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Finite, unrenormalized, nonperturbative solution to the Schwinger-Dyson equations of quantum electrodynamics

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An approximate solution to the unrenormalized Schwinger-Dyson equations of quantum electrodynamics is obtained for the vertex amplitude by using combined analytical and numerical techniques. The photon propagator is approximated by its form near the mass shell. The four-point diagram appearing in the vertex equation is related to lower-order diagrams by a generalization of the Ward identity. Under these approximations a functional form for the vertex function, $\Gamma^\lambda(\bar{p},\bar{p}+\bar{k})$, was obtained with a range of validity for all momenta extending from very near the mass shell to indefinitely large asymptotic values. No infinities were subtracted to obtain the solution.

I. INTRODUCTION

A number of researchers[1–12] have investigated the possibility that the infinities which occur in the evaluation of such physical observables as the Lamb shift and anomalous magnetic moment of the electron are the result of improper mathematical procedures rather than an inherent flaw in the theory of quantum electrodynamics. This reasoning has led them to avoid the usual perturbation techniques and look for alternative means of solving the Schwinger-Dyson[13] equations for the electron and photon propagators and their vertex function. One approach has been to make use of renormalization-group methods[1,2] to obtain the asymptotic forms for the propagators. A different approach, presented here, derives an approximate solution to the Schwinger-Dyson equations for the vertex amplitude with a range of validity for the electron four-momentum extending from near the mass shell to indefinitely large asymptotic values. In an earlier work[12] we initiated this approach with a solution to the electron propagator equation. In this paper we will describe how an extension of this method is used to evaluate the vertex function. In each case we have obtained solutions which are finite.

The discussion in this paper will be arranged in the following way. Section II will describe the general method for the solution of the Schwinger-Dyson equations. Section III will briefly review the results of that method as it was applied to the electron propagator. Section IV will describe the application of this procedure to the vertex equation. Section V presents a simplified equation which was used to give an initial solution to the more exact equation. Section VI gives a description of the final solution to the vertex equation. Section VII contains concluding remarks.

II. GENERAL PROCEDURE FOR A NONPERTURBATIVE SOLUTION TO THE SCHWINGER-DYSON EQUATIONS

The Schwinger-Dyson equations are stated here in the momentum representation. The Fourier transform of the photon propagator, $D_{\mu\nu}(\bar{k})$, satisfies the equation

$$D_{\mu\nu}(\bar{k})=D_{0\mu\nu}(\bar{k})+D_{0\mu\nu}(\bar{k})\Pi^{\alpha\beta}(\bar{k})D_{\beta\nu}(\bar{k}) , \tag{2.1}$$

where

$$\Pi^{\alpha\beta}(\bar{k})=\frac{i4\pi e_0{}^2}{(2\pi)^4}\int \mathrm{Tr}[\gamma^\alpha S(\bar{r})\Gamma^\beta(\bar{r},\bar{r}-\bar{k})S(\bar{r}-\bar{k})]d^4r .$$

The subscript 0 follows all bare quantities, and bars over letters indicate four-vectors. The Fourier transform of the electron propagator, $S(\bar{p})$, satisfies the second Schwinger-Dyson equation,

$$S^{-1}(\bar{p})=S_0{}^{-1}(\bar{p})-\Sigma(\bar{p}) , \tag{2.2}$$

where

$$\Sigma(\bar{p})=\frac{i4\pi e_0{}^2}{(2\pi)^4}\int D_{\lambda\beta}(\bar{k})\gamma^\beta S(\bar{p}+\bar{k})\Gamma^\lambda(\bar{p}+\bar{k},\bar{p})d^4k .$$

The last equation involves the vertex,

$$\Gamma(\bar{p}+\bar{k},\bar{p})=\gamma^\lambda+\Lambda^\lambda(\bar{p}+\bar{k},\bar{p}) , \tag{2.3}$$

where $\Lambda^\lambda(\bar{p}+\bar{k},\bar{p})$ represents the contributions of all nodeless diagrams to which are connected two external electron lines and one external photon line. It satisfies an equation containing a four-point diagram as we shall see.

The difficulty in obtaining a complete solution to the Schwinger-Dyson equations is basically that we are seeking to determine 16 scalar functions (two for the electron propagator, two for the photon propagator, and twelve for the vertex) through the simultaneous solution of three nonlinear integral equations (out of an infinite hierarchy), in a four-dimensional space. Furthermore, it is desirable to avoid the substitution of bare quantities for dressed quantities (a process that traditionally leads to infinities).

The procedure that we use to obtain a solution to these equations has three components to it. The first is the selection of an approximate photon propagator. The second component is the use of a generalization of the Ward identity for higher-order diagrams. These provide a means for truncating the infinite hierarchy. The last component consists of transforming the electron and vertex integral equations into differential equations by the action of the D'Alembertian on the photon propagator. We will now take up each of these three components in greater detail.

The first component, which approximates the photon propagator, is justified by the following observations. The vacuum polarization is known to make only a small contribution to the Lamb shift. Thus, our solution is not expected to be significantly altered by small misrepresentations in $D_{\mu\nu}(\bar{k})$. When the mass-shell-limiting form is used to approximate $D_{\mu\nu}(\bar{k})$, the substitution is most accurate where k^2 approaches zero. Where k^2 vanishes the integrands in Eqs. (2.2) and (2.3) are at their largest. Thus, this form is most accurate in the region of most importance. In Ref. 12, it was proved that, if this form is used, the Landau gauge must be selected if a solution to the approximate equations for $S(\bar{p})$ exists. Thus, we shall choose

$$D_{\mu\nu}(\bar{k})=Z_3(-g_{\mu\nu}+k_\mu k_\nu/k^2)/k^2 \; . \tag{2.4}$$

Finally, the approximation of $D_{\mu\nu}(\bar{k})$ with Eq. (2.4) was additionally motivated by the results of earlier investigations.[2,3]

The second element employs the Ward identity consistently at all levels. At the first level we have

$$(q_\lambda-p_\lambda)\Gamma^\lambda(\bar{q},\bar{p})=S^{-1}(\bar{q})-S^{-1}(\bar{p}) \; . \tag{2.5}$$

This can be seen to determine the longitudinal components of the vertex in terms of the electron propagator. The differential form of the same identity,

$$\Gamma^\lambda(\bar{p},\bar{p})=\frac{\partial S^{-1}(\bar{p})}{\partial p_\lambda} \; , \tag{2.6}$$

provides information about the complete vertex in the limit of vanishing photon momentum.

Just as the Fourier transform of the two-point diagrams, $D_{\mu\nu}(\bar{k})$ and $S(\bar{p})$, are related to the Fourier transform of the three-point diagram, $\Gamma^\lambda(\bar{q},\bar{p})$ (where $\bar{q}=\bar{p}+\bar{k}$), so the $\Gamma^\lambda(\bar{q},\bar{p})$ can be related to the Fourier transform of a four-point diagram, $E^{\lambda\alpha}$:

$$\Gamma^\lambda(\bar{q},\bar{p})=\gamma^\lambda+\Lambda^\lambda(\bar{q},\bar{p}) \; , \tag{2.7}$$

where

$$\Lambda^\lambda(\bar{q},\bar{p})=\frac{i4\pi e_0{}^2}{(2\pi)^4}\int D_{\alpha\beta}(\bar{q}-\bar{r})\gamma^\beta S(\bar{r})\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r}) \times S(\bar{p}-\bar{q}-\bar{r})\Gamma^\alpha(\bar{p}-\bar{q}+\bar{r},\bar{p})d^4r + \frac{i4\pi e_0{}^2}{(2\pi)^4}\int D_{\alpha\beta}(\bar{q}-\bar{r})\gamma^\beta S(\bar{r}) \times E^{\lambda\alpha}(\bar{r},\bar{q}-\bar{r},\bar{p})d^4r \; .$$

The function $E^{\lambda\alpha}$ can be represented as the sum of all nodeless diagrams connected externally to the two photon lines, one incoming electron line, and one outgoing electron line. Other equations can be constructed to relate Γ^μ to higher-order amplitudes.[14] However, the advantage of Eq. (2.7) lies in the fact that there exists a generalization of Wards's identity for these higher-order n-point diagrams (Green[15]). Just as we have Eq. (2.5) for the vertex function, so we have for the four-point diagram an equivalent relationship

$$(\bar{r}_\alpha-\bar{q}_\alpha)E^{\lambda\alpha}(\bar{r},\bar{q}-\bar{r},\bar{p})=\Gamma^\lambda(\bar{q},\bar{p})-\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r}) \; . \tag{2.8}$$

Similar relationships exist for the remaining n-point diagrams. These identities exactly define the longitudinal components of the n-point diagrams in terms of the $(n-1)$-point diagrams. They can be used to truncate the hierarchy of integral equations.

The last element in the procedure of reducing the Schwinger-Dyson equations to a tractable form was to convert the integral equations into a set of differential equations with boundary conditions. This method was developed in connection with the Bethe-Salpeter equation (Green[16]). It was first used for a study of the Schwinger-Dyson equations by Bose and Biswas.[17]

Equation (2.4) for the photon propagator can be restated in the following way:

$$D_{\mu\nu}(\bar{k})=Z_3\left[-\frac{1}{4}\frac{\partial}{\partial k_\nu}\frac{\partial}{\partial k_\mu}\ln(k^2)-\frac{1}{2}\frac{g_{\mu\nu}}{k^2}\right] . \tag{2.9}$$

When the D'Alembertian operator is applied to any integral over the photon propagator, we can make use of the following property:

$$\Box_p\frac{1}{p^2}=4\pi^2 i\delta(\bar{p}) \; , \tag{2.10}$$

where

$$\Box_p=\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p^\alpha} \; .$$

This delta function can be used to trivially evaluate integrals of the type

$$I^\lambda(\bar{p})=\frac{i4\pi e_0{}^2}{(2\pi)^4}\int D_{\mu\nu}(\bar{r}-\bar{p})\gamma^\mu F^{\lambda\nu}(\bar{r})d^4r \; . \tag{2.11}$$

Using the definition $e_0{}^2Z_3=e^2$ and applying the operator $\partial\Box_p$ to the above, we find

$$\partial_p\Box_p I^\lambda(\bar{p})=(e^2/\pi)[\tfrac{1}{2}\partial_p\gamma_\nu F^{\lambda\nu}(\bar{p})+(\partial/\partial p^\nu)F^{\lambda\nu}(\bar{p})] \; , \tag{2.12}$$

where $\partial=\gamma^\alpha\partial/\partial p^\alpha$.

The solution to the integral Eq. (2.11) is equivalent to the solution of the differential Eq. (2.12) when the appropriate boundary conditions are satisfied.

III. THE ELECTRON EQUATION

In an earlier work,[12] a description was presented of our solution to the Schwinger-Dyson equations at the lowest level of approximation. It is useful to summarize the results of that paper. The photon propagator was taken to have the form of Eq. (2.4). The vertex function was approximated through Ward's identity,

$$\Gamma^\mu(\bar{p}+\bar{k},\bar{p})\approx\Gamma^\mu(\bar{p},\bar{p})=\frac{\partial}{\partial p_\mu}S^{-1}(\bar{p}) \; . \tag{3.1}$$

The electron equation was converted into a pair of differential equations—one the scalar coefficient of the unit matrix, the other the scalar coefficient of $\not p$. These equations were solved through a combination of numerical and analytical techniques which lead to the identification

$$S^{-1}(\bar{p})=A(p^2)+\not p B(p^2) , \tag{3.2}$$

where

$$A(p^2)=-|p^2-1|^{-\tau(p^2-1)/p^2} ,$$

$$B(p^2)=1 ,$$

and

$$\tau=\frac{3}{4}+\left[\frac{3\alpha}{4\pi}\right]^2+3\left[\frac{3\alpha}{4\pi}\right]^3+\cdots .$$

It was found that a finite solution for the electron propagator existed only if the photon propagator is restricted to Landau gauge and the bare mass is zero. These results were in agreement with the findings of other researchers.[2] In addition, because the electron propagator had been determined for all values of the momentum, it was possible to take the mass-shell-limit of the solution and evaluate Z_2. Z_2 was found to be finite and equal to unity. The success of this procedure encouraged us to attempt the solution of the hierarchy at a higher level so that the vertex function could be evaluated.

IV. THE VERTEX EQUATION

The vertex function is a matrix function and can be written in terms of the Dirac γ matrices, the four-momentum of the electron, $\bar{p}$, and the four-momentum of photon, $\bar{k}$. Its most general form is

$$\begin{aligned}\Gamma^\lambda(\bar{p}+\bar{k},\bar{p})=&\, p^{\tilde\lambda}F+\gamma^{\tilde\lambda}G_0+p^{\tilde\lambda}\not p G_1+p^{\tilde\lambda}\not k G_2\\&+2i\sigma^{\beta\alpha}p_\alpha k_\beta p^{\tilde\lambda}H_0+2i\sigma^{\alpha\tilde\lambda}p_\alpha H_1\\&+2i\sigma^{\alpha\tilde\lambda}k_\alpha H_2+\epsilon^{\tilde\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta I\\&+\Gamma^\lambda_{\text{longitudinal}}(\bar{p}+\bar{k},\bar{p}) .\end{aligned} \tag{4.1}$$

The scalar functions F, G_0, G_1, G_2, H_1, H_2, and I are all functions of p^2, k^2, and u, where

$$p^2=p_\alpha p^\alpha ,$$
$$k^2=k_\alpha k^\alpha , \tag{4.2}$$

and

$$u=\frac{p_\alpha k^\alpha}{(p^2)^{1/2}(k^2)^{1/2}} .$$

The first eight functions in Eq. (4.1) define that part of the vertex function which is transverse to the photon momentum k_λ. The tilde over the λ indicates that the longitudinal components have been subtracted out, such that for instance,

$$p^{\tilde\lambda}=p^\lambda-k^\lambda p_\alpha k^\alpha/k^2 .$$

The longitudinal part of Γ^λ is exactly related to $S(\bar{p})$ by the Ward identity so that with our approximation,

$$\Gamma^\lambda_{\text{longitudinal}}(\bar{p}+\bar{k},\bar{p})=\frac{k^\lambda}{k^2}\{A[(\bar{p}+\bar{k})^2]-A(p^2)\}-\frac{k^\lambda\not k}{k^2} . \tag{4.3}$$

Equation (2.7) can be solved for the eight transverse scalar functions after the following preparations. First the four-point function, $E^{\lambda\alpha}$, is approximated by the differential form of Eq. (2.8). Thus,

$$E^{\lambda\alpha}(\bar{r},\bar{q}-\bar{r},\bar{p})=-\frac{\partial\Gamma^\lambda}{\partial r_\alpha}(\bar{r},\bar{p}-\bar{q}+\bar{r}) . \tag{4.4}$$

Second, we make use of the approximation

$$\begin{aligned}\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r})&S(\bar{p}-\bar{q}+\bar{r})\Gamma^\alpha(\bar{p}-\bar{q}+\bar{r},\bar{p})\\&\cong\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r})S(\bar{p}-\bar{q}+\bar{r})\\&\quad\times\Gamma^\alpha(\bar{p}-\bar{q}+\bar{r},\bar{p}-\bar{q}+\bar{r}) .\end{aligned} \tag{4.5}$$

Finally, we employ the identity

$$\begin{aligned}\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r})&S(\bar{p}-\bar{q}+\bar{r})\Gamma^\alpha(\bar{p}-\bar{q}+\bar{r},\bar{p}-\bar{q}+\bar{r})\\&=-\frac{\partial}{\partial r_\alpha}[\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r})S(\bar{p}-\bar{q}+\bar{r})]\\&\quad\times S(\bar{p}-\bar{q}+\bar{r})^{-1}+\frac{\partial\Gamma^\lambda}{\partial r_\alpha}(\bar{r},\bar{p}-\bar{q}+\bar{r}) .\end{aligned} \tag{4.6}$$

When Eqs. (4.4), (4.5), and (4.6) are substituted into Eq. (2.7), the vertex equation becomes

$$\Gamma^\mu(\bar{q},\bar{p})=\gamma^\mu-\frac{i4\pi e_0{}^2}{(2\pi)^4}\int D_{\alpha\beta}(\bar{q}-\bar{r})\gamma^\beta S(\bar{r})\frac{\partial}{\partial r_\alpha}[\Gamma^\lambda(\bar{r},\bar{p}-\bar{q}+\bar{r})S(\bar{p}-\bar{q}+\bar{r})]S(\bar{p}-\bar{q}+\bar{r})^{-1}d^4r . \tag{4.7}$$

By changing notation slightly and allowing $\bar{q}$ to be represented by $\bar{p}+\bar{k}$, it can be seen that the rules given in Eqs. (2.11) and (2.12) provide a formula for the action of $\partial_p\Box_p$ on Eq. (4.7). By this mechanism, Eq. (4.7) can be converted into the following differential equation for the vertex;

$$\partial_p\Box_p\Gamma^\lambda(\bar{p}+\bar{k},\bar{p})=-\frac{e^2}{\pi}\left[\tfrac{1}{2}\gamma_\nu\partial_p F^{\lambda\nu}(\bar{p}+\bar{k},\bar{p})+\frac{\partial}{\partial p_\nu}F^{\lambda\nu}(\bar{p}+\bar{k},\bar{p})\right] , \tag{4.8}$$

where

$$F^{\lambda\nu}(\bar{p}+\bar{k},\bar{p})=S(\bar{p}+\bar{k})\frac{\partial}{\partial p_\nu}[\Gamma^\lambda(\bar{p}+\bar{k},\bar{p})S(\bar{p})]S(\bar{p})^{-1} .$$

Since only the transverse functions are left to be determined, it is possible to subtract out the longitudinal parts of Eq. (4.8). The transverse remainder then contains eight linearly independent scalar equations which are coefficients of the matrices $p^{\tilde\lambda}$, $\gamma^{\tilde\lambda}$, $p^{\tilde\lambda}\not p$, $p^{\tilde\lambda}\not k$, $i\sigma^{\alpha\beta}p_\alpha k_\beta p^{\tilde\lambda}$, $i\sigma^{\tilde\lambda\alpha}p_\alpha$, $i\sigma^{\tilde\lambda\beta}k_\beta$, and $\epsilon^{\tilde\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta$. Into these eight differential equations were substituted Eq. (3.2) for the electron propagator and Eq. (2.4) for the photon propagator.

The eight differential equations were written down with the aid of a Fortran program designed to do the necessary bookkeeping. Despite the apparent simplicity of the matrix equation, Eq. (4.8), the eight scalar equations, expanded completely in terms of the two electron propagator functions and eight transverse vertex functions, are inordinately lengthy. For this reason they are not presented here.[18] In the next section, as a preliminary to finding the vertex function which satisfies Eq. (4.8), we consider a simpler vertex equation—called the mass-shell equation—and its solution.

V. AN APPROXIMATE SOLUTION

Applying the D'Alembertian operator to Eq. (2.7) and making use of Eq. (2.10) gives

$$\Box_p \Gamma^\lambda(\bar{q},\bar{p}) = -\epsilon(\tfrac{1}{2}\gamma_\nu F^{\nu\lambda} + \partial_\nu \partial^{-1} F^{\nu\lambda}) , \qquad (5.1)$$

where

$$\Box_p = \frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p^\alpha}, \quad \Box_p \partial^{-1} = \partial = \gamma^\alpha \frac{\partial}{\partial p_\alpha}, \quad \epsilon = \frac{e^2}{\pi}, \quad \bar{q} = \bar{p} + \bar{k},$$

and

$$F^{\nu\lambda} = S(\bar{q}) \frac{\partial}{\partial p_\nu} [\Gamma^\lambda(\bar{q},\bar{p}) S(\bar{p})] S^{-1}(\bar{p}) . \qquad (5.2)$$

This equation can be approximated by

$$F^{\nu\lambda} = S(\bar{q}) \Gamma^\lambda(\bar{q},\bar{p}) S(\bar{p}) \gamma^\nu . \qquad (5.3)$$

This amounts to neglecting

$$S(\bar{q}) \left\{ \frac{\partial}{\partial p_\nu} \Gamma^\lambda(\bar{q},\bar{p}) - \Gamma^\lambda(\bar{q},\bar{p}) S(\bar{p}) \left[\frac{\partial}{\partial p_\nu} S^{-1}(\bar{p}) - \gamma^\nu \right] \right\} . \qquad (5.4)$$

This expression vanishes if we neglect the derivative of Γ^λ and of A. We shall see that Γ^λ is well approximated by γ^λ over the entire range of momenta so that its derivatives are small. In addition, A in Eq. (3.2) is slowly varying almost everywhere. We shall obtain an approximate solution using the neglected-derivative (ND) expression in Eq. (5.3) and test how well it solves Eq. (5.1) using Eq. (5.2).

In Ref. 12, the asymptotic forms of the solutions were found by substituting the differential equations [Eqs. (2.20) and (2.22)] into the integral equations [Eqs. (2.18) and (2.19)] from which they were derived. This suggests substituting Eqs. (5.1) and (5.2) above into the integral equation (4.7). When this is done, it becomes clear that $\Gamma^{\tilde{\lambda}}(\bar{p}+\bar{k},\bar{p})$ approaches $[1+O(\alpha)]\gamma^{\tilde{\lambda}}$ as p^2 increases without limit where α is the fine-structure constant. Since Γ^λ approaches $Z_2{}^{-1}\gamma^\lambda$ in the vicinity of the mass shell, and since Z_2 is close to unity according to Ref. 12, we are motivated to assume that

$$\Gamma^{\tilde{\lambda}} \simeq \gamma^{\tilde{\lambda}} \qquad (5.5)$$

everywhere.

We take, then, γ^λ to be our first approximation. It satisfies the boundary conditions to order α and is consistent with our assumption that the derivatives of Γ^λ are negligible in deriving Eq. (5.3). Substituting γ^λ for Γ^λ in Eq. (5.3) and this $F^{\nu\lambda}$ into Eq. (5.1) gives us a result equivalent to

$$\Box_p C^{\tilde{\lambda}} = -\frac{6\epsilon A p^{\tilde{\lambda}}}{(q^2-A^2)(p^2-A^2)} , \qquad (5.6a)$$

$$\Box_p C_\mu^{\tilde{\lambda}} = 2\epsilon \frac{[(A^2-\bar{q}\cdot\bar{p})\delta_\mu^{\tilde{\lambda}} + (q_\mu+p_\mu)p^{\tilde{\lambda}} - \partial_\mu E^{\tilde{\lambda}}]}{(q^2-A^2)(p^2-A^2)} , \qquad (5.6b)$$

$$\Box_p \frac{\partial}{\partial p_\nu} C_{\mu\nu}^{\tilde{\lambda}} = \epsilon A \left[\delta_\mu^{\tilde{\lambda}} k_\alpha \frac{\partial}{\partial p_\alpha} - k_\mu \frac{\partial}{\partial p_{\tilde{\lambda}}} \right] \left[\frac{1}{(q^2-A^2)(p^2-A^2)} \right] , \qquad (5.6c)$$

$$\Box_p \frac{\partial}{\partial p_\rho} C_{\mu\nu\rho}^{\tilde{\lambda}} = -2\epsilon \left\{ \frac{\partial}{\partial p_{\tilde{\lambda}}} \left[\frac{q_\nu p_\rho - p_\nu q_\rho}{(q^2-A^2)(p^2-A^2)} \right] + \frac{\partial}{\partial p_\rho} \left[\frac{(q_\nu p_\rho - p_\nu q_\rho)\delta_\mu^{\tilde{\lambda}} + (q_\rho p_\mu - p_\rho q_\mu)\delta_\nu^{\tilde{\lambda}}}{(q^2-A^2)(p^2-A^2)} \right] \right\} , \qquad (5.6d)$$

where

$$\Box_p E^{\tilde{\lambda}} = \frac{\partial}{\partial p_\nu} \left[\frac{(A^2-\bar{q}\cdot\bar{p})\delta_\nu^{\tilde{\lambda}} + (q_\nu+p_\nu)p^{\tilde{\lambda}}}{(q^2-A^2)(p^2-A^2)} \right] .$$

The tensor functions are defined and thereby related to the eight scalar functions by the following:

$$C^{\tilde{\lambda}} = \tfrac{1}{4}\mathrm{tr}[\Gamma^{\tilde{\lambda}}] = p^{\tilde{\lambda}} F , \qquad (5.7a)$$

$$C_\mu^{\tilde{\lambda}} = \tfrac{1}{4}\mathrm{tr}[\Gamma^{\tilde{\lambda}}\gamma_\mu] = \delta_\mu^{\tilde{\lambda}} G_0 + p^{\tilde{\lambda}} p_\mu G_1 + p^{\tilde{\lambda}} k_\mu G_2 , \qquad (5.7b)$$

$$\begin{aligned} C_{\mu\nu}^{\tilde{\lambda}} &= \tfrac{1}{4}\mathrm{tr}[-2i\Gamma^{\tilde{\lambda}}\sigma_{\mu\nu}] \\ &= 2(p^{\tilde{\lambda}} p_\nu k_\mu - p^{\tilde{\lambda}} p_\mu k_\nu) H_0 \\ &\quad + 2(\delta_\nu^{\tilde{\lambda}} p_\mu - \delta_\mu^{\tilde{\lambda}} p_\nu) H_1 + 2(\delta_\nu^{\tilde{\lambda}} k_\mu - \delta_\mu^{\tilde{\lambda}} k_\nu) H_2 , \end{aligned} \qquad (5.7c)$$

$$\begin{aligned} C_{\mu\nu\rho}^{\tilde{\lambda}} &= \tfrac{1}{4}\mathrm{tr}[\Gamma^{\tilde{\lambda}} \epsilon_{\mu\nu\rho\phi}\gamma^5\gamma^\phi] \\ &= [\,\delta_\mu^{\tilde{\lambda}}(k_\rho p_\nu - k_\nu p_\rho) + \delta_\nu^{\tilde{\lambda}}(k_\mu p_\rho - k_\rho p_\mu) \\ &\quad + \delta_\rho^{\tilde{\lambda}}(k_\nu p_\mu - k_\mu p_\nu)] I . \end{aligned} \qquad (5.7d)$$

In the Appendix there is presented a set of integral transforms which can be employed to find approximate analytic solutions to Eqs. (5.6) if the variation of A is neglected. Once Eqs. (5.6) had been solved for the $C^{\tilde{\lambda}}$, $C_\mu^{\tilde{\lambda}}$, $C_{\mu\nu}^{\tilde{\lambda}}$ and $C_{\mu\nu\rho}^{\tilde{\lambda}}$ tensors, the determination of the eight scalar functions followed through Eqs. (5.7).

VI. THE VERTEX SOLUTION

In order to use the transforms in the appendix to solve Eq. (5.1), we find it necessary to ignore the variation of $A(p^2)$. However, it is clear that both $A(p^2)$ and $A(q^2)$ will appear in Eq. (5.2), and the question arises as to which one or what combination of the two should be used for the constant value. To answer this question, we returned to Eq. (4.8), programmed it on a computer, and substituted the solutions into it. By comparing the numerical values of the left-hand side, of Eq. (4.8) with those obtained for the right-hand side, we found that the linear combination

$$A=\tfrac{1}{2}[A(\bar{p}+\bar{k})+A(\bar{p})]\ , \tag{6.1}$$

gave the best solution over the largest range of values of $\bar{p}$ and $\bar{K}$.

Table I summarizes our determination of the eight transverse scalar vertex functions. These eight scalar functions satisfy Eq. (4.8), with less than a 3% difference between left- and right-hand sides over the following range of variables p^2, k^2, and u. For any value of k^2, restrict p^2 and u by

$$\begin{aligned} &10^{-6}<\left|\frac{p^2}{k^2}\right|<\infty\ ,\\ &0<u^2\left|\frac{k^2}{p^2}\right|<10^5\ . \end{aligned} \tag{6.2}$$

Thus, this solution to the vertex equation has a range of validity which extends from the vicinity of the mass shell out to arbitrarily large values of the electron momentum squared.

When the functions in Table I are evaluated in the region complementary to Eq. (6.2) we have difficulty performing the numerical integrations accurately. Where the ingoing electron momentum squared, p^2, is allowed to grow smaller in magnitude than the photon momentum squared, k^2, greater and greater precision is required. The need for precision is exacerbated by the fact that in order to test Eq. (4.8) we require not only an evaluation of the eight functions but also an equally precise evaluation of all possible mixed derivatives (with respect to the variables p^2 and u) up to third order. An assortment of numerical procedures of increasing complexity was applied to the computation of the functions and their derivatives. Each improvement in precision extended the region of validity of the solution represented by Table I. Equation (6.2) recognizes a practical limit.

It would be nice to be able to evaluate the functions in the region complementary to Eq. (6.2) without having to go to heroic lengths. Therefore, we observe that where the vertex $\Gamma^\lambda(\bar{p},\bar{q})$ has ingoing momentum $\bar{q}$, such that $|q^2|<10^{-6}|k^2|$, then reversed order vertex, $\Gamma^\lambda(\bar{q},\bar{p})$, with ingoing momentum $\bar{p}$ will be such that $|p^2|\approx|k^2|$. Thus, where one cannot evaluate $\Gamma^\lambda(\bar{p},\bar{q})$, one can evaluate $\Gamma^\lambda(\bar{q},\bar{p})$. So, if there exists a relationship between the vertices, we could evaluate the scalar functions of $\Gamma^\lambda(\bar{p},\bar{q})$ in terms of the available scalar functions of $\Gamma^\lambda(\bar{q},\bar{p})$. The scalar functions of $\Gamma^\lambda(\bar{p},\bar{q})$ are defined by

$$\begin{aligned}\Gamma^\lambda(\bar{p},\bar{q})=&\,q^{\bar{\lambda}}F(q^2,k^2,u_q)+\gamma^{\bar{\lambda}}G_0\\ &+q^{\bar{\lambda}}\not{q}G_1-q^{\bar{\lambda}}\not{k}G_2-2i\sigma^{\beta\alpha}q_\alpha k_\beta q^{\bar{\lambda}}H_0\\ &+2i\sigma^{\alpha\bar{\lambda}}q_\alpha H_1-2i\sigma^{\alpha\bar{\lambda}}k_\alpha H_2-\epsilon^{\bar{\lambda}\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha q_\beta I\\ &+\Gamma^\lambda_{\text{longitudinal}}(\bar{p},\bar{q})\ ,\end{aligned} \tag{6.3}$$

where $\bar{p}=\bar{q}-\bar{k}$ and where F, G_0, G_1, G_2, H_0, H_1, H_2, and I are all functions of q^2, k^2, and

$$u_q=\frac{q_\alpha k^\alpha}{(q^2)^{1/2}(k^2)^{1/2}}\ .$$

The scalar functions of $\Gamma^\lambda(\bar{q},\bar{p})$ were defined in Eq. (4.1) and can be evaluated by Table I.

A relationship between $\Gamma^\lambda(\bar{q},\bar{p})$ and $\Gamma^\lambda(\bar{p},\bar{q})$ can be demonstrated by considering the effect of charge conjugation on the vertex, the electron propagator and the photon propagator:

$$C\Gamma^\lambda(\bar{q},\bar{p})^TC^{-1}=-\Gamma^\lambda(-\bar{p},-\bar{q})\ , \tag{6.4}$$

$$CS^{-1}(\bar{p})^TC^{-1}=S^{-1}(-\bar{p})\ , \tag{6.5}$$

$$CD_{\mu\nu}(\bar{k})^TC^{-1}=D_{\mu\nu}(\bar{k})\ , \tag{6.6}$$

also

$$C\gamma^{\lambda T}C^{-1}=-\gamma^\lambda\ , \tag{6.7}$$

where $C=i\gamma^2\gamma^0$.

From Eq. (6.5) we have

$$C\Sigma(\bar{p})^TC^{-1}=\Sigma(-\bar{p})\ . \tag{6.8}$$

Thus,

$$\begin{aligned}C\left[\int D_{\mu\nu}(\bar{k})\Gamma^\mu(\bar{p},\bar{p}+\bar{k})S(\bar{p}+\bar{k})\gamma^\nu d^4k\right]^TC^{-1}\\ =\int D_{\alpha\beta}(\bar{k})\Gamma^\alpha(-\bar{p},-\bar{p}-\bar{k})S(-\bar{p}-\bar{k})\gamma^\beta d^4k\ .\end{aligned} \tag{6.9}$$

This yields the relationship

$$D_{\mu\nu}(\bar{k})\gamma^\nu S(\bar{q})\Gamma^\mu(\bar{q},\bar{p})=D_{\alpha\beta}(\bar{k})\Gamma^\alpha(\bar{p},\bar{q})S(\bar{q})\gamma^\beta\ . \tag{6.10}$$

By inserting Eq. (6.3) and Eq. (4.1) into the above we can solve for each of the scalar functions $F(q^2,k^2,u_q)$, etc., in terms of the scalar functions $F(p^2,k^2,u)$, etc. In this way a complete solution to Eq. (4.8) is obtained over the entire range of momenta excluding only a small region at the mass shell,

$$\left|\frac{p^2}{m^2}-1\right|<10^{-6}\ . \tag{6.11}$$

The limitation in Eq. (6.11) reflects back to our original determination of the electron propagator functions, A and B. Although these functions well represent the electron propagator at the mass shell, their derivatives are logarithmically divergent at the mass shell. Accordingly, since the two vertex functions, F and G_1, have the following limiting behavior at the mass shell,

$$F\to 2\frac{dA(\bar{p})}{dp^2}\ ,$$

$$G_1\to 2\frac{dB(\bar{p})}{dp^2}\ ,$$

TABLE I. The eight transverse scalar vertex functions.

$$F=\tfrac{3}{8}\epsilon(A_1+A_2)\int_{-1}^{1}I_1 d\beta\,,$$

$$I=\frac{\epsilon}{4}\int_{-1}^{1}I_1 d\beta\,,$$

$$G_0=1+\frac{\epsilon}{4}\int_{-1}^{1}I_2 d\beta-\frac{\epsilon}{4}CK-\frac{\epsilon}{4}\int_{-1}^{1}I_3 d\beta\,,$$

$$G_1=\frac{\epsilon}{2}\int_{-1}^{1}I_3^x d\beta\,,$$

$$G_2=\frac{\epsilon}{4}\left[1+2u\left[\frac{p^2}{k^2}\right]^{1/2}\right]\left[\int_{-1}^{1}I_3^x d\beta+\int_{-1}^{1}I_4^x d\beta\right]-\frac{\epsilon}{4}\int_{-1}^{1}I_3^x d\beta,$$

$$H_0=-\frac{\epsilon}{16}(A_1+A_2)\int_{-1}^{1}I_5^{xx}d\beta\,,$$

$$H_1=\frac{\epsilon}{32}(A_1+A_2)\left\{-k^2\int_{-1}^{1}I_5^{xx}\beta d\beta-2\left[(p^2)^{1/2}(k^2)^{1/2}u+\frac{k^2}{2}\right]\int_{-1}^{1}I_5^{xx}d\beta\right\},$$

$$H_2=\frac{\epsilon}{16}(A_1+A_2)\left\{\int_{-1}^{1}I_5^x d\beta+\tfrac{1}{2}\int_{-1}^{1}I_6 d\beta+\left[(p^2)^{1/2}(k^2)^{1/2}\frac{u}{2}+\frac{k^2}{4}\right]\left[\int_{-1}^{1}I_5^{xx}d\beta+\int_{-1}^{1}I_5^{xx}\beta d\beta\right]+\tfrac{1}{4}k^2\int_{-1}^{1}I_5^{xx}\beta^2 d\beta\right\},$$

where

$$u=\frac{p_\alpha k^\alpha}{(p^2)^{1/2}(k^2)^{1/2}},\quad A_1=A((\bar p+\bar k)^2),\quad A_2=A(p^2)\,,$$

and

$$\epsilon=\frac{e^2}{4\pi^2}\,,$$

and where

$$I_1=\frac{1}{x_\beta}\left[\frac{\mu^\beta}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+1\right],$$

$$I_2=\left[1-\frac{\mu_\beta}{x_\beta}-\frac{k^2}{2x_\beta}\right]\ln\left[1-\frac{x_\beta}{\mu_\beta}\right],$$

$$I_3=(A_1A_2-\mu_\beta)\frac{1}{x_\beta}\left[\left[\frac{\mu_\beta}{x_\beta}-1\right]\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+1\right],$$

$$I_4=(A_1A_2-\mu_\beta)\frac{1}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right],$$

$$CK=\left[1-\frac{A_1A_2}{{p_1}^2}\right]\ln\left[1-\frac{{p_1}^2}{m^2}\right]+\left[1-\frac{A_1A_2}{{p_2}^2}\right]\ln\left[1-\frac{{p_2}^2}{m^2}\right],$$

$$I_3^x=(A_1A_2-\mu_\beta)\left[-\frac{2\mu_\beta}{{x_\beta}^3}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+\frac{1}{{x_\beta}^2}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]-\frac{2}{{x_\beta}^2}\right],$$

$$I_4^x=(A_1A_2-\mu_\beta)\left[-\frac{1}{{x_\beta}^2}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]-\frac{1}{\mu_\beta x_\beta}\frac{1}{\left[1-\dfrac{x_\beta}{\mu_\beta}\right]}\right],$$

$$I_5=L_2\left[\frac{x_\beta}{\mu_\beta}\right]+\left[1-\frac{\mu_\beta}{x_\beta}\right]\ln\left[1-\frac{x_\beta}{\mu_\beta}\right],$$

$$L_2(z)=-\int_0^z\ln(1-z)\frac{dz}{z}\,,$$

$$I_5^x=\frac{1}{\mu_\beta}\left[-\frac{\mu_\beta}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+\frac{{\mu_\beta}^2}{{x_\beta}^2}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+\frac{\mu_\beta}{x_\beta}\right],$$

TABLE I. (*Continued*).

$$I_5^{xx}=\frac{1}{\mu_\beta^2}\left[\frac{\mu_\beta^2}{x_\beta^2}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+\frac{\mu_\beta}{x_\beta}\frac{1}{\left[1-\frac{x_\beta}{\mu_\beta}\right]}-2\frac{\mu_\beta^3}{x_\beta^3}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]-\frac{\mu_\beta^2}{x_\beta^2}\frac{1}{\left[1-\frac{x_\beta}{\mu_\beta}\right]}-\frac{\mu_\beta^2}{x_\beta^2}\right],$$

$$I_6=\frac{1}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]$$

for

$$x_\beta=[\bar{p}+\tfrac{1}{2}(\beta+1)\bar{k}]^2,$$

$$\mu_\beta=A_1A_2-\tfrac{1}{4}(1-\beta^2)k^2.$$

they have the same $\ln(p^2/m^2-1)$ dependence. It is not surprising to find that F and G_1 fail to satisfy the vertex equation in very close proximity to the mass shell. Work is currently underway to expand the vertex equation around the mass shell in order to establish the correct limiting behavior of F and G_1.

VII. CONCLUSION

By the implementation of a generalization of the Ward identity, we were able to truncate the infinite hierarchy of integral equations at the second level and apply the understanding gained at the previous level to facilitate the solution of the resulting equations. This procedure can be generalized to allow truncation of any higher level. An earlier paper suggested that the renormalization constant of the electron propagator is approximately unity. Since the renormalization constants for the propagator and the vertex are the same when Ward's identity is satisfied, the convergence of the vertex integral is guaranteed. Thus, by maintaining Ward's identity at all levels in the calculation, we guarantee that the vertex renormalization remains finite. A complete solution to the vertex equation has been found for all values of momenta more than a finite distance from the mass shell,

$$\left|\frac{p^2}{m^2}-1\right|>10^{-6}$$

and correct up to second order in the coupling constant. The solution can be refined by repeating the procedure to the next order in the coupling constant. This solution is represented in Eq. (6.9) with the eight scalar functions defined in Table I.

Encouraged by the success of the process to this point, we look forward to returning to the initial assumption which fixed the photon propagator in its near-the-mass-shell form, feeding it into a general form, and using the knowledge already acquired to make possible a simultaneous solution to all three equations.

APPENDIX

The solution of Eqs. (5) and (6) is facilitated by the observation of the following general relationships. Let

$$x_\beta=[\bar{p}+(\beta+1)\tfrac{1}{2}\bar{k}]^2$$

and

$$\mu_\beta=m^2-\tfrac{1}{4}(1-\beta^2)k^2.$$

Notice that where $\beta=1$, $x_\beta=p^2$ and where $\beta=-1$, $x_\beta=(\bar{p}+\bar{k})^2=q^2$. If

$$\Box_p\phi_A=\frac{1}{x_\beta-m^2},$$

then

$$\phi_A=\tfrac{1}{4}\left[1-\frac{m^2}{x_\beta}\right]\ln\left[1-\frac{x_\beta}{m^2}\right]. \tag{A1}$$

If

$$\Box_p\phi_B=\frac{1}{(q^2-m^2)(p^2-m^2)}=\tfrac{1}{2}\int_{-1}^{1}\frac{1}{(x_\beta-\mu_\beta)^2}d\beta,$$

then

$$\phi_B=-\tfrac{1}{8}\int_{-1}^{1}\frac{1}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]d\beta. \tag{A2}$$

If

$$\Box_p\phi_c=\frac{\ln\left[\dfrac{q^2-m^2}{p^2-m^2}\right]}{(q^2-p^2)}=\tfrac{1}{2}\int_{-1}^{1}\frac{d\beta}{(x_\beta-\mu_\beta)},$$

then

$$\phi_c=\tfrac{1}{2}\int_{-1}^{1}\phi_A d\beta=\tfrac{1}{8}\int_{-1}^{1}\left[1-\frac{\mu_\beta}{x_\beta}\right]\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]d\beta. \tag{A3}$$

If

$$\Box_p\phi_D=\int_{-1}^{1}\frac{1}{x_\beta}(m^2-\mu_\beta)\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]d\beta,$$

then

$$\phi_D=\tfrac{1}{4}\int_{-1}^{1}(m^2-\mu_\beta)\left[L\left[\frac{x_\beta}{\mu_\beta}\right]-\left[1-\frac{\mu_\beta}{x_\beta}\right]\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]\right], \tag{A4}$$

where

$$L(x)=-\int_0^x \frac{1}{x}\ln(1-x)dx$$
$$=-xL(x)-(x-1)\ln(1-x)+x\ .$$

As an illustration of the procedure, we will show how the above transformation rules will lead to a simple solution to Eqs. (5) and (6a). Equations (5) and (6a) can be reexpressed as

$$\Box_p C^\lambda=3\epsilon m\frac{\partial}{\partial p_\lambda}\left[\frac{\ln\left[\dfrac{q^2-m^2}{p^2-m^2}\right]}{q^2-p^2}\right]. \tag{A5}$$

Using the definition of ϕ_c in Eq. (A3) and letting $\phi'(z)=(\partial/\partial z)\phi(z)$ we can conclude that

$$C^\lambda=3\epsilon m\frac{\partial}{\partial p_\lambda}\phi_c$$
$$=\tfrac{3}{4}\epsilon m p^\lambda\int_{-1}^{1}\frac{1}{\mu_\beta}\phi'_A\left[\frac{x_\beta}{\mu_\beta}\right]d\beta$$
$$=\tfrac{3}{4}\epsilon m p^\lambda\int_{-1}^{1}\frac{1}{x_\beta}\left[\frac{\mu_\beta}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+1\right]d\beta\ . \tag{A6}$$

Through Eqs. (5) and (7a) the identification of the first of the eight scalar functions is shown to be

$$F=\tfrac{3}{4}\epsilon m\int_{-1}^{1}\frac{1}{x_\beta}\left[\frac{\mu_\beta}{x_\beta}\ln\left[1-\frac{x_\beta}{\mu_\beta}\right]+1\right]d\beta\ . \tag{A7}$$

The remaining seven scalar functions were determined by a similar utilization of the integral transform rules. The evaluation of the eight scalar functions was done by a numerical integration over beta.

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[18]See AIP document no. PAPS PRVDA-30-1742-9 for 9 pages containing the eight equations which were solved for the transverse scalar functions F, G_0, G_1, G_2, H_0, H_1, H_2, and I. The eight equations represent the coefficients of the matrices $p^{\bar\lambda}$, $\gamma^{\bar\lambda}$, $p^{\bar\lambda}p$, $p^{\bar\lambda}k$, $i\sigma^{\alpha\beta}p_\alpha k_\beta p$, $i\sigma^{\bar\lambda\alpha}p_\alpha$, $i\sigma^{\bar\lambda\beta}k_\beta$, and $\epsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta$ in the matrix Eq. (4.8). Order by PAPS number and journal reference from American Institute of Physics, Physics Auxiliary Publication Service, 335 East 45th Street, New York, NY 10017. The price prepaid is \$1.50 for a microfiche (98 pages), or \$5.00 for a photocopy. Airmail additional.