Field-strength formulation of gauge theories. The Hamiltonian approach in the Abelian theory

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(Received 12 March 1984)

We develop a Hamiltonian approach to the field-strength or dual formation of the Abelian gauge theory in which the potential A^μ is eliminated as a dynamical variable. Our work is based on the covariant gauge $x^\mu A_\mu(x)=0$ which allows a simple elimination of A^μ in terms of the field strengths $F^{\mu\nu}$. We obtain complete results for the generating functional for the Green's functions of the theory, $Z=Z[\vec{f},\vec{g}]$, where $\vec{f}$ and $\vec{g}$ are nonlocal currents coupled to $\vec{E}$ and $\vec{B}$, and illustrate some unfamiliar aspects of the new formalism.

I. INTRODUCTION

It was pointed out by Halpern[1] that gauge field theories can be written in a field-strength formulation in which the usual gauge potentials $A^\mu(x)$ are eliminated in favor of the field strengths $F^{\mu\nu}(x)$. This formulation has considerable potential advantages for the construction of confining states in non-Abelian theories, the calculation of Wilson loop integrals, and the study of monopole configurations. In a recent paper,[2] we rederived and generalized the field-strength formulation for gauge theories written initially in the coordinate gauge[3,4] $x^\mu A_\mu=0$. This choice leads to a substantially simpler and more symmetrical formalism than that obtained by Halpern. Results similar to ours have also been obtained by Itabashi.[5]

In the present paper, we continue our investigation of the field-strength formulation, restricted this time to the Abelian case, and develop the theory from a Hamiltonian rather than Lagrangian point of view. This has the advantage of clarifying the constraints on the new canonical variables, and providing (modified) canonical quantization conditions. Not surprisingly, we obtain complete results for the generating functional for the Green's functions of the theory, $Z=Z[\vec{f},\vec{g}]$, where $\vec{f}$ and $\vec{g}$ are nonlocal currents coupled to $\vec{E}$ and $\vec{B}$. We illustrate the unfamiliar aspects of the formalism by using $Z[\vec{f},\vec{g}]$ to obtain the classical equations of motion and various Green's functions for the field strengths.

II. THE HAMILTONIAN FORMALISM

A. The Lagrangian expression for $Z[J]$

The generating functional $Z[J]$ for an Abelian gauge theory with an external current $J^\mu(x)$ is given in the coordinate gauge $x^\mu A_\mu(x)=0$ by the functional integral

$$Z[J]=\int \mathscr{D}A \exp\left[-i\int d^4x(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}[A]+A_\mu J^\mu)\right] \times \prod_x \delta(x_\mu A^\mu)\ , \tag{1}$$

where we have suppressed an (infinite) overall normalization constant. Here $A^\mu(x)$ is the gauge potential, and the field strengths $F^{\mu\nu}[A]$ are defined by

$$F^{\mu\nu}[A]=\partial^\mu A^\nu-\partial^\nu A^\mu\ . \tag{2}$$

We showed in I that the potential A^μ for a coordinate-gauge field can be expressed relatively simply in terms of $F^{\mu\nu}$ by[2,6–9]

$$A^\mu[F]=\int_0^1 d\alpha\, \alpha x_\nu F^{\nu\mu}(\alpha x)\ . \tag{3}$$

That is, Eq. (2) can be inverted. The inverse is unique for potentials less singular than x^{-1} on the line $[0,x]$. We also showed that a tensor field $F^{\mu\nu}(x)$ is a coordinate-gauge field strength if and only if F satisfies the identities

$$\epsilon^{\mu\nu\sigma\lambda}x_\sigma\partial^\rho {}^*F_{\rho\lambda}=0\ , \tag{4}$$

where

$${}^*F_{\rho\lambda}=\tfrac{1}{2}\epsilon_{\rho\lambda\alpha\beta}F^{\alpha\beta}\ . \tag{5}$$

The conditions in Eq. (4) are just the restriction of the usual Bianchi identities

$$\partial^\rho {}^*F_{\rho\lambda}=0 \tag{6}$$

to the hypersurface orthogonal to x^μ. If these conditions are satisfied, $A[F]$ is a potential for F and the remaining Bianchi identity

$$x^\lambda\partial^\rho {}^*F_{\rho\lambda}=0 \tag{7}$$

is satisfied automatically.

The results above may be used to eliminate A in terms of F in Eq. (1), leading to an expression for $Z[J]$ in terms of the field strengths,[2]

$$Z[J]=\int \mathscr{D}F\,\mathscr{D}\lambda \exp\left[-i\int d^4x(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu[F]J^\mu+\lambda^\sigma\partial^\rho {}^*F_{\rho\sigma})\right]\ . \tag{8}$$

The functional integrations involve the six independent field strengths $F^{\mu\nu}$ with $\nu<\mu$ and four Lagrange multiplier fields λ^σ used to enforce the Bianchi constraints. In fact, only the three constraints described by Eq. (4) need to be enforced; the fourth Bianchi identity is then satisfied automatically. The component of λ parallel to x is therefore superfluous, and we can impose the constraint $x^\mu\lambda_\mu(x)=0$ without changing the physical content of Eq. (8). (This point is discussed in detail in I.) We will use this freedom later.

B. The Hamiltonian formalism

The action in the functional integral for $Z[J]$ can be written as[2]

$$S[J]=\int d^4x\,\mathscr{L}[F,\lambda,J]$$
$$=-\int d^4x\left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}J_{\mu\nu}F^{\mu\nu}+\lambda^\sigma\partial^\rho{}^*F_{\rho\sigma}\right)\,, \quad (9)$$

where $J^{\mu\nu}$ is a nonlocal current density

$$J^{\mu\nu}(x)=\int_1^\infty d\gamma\,\gamma^2[x^\mu J^\nu(\gamma x)-x^\nu J^\mu(\gamma x)]\,. \quad (10)$$

It will be convenient for our development of the Hamiltonian formalism (in which time plays a special role) to introduce the noncovariant notation

$$\lambda^\mu=(\beta,\vec{\lambda})\,, \quad (11)$$

$$E^k=F^{k0},\quad B^k=-\tfrac{1}{2}\epsilon^{0ijk}F^{ij}\,, \quad (12)$$

$$f^k=J^{k0},\quad g^k=-\tfrac{1}{2}\epsilon^{0ijk}J^{ij}\,. \quad (13)$$

With these conventions

$$S[J]=\int d^4x\Big[\tfrac{1}{2}(\vec{E}^2-\vec{B}^2)-\vec{\lambda}\cdot(\partial_0\vec{B}+\nabla\times\vec{E})-\beta\nabla\cdot\vec{B}$$
$$+\vec{f}\cdot\vec{E}-\vec{g}\cdot\vec{B}\Big]\,, \quad (14)$$

where

$$\vec{f}=\int_1^\infty d\gamma\,\gamma^2[\vec{x}J^0(\gamma x)-x^0\vec{J}(\gamma x)]\,, \quad (15)$$

and

$$\vec{g}=\int_1^\infty d\gamma\,\gamma^2\vec{J}(\gamma x)\times\vec{x}\,. \quad (16)$$

To introduce the Hamiltonian formalism, we initially regard $\vec{E}$, $\vec{B}$, $\vec{\lambda}$, and β as canonical variables, and define the canonical momenta as

$$\vec{\pi}_B(x)=\delta S/\delta\dot{\vec{B}}(x)=-\vec{\lambda}(x)\,, \quad (17a)$$

$$\vec{\pi}_E(x)=\delta S/\delta\dot{\vec{E}}(x)\approx 0\,, \quad (17b)$$

$$\vec{\pi}_\beta(x)=\delta S/\delta\dot{\beta}(x)\approx 0\,. \quad (17c)$$

The vanishing of the momenta $\vec{\pi}_E$ and π_β indicates that the Lagrangian is singular: there is no unique solution for the velocities in terms of the canonical coordinates and momenta. To deal with this problem, we will use Dirac's method for treating constrained Hamiltonian systems.[10,11] In this approach, Eqs. (17b) and (17c) appear as primary constraints which restrict the solution space to a submanifold of the full phase space. The constraints are incompatible with the canonical Poisson brackets, hence are said to be only weakly zero (≈ 0). The objective of Dirac's procedure is the construction of a complete set of constraints and gauge conditions, and of modified (Dirac) brackets for the remaining physical variables which are consistent with the constraints. The constraints can then be set strongly equal to zero ($=0$), and the theory expressed entirely in terms of the physical variables, the Hamiltonian, and the Dirac bracket relations. The transition to the quantum theory is effected by replacing the Dirac brackets by commutators.

The canonical Hamiltonian density

$$\mathscr{H}_c=\vec{\pi}_B\cdot\dot{\vec{B}}-\mathscr{L} \quad (18)$$

is independent of the weakly zero canonical momenta $\vec{\pi}_E$ and π_β, hence gives no information on $\dot{\vec{E}}$ and $\dot{\beta}$. However, it can be generalized to another weakly equivalent Hamiltonian by adding arbitrary multiples of $\vec{\pi}_E$ and π_β:

$$\mathscr{H}=\vec{\pi}_B\cdot\dot{B}+\vec{\pi}_E\cdot\vec{u}+\pi_\beta v-\mathscr{L}$$
$$=\tfrac{1}{2}(\vec{B}^2-\vec{E}^2)-\vec{\pi}_B\cdot\nabla\times\vec{E}+\beta\nabla\cdot\vec{B}-\vec{f}\cdot\vec{E}+\vec{g}\cdot\vec{B}$$
$$+\vec{\pi}_E\cdot\vec{u}+\pi_\beta v\,, \quad (19)$$

where the fields $\vec{u}$ and v coupled to $\vec{\pi}_E$ and π_β are to be determined.

The equations of motion are easily determined using the Hamiltonian

$$H=\int d^3x\,\mathscr{H} \quad (20)$$

and the canonical Poisson brackets,

$$\{B_i(x),\pi_{Bj}(x')\}_{x^0=x'^0}=\delta(\vec{x}-\vec{x}\,')\delta_{ij}\,,$$
$$\{B_i(x),B_j(x')\}_{x^0=x'^0}=\{\pi_{Bi}(x),\pi_{Bj}(x')\}_{x^0=x'^0}=0\,, \quad (21)$$

etc. The result is

$$\dot{\vec{B}}+\nabla\times\vec{E}\approx 0\,, \quad (22a)$$

$$\dot{\vec{\pi}}_B+\vec{B}-\nabla\beta+\vec{g}\approx 0\,, \quad (22b)$$

$$\dot{\vec{\pi}}_E+\vec{E}+\nabla\times\vec{\pi}_B+\vec{f}\approx 0\,, \quad (22c)$$

$$\dot{\pi}_\beta+\nabla\cdot\vec{B}\approx 0\,, \quad (22d)$$

$$\dot{\vec{E}}-\vec{u}\approx 0\,, \quad (22e)$$

$$\dot{\beta}-v\approx 0\,, \quad (22f)$$

subject to the constraints

$$\vec{\pi}_E\approx 0\,, \quad (23a)$$

$$\pi_\beta\approx 0\,. \quad (23b)$$

The constraints $\vec{\pi}_E\approx 0$, $\pi_\beta\approx 0$ must hold at all times, hence $\dot{\vec{\pi}}_E\approx 0$ and $\dot{\pi}_\beta\approx 0$, and those terms can be dropped in Eqs. (22c) and (22d). The resulting equations are not dynamical, but are simply new equations of constraint (secondary constraints in Dirac's nomenclature[10,11]). In particular, Eq. (22c) defines $\vec{E}$ in terms of $\vec{\pi}_B$ and $\vec{f}$. We show in the Appendix that Eqs. (22a)–(22d) and the primary constraints $\vec{\pi}_E \approx 0$, $\pi_\beta \approx 0$ lead to the usual equations of electrodynamics.

C. Extra constraints or gauge conditions

In order to use Dirac's procedure to construct the modified (Dirac) brackets for our constrained system, we need a complete set of constraints, each of which has a nonvanishing Poisson bracket with at least one other. The constraint $\pi_\beta \approx 0$ has vanishing brackets with the other constraints determined so far. This means that the constraints which follow from the form of the Lagrangian and Hamilton's equations are not complete and, not surprisingly, must be supplemented by at least two independent gauge conditions (the total number of constraints must be even[10]). We can, in fact, see quite easily that there are still two redundant variables in our system.

The first redundant variable is simply the component $x_\mu \lambda^\mu$ of $\lambda^\mu = (\beta, -\vec{\pi}_B)$ parallel to x^μ. As remarked above and proved in I, it is necessary to enforce only the Bianchi identities on the hypersurface perpendicular to x^μ, Eq. (4). The "reconstruction theorem" in I then shows that F can be expressed in the standard way in terms of the potential $A[F]$, Eq. (3), and the last Bianchi identity is just that, an identity rather than a constraint. Since the component of λ^μ parallel to x^μ appears in Eq. (8) as the Lagrange multiplier field which enforces the redundant Bianchi identity, it is itself a redundant variable, and may be eliminated by requiring that $x_\mu \lambda^\mu \approx 0$.

The second degree of arbitrariness in the field λ^μ is associated with the invariance of the action under the transformation

$$\lambda^\mu \rightarrow \lambda'^\mu = \lambda^\mu + \partial^\mu \chi \ . \tag{24}$$

This transformation is consistent with the gauge condition $x_\mu \lambda^\mu \approx 0$ provided the gauge function χ is homogeneous of degree zero, $x_\mu \partial^\mu \chi = 0$. A second gauge condition is therefore required.

We will choose our first gauge condition as

$$x_\mu \lambda^\mu = \vec{x}\cdot\vec{\pi}_B + x^0 \beta \approx 0 \ . \tag{25}$$

For this condition to be consistent, its time derivative must vanish,

$$\vec{x}\cdot\dot{\vec{\pi}}_B + \beta + x^0 \dot{\beta} \approx 0 \ . \tag{26}$$

The function $v = \dot{\beta}$ which appears in $\mathscr{H}$, Eq. (19), is so far unspecified. We will choose v to correspond to the Lorentz gauge condition $\partial_\mu \lambda^\mu \approx 0$, or in our present notation,

$$v = \dot{\beta} \approx \nabla\cdot\vec{\pi}_B \ . \tag{27}$$

If we use this condition and Eq. (22f) to eliminate $\dot{\beta}$ and $\dot{\vec{\pi}}_B$, and note that $\vec{x}\cdot\vec{g} = 0$, Eq. (26) reduces to the condition

$$(1 + \vec{x}\cdot\nabla)\beta + x^0 \nabla\cdot\vec{\pi}_B - \vec{x}\cdot\vec{B} \approx 0 \ . \tag{28}$$

This equation can be solved for fields $\vec{B}$, $\vec{\pi}_B$ less singular than x^{-2} and x^{-1} at the origin, that is, for fields consistent with the coordinate gauge. We choose the solution

$$\beta - \int_0^1 d\alpha [\alpha \vec{x}\cdot\vec{B}(\alpha x) - x^0 \nabla\cdot\vec{\pi}_B(\alpha x)] \approx 0 \tag{29}$$

as our second gauge condition. This condition completely fixes β. Equation (25) then fixes $\vec{x}\cdot\vec{\pi}_B$.

It remains only for us to show that the second gauge condition can be maintained at all times. Using Eqs. (22a)–(22d) and (25), we find after some algebra that the time derivative of the left-hand side of Eq. (29) is given by

$$\begin{aligned}
\dot{\beta} - \int_0^1 d\alpha [\alpha^2 \vec{x}\cdot\dot{\vec{B}}(\alpha x) - \nabla\cdot\vec{\pi}_B(\alpha x) - \alpha x^0 \nabla\cdot\dot{\vec{\pi}}_B(\alpha x)] \\
\approx \nabla\cdot\vec{\pi}_B - \int_0^1 d\alpha [-\alpha \vec{x}\cdot\nabla\times\vec{E}(\alpha x) - \nabla\cdot\vec{\pi}_B(\alpha x) + \alpha x^0 \nabla\cdot\vec{B}(\alpha x) - x^0 \nabla^2 \beta(\alpha x) + \alpha x^0 \nabla\cdot\vec{g}(\alpha x)] \\
\approx \nabla\cdot\vec{\pi}_B - (1 + \vec{x}\cdot\nabla)\nabla\cdot \int_0^1 d\alpha\, \vec{\pi}_B(\alpha x) - \int_0^1 d\alpha\, \alpha (\vec{x}\cdot\nabla\times\vec{f} + x^0 \nabla\cdot\vec{g})(\alpha x) \ .
\end{aligned} \tag{30}$$

The second term in the final expression can be reduced to $\nabla\cdot\vec{\pi}_B(x)$ by using the general relation

$$\begin{aligned}
(n + \vec{x}\cdot\nabla) \int_a^b d\alpha\, \alpha^{n-1} f(\alpha x) &= \int_a^b d\alpha (n\alpha^{n-1} + \alpha^n \vec{x}\cdot\nabla_{\alpha x}) f(\alpha x) \\
&= \int_a^b d\alpha \frac{d}{d\alpha} \alpha^n f(\alpha x) \\
&= \alpha^n f(\alpha b) - \alpha^n f(\alpha a) \ ,
\end{aligned} \tag{31}$$

and the first two terms therefore cancel. The last term vanished by Eqs. (15) and (16). We therefore find that the entire expression in Eq. (30) is weakly zero, and the constraint in Eq. (29) has a vanishing time derivative as required for consistency.

At this point we have the following set of constraints on the dynamical variables:

$$\vec{\phi}_1 = \vec{\pi}_E \approx 0 \ , \tag{32a}$$

$$\vec{\phi}_2 = \vec{E} + \nabla\times\vec{\pi}_B + \vec{f} \approx 0 \ , \tag{32b}$$

$$\phi_3 = \pi_\beta \approx 0 \ , \tag{32c}$$

$$\phi_4 = \vec{x}\cdot\vec{\pi}_B + x^0 \beta \approx 0 \ , \tag{32d}$$

$$\phi_5 = \nabla\cdot\vec{B} \approx 0 \ , \tag{32e}$$

$$\phi_6 = \beta - \int_0^1 d\alpha [\alpha \vec{x}\cdot\vec{B}(\alpha x) - x^0 \nabla\cdot\vec{\pi}_B(\alpha x)] \approx 0 \ . \tag{32f}$$

This set is complete. The ten constraints reduce the number of degrees of freedom from fourteen ($\vec{E}, \vec{\pi}_E, \vec{B}, \vec{\pi}_B, \beta, \pi_\beta$) to the expected four. Furthermore, the constraints are all second class (that is, each constraint has a vanishing Poisson bracket with at least one other constraint), and can therefore be used to construct the Dirac brackets for the system.[10,11]

D. The Dirac brackets

As emphasized above, the usual Poisson brackets $\{,\}$ which serve as the basis for the canonical quantization of unconstrained Hamiltonian systems, are inconsistent with the constraints in Eqs. (32). Dirac[10] showed, however, that given a set of second class constraints, it is always possible to construct a set of modified brackets $\{,\}^*$ which are consistent with the constraints. These can be used to quantize the system through the usual transition to commutators, $\{,\}^* \to -i[,]$.

The Dirac brackets are defined in terms of the ordinary Poisson brackets by[10,11]

$$\{A,B\}^* = \{A,B\} - \sum_{\alpha\beta} \int d^3x\, d^3x' \{A,\phi_\alpha(\vec{x})\} C^{-1}{}_{\alpha\beta}(\vec{x},\vec{x}\,') \times \{\phi_\beta(\vec{x}\,'),B\}\ , \tag{33}$$

where

$$C_{\alpha\beta}(\vec{x},\vec{x}\,') = \{\phi_\alpha(\vec{x}),\phi_\beta(\vec{x}\,')\} \tag{34}$$

and

$$\sum_\gamma \int d^3x'' C_{\alpha\gamma}(\vec{x},\vec{x}\,'')C^{-1}{}_{\gamma\beta}(\vec{x}\,'',\vec{x}\,') = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}\,')\ . \tag{35}$$

All brackets are calculated at equal times and the sums run over all the constraints. The Dirac bracket of any of the constraints with another variable vanishes by construction, and the constraints may therefore be set strongly to zero $(=0)$ wherever they appear if the Dirac brackets are used. In effect, the modified brackets pick out the independent dynamical variables on the surface of constraint in phase space.

The matrix $C(\vec{x},\vec{x}\,')$ is not block symplectic for the constraints in Eqs. (32), and is difficult to invert directly. We will therefore make use of the "nesting" property of the Dirac brackets,[10,11] calculate modified brackets for a subset of the constraints, and use those in the definition of the modified brackets for the rest of the constraints. (This is equivalent to setting the first set of constraints strongly to zero in the remainder of the calculation.)

We begin by defining a matrix $R(\vec{x},\vec{x}\,')$ of the Poisson brackets of the eight constraints ϕ_{1i}, ϕ_{2i}, ϕ_3, ϕ_4, $i=1,2,3$. This matrix is block symplectic, with three repetitions of the 2×2 symplectic matrix

$$r(\vec{x},\vec{x}\,') = \begin{bmatrix} 0 & -\delta(\vec{x}-\vec{x}\,') \\ \delta(\vec{x}-\vec{x}\,') & 0 \end{bmatrix} \tag{36}$$

on the diagonal in the 1,2 (constraint) space, one for each value of i, and $x^0 r(\vec{x},\vec{x}\,')$ on the diagonal in the 3,4 space. $R^{-1}(\vec{x},\vec{x}\,')$ is obtained by replacing $r(\vec{x},\vec{x}\,')$ by $-r(\vec{x},\vec{x}\,')$ and x_0 by $x_0{}^{-1}$,

$$R^{-1}(\vec{x},\vec{x}\,') = -\begin{bmatrix} \mathbb{1} & 0 \\ 0 & x^{0-1} \end{bmatrix} r(\vec{x},\vec{x}\,')\ , \tag{37}$$

where $\mathbb{1}$ is the 3×3 unit matrix. The modified bracket $\{,\}^R$ which is consistent with setting the constraints $\vec{\phi}_1$, $\vec{\phi}_2$, ϕ_3, and ϕ_4 strongly to zero is given by the construction in Eq. (33),

$$\begin{aligned}\{A,B\}^R &= \{A,B\} - \sum_{\alpha\beta}\int d^3x\, d^3x' \{A,\phi_\alpha(\vec{x})\} R^{-1}{}_{\alpha\beta}(\vec{x},\vec{x}\,')\{\phi_\beta(\vec{x}\,'),B\} \\ &= \{A,B\} - \int d^3x \left[\{A,\vec{\phi}_1(\vec{x})\}\cdot\{\vec{\phi}_2(\vec{x}),B\} - \{A,\vec{\phi}_2(\vec{x})\}\cdot\{\vec{\phi}_1(\vec{x}),B\} \right. \\ &\qquad \left. + \frac{1}{x^0}\{A,\phi_3(\vec{x})\}\{\phi_4(\vec{x}),B\} - \frac{1}{x^0}\{A,\phi_4(\vec{x})\}\{\phi_3(\vec{x}),B\} \right]\ . \end{aligned} \tag{38}$$

It is useful to note that the R bracket and the ordinary Poisson bracket are equivalent for the fields $\vec{B}$ and $\vec{\pi}_B$ since these quantities have vanishing Poisson brackets with $\vec{\phi}_1$ and ϕ_3, e.g.,

$$\{B_i(x),\pi_{Bj}(x')\}^R_{x^0=x'^0} = \{B_i(x),\pi_{Bj}(x')\}_{x^0=x'^0} = \delta(\vec{x}-\vec{x}\,')\delta_{ij},\ \text{etc.} \tag{39}$$

The Poisson brackets of the remaining constraints ϕ_5 and ϕ_6 define a matrix $S(\vec{x},\vec{x}\,')$,

$$S(\vec{x},\vec{x}\,') = \begin{bmatrix} 0 & -x^0\nabla^2 \int_0^1 d\alpha\, \alpha\delta(\vec{x}-\alpha\vec{x}\,') \\ x^0\nabla'^2 \int_0^1 d\alpha\,\alpha\delta(\alpha\vec{x}-\vec{x}\,') & 0 \end{bmatrix} \tag{40}$$

with the inverse

$$S^{-1}(\vec{x},\vec{x}\,') = \begin{bmatrix} 0 & -\vec{x}\cdot\nabla \dfrac{1}{4\pi x^0|\vec{x}-\vec{x}\,'|} \\ \vec{x}\,'\cdot\nabla' \dfrac{1}{4\pi x^0|\vec{x}-\vec{x}\,'|} & 0 \end{bmatrix}\ . \tag{41}$$

The Dirac brackets $\{,\}^*$ consistent with the strong vanishing of all constraints are now

$$\{A,B\}^* = \{A,B\}^R - \sum_{\alpha\beta}\int d^3x\, d^3x' \{A,\phi_\alpha(\vec{x})\}^R S^{-1}{}_{\alpha\beta}(\vec{x},\vec{x}\,')\{\phi_\beta(\vec{x}\,'),B\}^R\ , \tag{42}$$

where the sums run over $\alpha,\beta=5,6$.

It is clear from the constraints $\vec{\pi}_E=\vec{\phi}_1=0$, $\pi_\beta=\phi_3=0$ and the explicit expressions for $\vec{E}$ and β given by the conditions $\vec{\phi}_2=0$, $\phi_4=0$, that none of these quantities is a real dynamical variable. The only interesting Dirac brackets are therefore those for $\vec{B}$ and $\vec{\pi}_B$,

$$\{B_i(x),B_j(x')\}^*=0\ , \tag{43a}$$

$$\{\pi_{Bi}(x),\pi_{Bj}(x')\}^*=(x_i\nabla'_j-x'_j\nabla_i)\frac{1}{4\pi x^0|\vec{x}-\vec{x}'|}+\int_1^\infty d\gamma\,\gamma\left[x_i\nabla'_j\frac{1}{4\pi x^0|\vec{x}-\gamma\vec{x}'|}-x'_j\nabla_i\frac{1}{4\pi x^0|\gamma\vec{x}-\vec{x}'|}\right], \tag{43b}$$

$$\{B_i(x),\pi_{Bj}(x')\}^*=\delta_{ij}\delta(\vec{x}-\vec{x}')+\nabla_i\nabla_j\frac{1}{4\pi|\vec{x}-\vec{x}'|}\ , \tag{43c}$$

where all brackets are evaluated for $x^0=x'^0$. From these equations, we observe that the transverse components of $\vec{B}$ and $\vec{\pi}_B$ satisfy the canonical bracket relations,

$$\{B_i^T(x),B_j^T(x')\}^*=0\ , \tag{44a}$$

$$\{\pi_{Bi}^T(x),\pi_{Bj}^T(x')\}^*=0\ , \tag{44b}$$

$$\{B_i^T(x),\pi_{Bj}^T(x')\}^*=\delta_{ij}\delta(\vec{x}-\vec{x}')+\nabla_i\nabla_j\frac{1}{4\pi|\vec{x}-\vec{x}'|}\ , \tag{44c}$$

for $x^0=x'^0$ where

$$\vec{B}^T(x)=\vec{B}(x)+\nabla\int d^3x'\frac{1}{4\pi|\vec{x}-\vec{x}'|}\nabla\cdot\vec{B}(x')\ , \tag{45a}$$

$$\vec{\pi}_B^T(x)=\vec{\pi}_B(x)+\nabla\int d^3x'\frac{1}{4\pi|\vec{x}-\vec{x}'|}\nabla\cdot\vec{\pi}_B(\vec{x}')\ . \tag{45b}$$

The longitudinal part of $\vec{B}$ vanishes by Eq. (32c),

$$\vec{B}_L(x)=-\nabla\int d^3x'\frac{1}{4\pi|\vec{x}-\vec{x}'|}\nabla\cdot\vec{B}(x')=0\ , \tag{46}$$

as the constraint $\nabla\cdot\vec{B}=0$ can now be considered to hold strongly. The longitudinal part of $\vec{\pi}_B$ is not an independent dynamical variable, but is determined in principle by the remaining equations,

$$\vec{\pi}_L(x)=-\nabla\int d^3x'\frac{1}{4\pi|\vec{x}-\vec{x}'|}\nabla\cdot\vec{\pi}_B(\vec{x}')\ , \tag{47}$$

where from Eqs. (28) and (32d)

$$\left[x^0-\frac{1}{x^{02}}(1+\vec{x}\cdot\nabla)\vec{x}\cdot\nabla\frac{1}{\nabla^2}\right]\nabla\cdot\vec{\pi}_B=\vec{x}\cdot\vec{B}\ . \tag{48}$$

Fortunately, $\vec{\pi}_B^L$ is never needed.

To complete the transition to a canonical description of the Abelian gauge theory, we rewrite the Hamiltonian using the constraints in Eqs. (32) to eliminate the dependent variables in $\mathscr{H}$, Eq. (19). The modified Hamiltonian is

$$\begin{aligned}H^*&=\int d^3x\,\mathscr{H}^*(\vec{B}^T,\vec{\pi}^T,\vec{f},\vec{g})\\&=\int d^3x[\tfrac{1}{2}\vec{B}^{T2}+\tfrac{1}{2}(\nabla\times\vec{\pi}^T+\vec{f})^2+\vec{g}^T\cdot\vec{B}^T]\ ,\end{aligned} \tag{49}$$

where we have dropped the now-superfluous index B on $\vec{\pi}_B$. The theory is now completely specified by H^* and the Dirac brackets $\{,\}^*$ in Eqs. (44). $\vec{E}$ is simply an auxiliary variable defined by Eq. (32b). As expected, the present form of the theory based on $\vec{B}^T$ and $\vec{\pi}^T$ is dual in the absence of currents to the usual form in which the canonical variables are $\vec{E}^T$ and $\vec{A}^T$.[12,13] Thus, for $\vec{f}=\vec{g}=0$, $\vec{\pi}$ is a magnetic potential such that $\vec{B}=-\dot{\vec{\pi}}^T$, $\vec{E}^T=-\nabla\times\vec{\pi}^T$. However, the complete theory is more general.

III. THE QUANTIZED THEORY

A. Calculation of $Z[\vec{f},\vec{g}]$

The Dirac bracket relations derived above provide a basis for the canonical quantization of the (dual) theory through the usual transition $\{,\}^*\to -i[,]$. However, we will instead calculate the generating functional $Z[\vec{f},\vec{g}]$ for the Green's functions of the theory. This can be obtained in closed but unfamiliar form, and we will illustrate its use by calculating the field-field Green's functions.

$Z[\vec{f},\vec{g}]$ is defined in terms of the usual functional integral over the unconstrained phase space of the physical variables $\vec{B}^T$, $\vec{\pi}^T$,[14]

$$\begin{aligned}Z[\vec{f},\vec{g}]&=\int \mathscr{D}\vec{\pi}^T\mathscr{D}\vec{B}^T\exp\left[i\int d^4x[\vec{\pi}^T\cdot\dot{\vec{B}}^T-\mathscr{H}^*(\vec{B}^T,\vec{\pi}^T,\vec{f},\vec{g})]\right]\\&=\int \mathscr{D}\vec{\pi}^T\mathscr{D}\vec{B}\exp\left[i\int d^4x[\vec{\pi}^T\cdot\dot{\vec{B}}^T-\tfrac{1}{2}\vec{B}^{T2}-\tfrac{1}{2}(\nabla\times\vec{\pi}^T+\vec{f})^2-\vec{g}^T\cdot\vec{B}^T]\right]\ .\end{aligned} \tag{50}$$

The exponent in this expression is quadratic in $\vec{\pi}^T$ and $\vec{B}^T$, and the functional integrals can be performed completely. Integration over $\vec{B}^T$ gives

$$Z[\vec{f},\vec{g}]=\int \mathscr{D}\vec{\pi}^T \exp\left[i\int d^4x[\tfrac{1}{2}(\dot{\vec{\pi}}^T+\vec{g}^T)^2-\tfrac{1}{2}(\nabla\times\vec{\pi}^T+\vec{f})^2]\right]$$

$$=\int \mathscr{D}\vec{\pi}^T \exp\left[i\int d^4x[-\tfrac{1}{2}\vec{\pi}^T\cdot\Box\vec{\pi}^T-\vec{\pi}^T\cdot(\dot{\vec{g}}^T+\nabla\times\vec{f}^T)+\tfrac{1}{2}\vec{g}^{T2}-\tfrac{1}{2}\vec{f}^2]\right] . \quad (51)$$

Performing the final Gaussian integration using time-ordered boundary conditions for $\Box^{-1}$, we find that

$$Z[\vec{f},\vec{g}]=\exp\left[i\int d^4x\tfrac{1}{2}[\vec{g}^{T2}-\vec{f}^2+(\dot{\vec{g}}^T+\nabla\times\vec{f}^T)\cdot\Box^{-1}(\dot{\vec{g}}^T+\nabla\times\vec{f}^T)]\right]$$

$$=\exp\left[i\int d^4x\tfrac{1}{2}[\vec{g}^{T2}-\vec{f}^2+(\dot{\vec{g}}^T+\nabla\times\vec{f}^T)(x)\int d^4y\, D_F(x-y)(\dot{\vec{g}}^T+\nabla\times\vec{f}^T)(y)]\right] , \quad (52)$$

where

$$D_F(x)=-\frac{i}{(2\pi)^2}\frac{1}{x^2-i\epsilon} . \quad (53)$$

B. Expectation values and field-strength propagators

Our explicit expression for $Z[\vec{f},\vec{g}]$ allows us to calculate the vacuum expectation values of the fields and the field-strength Green's functions of the theory in the presence of external sources without any intermediate reference to the usual potential A. For example, $\vec{B}^T$ is coupled directly to the source $\vec{g}^T$ in Eq. (50). Its vacuum expectation value is therefore given by the functional derivative of $Z[\vec{f},\vec{g}]$ with respect to $\vec{g}^T$,

$$\langle 0|\vec{B}^T(x)|0\rangle=Z^{-1}\int \mathscr{D}\vec{\pi}^T\mathscr{D}\vec{B}^T\vec{B}^T(x)e^{iS[\vec{B}^T,\vec{\pi}^T,\vec{f},\vec{g}]}$$
$$=iZ^{-1}\frac{\delta}{\delta\vec{g}^T(x)}Z[\vec{f},\vec{g}]$$
$$=-\vec{g}^T(x)+\partial_0\int d^4y\, D_F(x-y)\times(\dot{\vec{g}}^T+\nabla\times\vec{f}^T)(y). \quad (54)$$

$\vec{B}^L$ is of course identically zero. The expectation value of $\vec{E}$ can be derived using the definition

$$\vec{E}=-\nabla\times\vec{\pi}-\vec{f} , \quad (55)$$

that is, the constraint on $\vec{E}$ on Eq. (32b), and differentiating $Z[\vec{f},\vec{g}]$ with respect to $\vec{f}$,

$$\langle 0|\vec{E}(x)|0\rangle=Z^{-1}\int \mathscr{D}\vec{\pi}^T\mathscr{D}\vec{B}^T[-\nabla\times\vec{\pi}-\vec{f}](x)\times e^{iS[\vec{B}^T,\vec{\pi}^T,\vec{f},\vec{g}]}$$
$$=-iZ^{-1}\frac{\delta}{\delta\vec{f}(x)}Z[\vec{f},\vec{g}]$$
$$=-\vec{f}+\nabla\times\int d^4y D_F(x-y)(\dot{\vec{g}}+\nabla\times\vec{f})(y) , \quad (56)$$

where the second step follows from Eq. (50).

It is straightforward using manipulations similar to those in the Appendix to show that the expectation values of $\vec{B}$ and $\vec{E}$ satisfy Maxwell's equations in the presence of currents provided $\vec{f}$ and $\vec{g}$ are of the form in Eqs. (15) and (16). We can also put the expectation values in more familiar form by using the identities

$$\nabla\cdot\vec{f}(x)=-J^0(x) , \quad (57)$$

$$(\dot{\vec{f}}-\nabla\times\vec{g})(x)=\vec{J}(x) , \quad (58)$$

which are derived in the Appendix, Eqs. (A7) and (A9). (We would emphasize, however, that we need not make this connection.[15]) Thus, integrating by parts and using the relation $\Box D_F(x-y)=\delta(x-y)$ in Eq. (54), we find that

$$\langle 0|\vec{B}^T(x)|0\rangle=\int d^4D_F(x-y)[\nabla^2\vec{g}^T+\nabla\times\dot{\vec{f}}^T](y)$$
$$=\int d^4y\, D_F(x-y)[-\nabla\times(\nabla\times\vec{g}^T)+\nabla\times\dot{\vec{f}}](y)$$
$$=\nabla\times\int d^4y\, D_F(x-y)[\dot{\vec{f}}-\nabla\times\vec{g}^T](y)$$
$$=\nabla\times\int d^4y\, D_F(x-y)\vec{J}(y) . \quad (59)$$

(In the second step, we have used the facts that $\nabla\times\vec{f}=\nabla\times\vec{f}^T$ and $\nabla\cdot\vec{g}^T=0$.) Similarly,

$$\begin{aligned}\langle 0|\vec{E}(x)|0\rangle &= -\vec{f}+\int d^4y\,D_F(x-y)[\nabla\times\dot{\vec{g}}^{\,T}+\nabla\times(\nabla\times\vec{f})](y)\\ &= -\vec{f}+\int d^4y\,D_F(x-y)[\nabla\times\dot{\vec{g}}^{\,T}-\nabla^2\vec{f}+\nabla(\nabla\cdot\vec{f})](y)\\ &= \nabla\int d^4y\,D_F(x-y)\nabla\cdot\vec{f}(y)+\partial^0\int d^4y\,D_F(x-y)[\nabla\times\vec{g}^{\,T}-\dot{\vec{f}}](y)\\ &= -\nabla\int d^4y\,D_F(x-y)J^0(y)-\partial^0\int d^4y\,D_F(x-y)\vec{J}(y)\ .\end{aligned} \tag{60}$$

Equations (59) and (60) are standard representations for the classical fields in terms of causal Green's functions.

As a second illustration of the use of $Z[\vec{f},\vec{g}]$, we will derive the free propagators for $\vec{B}$ and $\vec{E}$. From Eq. (50),

$$\begin{aligned}\langle 0|T(B_j^T(x),B_k^T(x'))|0\rangle &= Z^{-1}\int \mathscr{D}\vec{\pi}^{\,T}\mathscr{D}\vec{B}^{\,T}T(B_j^T(x),B_k^T(x'))e^{iS[\vec{B}^{\,T},\vec{\pi}]}\\ &= Z^{-1}i\frac{\delta}{\delta g_j^T(x)}i\frac{\delta}{\delta g_k^T(x')}Z[\vec{f},\vec{g}]\Big|_{\vec{f}=\vec{g}=0}\ .\end{aligned} \tag{61}$$

Using Eq. (54) and the relations

$$\frac{\delta g_k^T(x')}{\delta g_j^T(x)}=\left[\delta_{jk}\delta(\vec{x}-\vec{x}\,')-\nabla_j\nabla'_k\frac{1}{4\pi|\vec{x}-\vec{x}\,'|}\right]\delta(x^0-x'^0)\ , \tag{62}$$

$$\Box D_F(x-y)=\delta(x-y)\ , \tag{63}$$

we obtain

$$\begin{aligned}\langle 0|T(B_j^T(x),B_k^T(x'))|0\rangle &= Z^{-1}\frac{i\delta}{\delta g_j^T(x)}\left[\left[-g_k^T(x')+\partial'_0\int d^4y\,D(x'-y)\dot{\vec{g}}{}_k^T(y)\right]Z\right]_{\vec{f}=\vec{g}=0}\\ &= i\left[\delta_{jk}[-\delta(x-x')+\partial_0'^2D_F(x'-x)]\right.\\ &\qquad\left.+\int d^3y\left[\nabla_{jx}\nabla_{ky}\frac{1}{4\pi|\vec{x}-\vec{y}|}\right][\delta(x'-y)-\partial_0'^2D_F(x'-y)]_{y^0=x^0}\right]\\ &= i(\delta_{jk}\nabla^2+\nabla_j\nabla'_k)D_F(x'-x)\ .\end{aligned} \tag{64}$$

This is identical to the result obtained as a derived quantity by quantizing the vector potential, but appears here as a primary result.

We can obtain the electric field propagator by a similar calculation using the definition in Eq. (55), but must be careful when expressing the propagator in terms of functional derivatives because of the quadratic dependence of the exponent in $Z[\vec{f},\vec{g}]$ on $\vec{f}$. This leads in the second functional derivative to an extra delta function contribution which is not part of the propagator and must be removed.[16] The proper result is

$$\begin{aligned}\langle 0|T(E_j(x),E_k(x'))|0\rangle &= Z^{-1}\int \mathscr{D}\vec{\pi}^{\,T}\mathscr{D}\vec{B}^{\,T}T(E_j(x),E_k(x'))e^{iS[\vec{B}^{\,T},\vec{\pi}^{\,T}]}\\ &= Z^{-1}\frac{-i\delta}{\delta f_j(x)}\frac{-i\delta}{\delta f_k(x')}Z[\vec{f},\vec{g}]\,|_{\vec{f}=\vec{g}=0}-i\delta_{jk}\delta(x-x')\\ &= Z^{-1}\frac{i\delta}{\partial f_j(x)}\left[\left[f_k(x')+\epsilon^{ijk}\int d^4y\,\partial'_jD_F(x'-y)(\nabla\times\vec{f})_i(y)\right]Z\right]_{\vec{f}=\vec{g}=0}-\delta_{jk}\delta(x-x')\\ &= i(\delta_{jk}\nabla^2+\nabla_j\nabla'_k)D_F(x'-x)\ ,\end{aligned} \tag{65}$$

again in accord with the vector potential formulation. These results can of course be generalized to nonzero values for the external sources $\vec{f}$ and $\vec{g}$.

IV. SUMMARY

Our purpose in this paper has been to develop the field-strength or dual formulation of the Abelian gauge theory from a Hamiltonian point of view. We have based our discussion on the form of the action obtained by eliminating the potential A^μ in the usual action in terms of $F^{\mu\nu}$ using the coordinate gauge $x_\mu A^\mu=0$. This replacement requires that the Bianchi identities (homogeneous Maxwell equations) be enforced as constraints, and leads to extra terms in the action. We were able to obtain a complete Hamiltonian formulation of the theory with $\vec{B}$ and its conjugate momentum $\vec{\pi}$ as canonical variables satisfying the Dirac bracket relations in Eqs. (43). The unconstrained variables are just the transverse components of $\vec{B}$ and $\vec{\pi}$, which satisfy the canonical Poisson bracket relations. In terms of these variables, the Hamiltonian of the system is

$$H^* = \int d^3x[\tfrac{1}{2}\vec{B}^{T2}+\tfrac{1}{2}(\nabla\times\vec{\pi}^T+\vec{f})^2-\vec{g}^T\cdot\vec{B}^T] , \quad (66)$$

where $\vec{f}$ and $\vec{g}$ are nonlocal currents and

$$\vec{E}=-\nabla\times\vec{\pi}-\vec{f} . \quad (67)$$

This form for H^* and the canonical bracket relations lead to the equations of motion

$$\dot{\vec{B}}^T-\nabla\times(\nabla\times\vec{\pi}^T+\vec{f})=0 , \quad (68)$$

$$\dot{\vec{\pi}}^T+\vec{B}^T+\vec{g}^T=0 . \quad (69)$$

We remark that Eqs. (67) and (69) can be interpreted as defining $\vec{E}$ and $\vec{B}$ in terms of $\vec{f}$, $\vec{g}^T$, and a dual (magnetic) potential $\vec{A}_B=\vec{\pi}$. The results can clearly be generalized to incorporate magnetic as well as electric currents.

Because H^* is quadratic, we were able to integrate the functional integral for $Z[\vec{f},\vec{g}]$ explicitly, and to calculate the field-strength Green's functions of the theory by functional differentiation. This provides an alternative to the usual quantization scheme based on the potential A^μ, and appears to be more natural when magnetic quantities are to be emphasized.

We remark finally that the transition from the Hamiltonian form of the theory back to the Lagrangian form in Eq. (8) can be accomplished by introducing auxiliary noncanonical variables such as $\vec{B}^L$ and $\vec{E}$, and appropriate Lagrange multiplier fields to enforce the necessary constraints, e.g., $\nabla\cdot\vec{B}=0$. The equivalence of the two forms has been demonstrated elsewhere.[17] The situation is different for non-Abelian gauge theories. The Lagrangian field-strength form of the non-Abelian theory was established and studied in detail in I.[2] However, the covariant derivatives $D=\partial+[A^F,\cdot]$ which appear are intrinsically nonlocal in time for A in the (covariant) coordinate gauge $x^\mu A_\mu=0$, and prevent the use of Dirac's methods to establish a Hamiltonian formulation of the theory.

ACKNOWLEDGMENTS

We would like to thank Dr. C. Zachos for useful conversations on the early part of this work. One of the authors (L.D.) would also like to thank the Fermilab Theory Group for the hospitality and support accorded him while parts of this work were done. This work was supported in part by the U.S. Department of Energy and the University of Wisconsin-Madison with funds granted by the Wisconsin Alumni Research Foundation.

APPENDIX

To derive Maxwell's equations, we start with the dynamical equations

$$\dot{\vec{B}}+\nabla\times\vec{E}=0 , \quad (A1)$$

$$\dot{\vec{\pi}}+\vec{B}-\nabla\beta+\vec{g}=0 , \quad (A2)$$

and the equations of constraint

$$\nabla\cdot\vec{B}=0 , \quad (A3)$$

$$\vec{E}+\nabla\times\vec{\pi}_B+\vec{f}=0 , \quad (A4)$$

where $\vec{f}$ and $\vec{g}$ are defined in Eqs. (15) and (16). Equations (A1) and (A3) are the homogeneous Maxwell equations. Equation (A4) gives

$$\nabla\cdot\vec{E}=-\nabla\cdot\vec{f} \quad (A5)$$

while Eqs. (A2) and (A4) give

$$\nabla\times\vec{B}-\dot{\vec{E}}=\dot{\vec{f}}-\nabla\times\vec{g} . \quad (A6)$$

We therefore need only to show that the right-hand sides of Eqs. (A5) and (A6) are equal to the charge and current densities $J^0(x)$ and $\vec{J}(x)$, respectively.

From Eq. (15),

$$\begin{aligned}-\nabla\cdot\vec{f}(x)&=-\int_1^\infty d\gamma\,\gamma^2\nabla\cdot[\vec{x}J^0(\gamma x)-x^0\vec{J}(\gamma x)]\\ &=-\int_1^\infty d\gamma\,\gamma^2[(3+\vec{x}\cdot\nabla)J^0(\gamma x)+x^0\partial_0 J^0(\gamma x)]\\ &=-\int_1^\infty d\gamma\frac{d}{d\gamma}[\gamma^3 J^0(\gamma x)]\\ &=J^0(x) , \end{aligned} \quad (A7)$$

where we have used current conservation in the second step,

$$\partial_0 J^0(x)+\nabla\cdot\vec{J}(x)=0 . \quad (A8)$$

Similarly, from Eqs. (25) and (16),

$$\begin{aligned}\dot{\vec{f}}(x)-\nabla\times\vec{g}(x)&=\int_1^\infty d\gamma\,\gamma^2\{\partial_0[\vec{x}J^0(\gamma x)-x^0\vec{J}(\gamma x)]\\ &\qquad+\nabla\times[\vec{x}\times\vec{J}(\gamma x)]\}\\ &=-\int_1^\infty d\gamma\,\gamma^2[3+\vec{x}\cdot\nabla+x^0\partial_0]\vec{J}(\gamma x)\\ &=-\int_1^\infty d\gamma\frac{d}{d\gamma}[\gamma^3\vec{J}(\gamma x)]\\ &=\vec{J}(x) , \end{aligned} \quad (A9)$$

where we have again used current conservation. Equations (A7) and (A9) are the identities needed above, and Maxwell's equations hold.

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[3]This is also known as the homogeneous gauge since gauge transformations $A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U$ with U homogeneous of degree zero in x maintain the condition $x^\mu A_\mu = 0$.

[4]The coordinate gauge is a special choice $f^\mu = x^\mu$ in the class of gauges $f_\mu(x)A^\mu(x)=0$, with f^α a conformal Killing vector,

$$\partial^\mu f^\nu + \partial^\nu f^\mu = \tfrac{1}{2} g^{\mu\nu}\partial_\alpha f^\alpha \ .$$

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[11]A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1976).

[12]The similarity of the two versions of the theory can be enhanced by interchanging the roles of $\vec{B}^T$ and $\vec{\pi}^T$ as canonical coordinates and momenta by the canonical transformation $\vec{B}^T \to \vec{B}^T$, $\vec{\pi}^T \to -\vec{A}_B^T$. $\vec{A}_B^T$ and $\vec{B}^T$ then have the usual bracket relations for a transverse potential and its canonical momentum.

[13]The importance of the duality transformation for the strong coupling limit of non-Abelian gauge theories is reviewed by R. Savit, Rev. Mod. Phys. **52**, 453 (1980). See also Refs. 1 and 2.

[14]See, e.g., L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Benjamin/Cummings, New York, 1980).

[15]All that is actually necessary is that $\vec{f}$ and $\vec{g}$ be consistent with the constraints we have used. This requires that

$$\vec{x}\cdot\vec{g}=0, \quad \vec{x}\cdot\nabla\times\vec{f}+x^0\nabla\cdot\vec{g}=0 \ .$$

[These conditions were used in Eqs. (28) and (30), respectively.] The expressions in Eqs. (57) and (58) then *define* $J^0(x)$ and $\vec{J}(x)$. Current conservation holds automatically.

[16]We can write the relevant part of the exponent in Eq. (50) as

$$-\tfrac{1}{2}(\nabla\times\vec{\pi}^T+\vec{f})^2 = -\tfrac{1}{2}(\nabla\times\vec{\pi}^T)^2+\vec{E}\cdot\vec{f}+\tfrac{1}{2}\vec{f}^2 \ .$$

The last is the offending term. It could be removed by adding $\frac{1}{2}\vec{f}^2$ to $\mathscr{H}^*$, but this would be inappropriate for $\vec{f}$, a dynamical rather than external field.

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