{A21}$$

We have explicitly restored the factor $\mu^\epsilon$ required to keep $g^2$ dimensionless for arbitrary dimension.

## APPENDIX B

In this appendix we show that if the $1/N$ expansion of the Gross-Neveu model can be renormalized by local counterterms, then only two counterterms are required for graphs with external fermion lines. For a general graph let $E_\sigma$ and $E_F$ denote the number of external $\sigma$ and fermion lines, respectively. From the Feynman rules of Sec. I the superficial degree of divergence of such a graph is

$$D=2-E_\sigma-E_F/2\,. \tag{B1}$$

This implies that the theory requires the counterterms given by

$$\begin{aligned}\mathscr{L}_{\rm CT}=a\sigma+\frac{N}{2g^2}b\sigma^2+c\bar\psi i\,\partial\psi+dm\,\bar\psi\psi\\ +e\sigma\bar\psi\psi+f(g^2/2N)(\bar\psi\psi)^2\,.\end{aligned} \tag{B2}$$

As shown by Schonfeld,[9] the only counterterm needed for the four-point function is $(\bar\psi\psi)^2$. In the symmetric phase, invariance under $\psi\to\gamma_5\psi$ and $\sigma\to-\sigma$ guarantees that the $a$ and $d$ terms are absent. If we restrict ourselves to graphs with only external fermion lines, then we can rescale $\sigma$ at will in the functional integral to eliminate the $e$ term. The generating functional for fermion graphs takes the form

$$Z(\eta,\bar\eta)=\int [d\psi\, d\bar\psi\, d\sigma]\exp\left[i\int\left[\bar\psi i\,\partial\psi-\frac{N}{2g^2}\sigma^2-\sigma\bar\psi\psi+\frac{N}{2g^2}b\sigma^2-c\bar\psi i\,\partial\psi+f(g^2/2N)(\bar\psi\psi)^2+\bar\eta\psi+\bar\psi\eta\right]\right]. \tag{B3}$$

The exponent in the integral can be written as

$$\begin{aligned}&\bar\psi i\,\partial\psi Z_1+(g^2/2N)(\bar\psi\psi)^2[f+1/(1-b)]\\ &\quad-(N/2g^2)[\sqrt{1-b}\,\sigma+(g^2/N)(1/\sqrt{1-b}\,)(\bar\psi\psi)]^2\,.\end{aligned} \tag{B4}$$

By rescaling and translation of the integration variable $\sigma$ we can let

$$\begin{aligned}&\sqrt{1-b}\,\sigma+(g^2/N)(1/\sqrt{1-b})(\bar\psi\psi)\\ &\qquad\to Z_2{}^{1/2}\sigma+Z_2{}^{-1/2}(g^2/N)(\bar\psi\psi)\,,\end{aligned} \tag{B5}$$

$$Z_2{}^{-1}=f+1/(1-b)\,.$$

Then we can write

$$\begin{aligned}Z(\eta,\bar\eta)=\int [d\psi\, d\bar\psi\, d\sigma]\exp\Big[i\int[\,\bar\psi i\,\partial\psi Z_1-\sigma\bar\psi\psi\\ -(N/2g^2)\sigma^2 Z_2\\ +\bar\eta\psi+\bar\psi\eta]\Big]\,.\end{aligned} \tag{B6}$$

Therefore, the theory is renormalized by two constants which can be written in terms of multiplicative constants as

$$\begin{aligned}&\psi_0=\sqrt{Z_\psi}\,\psi,\quad Z_\psi=Z_1\,,\\ &\sigma_0=\sigma/Z_1\,,\\ &g_0{}^2=g^2\mu^\epsilon Z_g,\quad Z_2=1/(Z_gZ_1{}^2)\,.\end{aligned} \tag{B7}$$

At the last step we have inserted the factor $\mu^\epsilon$ required to keep $g^2$ dimensionless.

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[10]The general formulation of the $1/N$ expansion and the leading-order results given here and in Sec. II B are developed in the previously cited references. We include this material for coherence of our exposition.

[11]N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1960); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).

[12]See, e.g., Sec. 9.132, formula 2 on p. 1043 of I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products,* fourth edition, translated by Yu. V. Geronimus and M. Yu. Tseytlin (Academic, New York, 1965).

[13]G. Källén and J. Toll, J. Math. Phys. **6**, 299 (1965).

[14]If one evaluates the integral in two dimensions using a naive ultraviolet cutoff, $\Lambda$, one obtains a $\ln(\ln\Lambda)$ dependence. See Schonfeld, Ref. 9.

[15]S. Hikami and E. Brezin, J. Phys. A **11**, 1141 (1978). See also Brezin and Zinn-Justin, Ref. 6. The perturbative calculations include a "magnetic field" coupled to one component of the $\eta$ field which removes infrared divergences. The perturbative renormalization constants therefore, are generated only by ultraviolet singularities as in the $1/N$ expansion.

[16]We calculate these renormalization effects with a mass term $m\bar{\psi}\psi$ added to the Lagrangian to eliminate infrared divergences.

[17]For a pedagogical development of this and related properties of the renormalization group, see D. J. Gross, in *Methods in Field Theory,* 1975 Les Houches Lectures, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[18]By a finite renormalization we can obtain the form given in Eq. (56) as the exact $\beta$ function for the NLSM. This $\beta(\alpha)$ has a pole at $\alpha_c = 1/b$. If we start with $\alpha < \alpha_c$ for some renormalization scale $\mu_0^2$, then the running coupling constant $\bar{\alpha}(t)$, $t = \ln(\mu^2/\mu_0^2)$, reaches $\alpha_c$ for a finite negative $t$, and $\bar{\alpha}$ becomes complex as we continue further into the infrared region. Similarly, if we start with $\alpha > \alpha_c$ so that $\beta(\alpha) > 0$, the running coupling constant becomes complex in the ultraviolet region. But this behavior has emerged as an artifact of our renormalization scheme and should not be related directly to the physics of the long- or short-distance behavior of the model. A $\beta$ function of precisely the form of Eq. (56) has been calculated as the exact result for $N=2$ supersymmetric Yang-Mills theory; D. R. T. Jones, Phys. Lett. **123B**, 45 (1983). Again, since $\beta$ depends on the renormalization scheme beyond two-loop order, the attribution of physical consequences to the singularity is suspect.

[19]D. R. T. Jones, Nucl. Phys. **B75**, 531 (1974); W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974).

[20]Progress in proving the convergence of the sum of all renormalized planar diagrams has been made by G. 't Hooft, Commun. Math. Phys. **86**, 449 (1982); **88**, 1 (1983).