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Quantum field theory on discrete space-time

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Classical dynamics and classical and quantum field theories are formulated on discrete space-time, the idea of which was originally introduced by Yukawa and was called an elementary domain. The theory given here is, so to speak, a naive realization of the elementary domain. All the fundamental equations are written in the form of difference equations instead of differential equations. The equations are solved exactly in special cases. The field is quantized canonically and the propagator of the field is obtained. In the case of an interacting field any matrix element is proved to be convergent as long as the field is massive.

I. INTRODUCTION AND FUNDAMENTAL ASSUMPTIONS

In the ancient Greek period, Hellenic philosophers (Leukippos and Demokritos) considered the construction of matter in nature and arrived at the concept of the "atom," which meant at that time an indivisible element of matter. It was, of course, a product of the imagination. In modern times Yukawa considered space-time in the same way and concluded that it should also consist of small parts, which are by no means divided into smaller pieces.[1] This constituent element was called an "elementary domain." The philosophical background of this idea is that the continuity of space-time is the very origin of the divergence difficulty encountered in field theory. To put this more strongly, matter cannot exist with a finite quantity in continuous space-time. Yukawa and his collaborators attempted to formulate his idea in a mathematical form, but did not succeed in establishing it for a widely accepted theory.[2] The most difficult point in this case is how one can make the elementary domain conform to relativity. After Yukawa's death the idea was left unnoticed and no development has been observed in this field. However, it offers a fundamental problem on our view of nature, which should be settled in the future.

The purpose of this note is to give a naive mathematical form to the idea of the elementary domain, which is completely different from the original one of Yukawa. As was suspected by Yukawa,[3] many contemporary physicists have vague doubts about the space-time concept of special relativity when applied to the fundamental nature of elementary particles. The formulation proposed here does not necessarily conform to relativity in the rigorous sense, but it might be acceptable with the above-mentioned interpretation.

We now consider the following three assumptions.

(1) "There exists an absolute minimum distance for discrimination in space-time." That is to say, two points that are closer to each other than this distance are by no means distinguishable. When λ and τ denote these minimum values of space and time, respectively, it might be reasonable to assume $\lambda = c\tau$, where c is the velocity of light (c and $\hbar$ are set equal to 1 hereafter).

(2) "The value of this minimum distance is independent of Lorentz frame." The introduction of a length which is not subjected to the Lorentz transformation might violate the principles of relativity. We will come back to this point a little latter and consider it more precisely. If we accept these assumptions, there is no meaning in considering a smaller length than the value λ. Therefore, all differential equations with respect to space-time should be replaced by *difference* equations. We call this replacement "quantization of space-time." The difference equations should tend to the original differential equations, if λ and τ go to zero. As is easily seen, we can consider many difference equations which correspond to one differential equation. For example,

$$\frac{df(t)}{dt} \Longrightarrow \Delta_\tau f(t) \equiv \frac{1}{(m-n)\tau}[f(t+m\tau)-f(t+n\tau)] \, , \tag{1.1}$$

where m and n are arbitrary integers. However, we will consider only the cases where m and n are equal to zero or ± 1 for simplicity.

(3) "What describes nature correctly is the difference equation." The differential equation is just an approximation. This situation is similar to the relation between classical theory and quantum theory. As we obtain classical theory from quantum theory by setting $\hbar \rightarrow 0$, we also obtain the original (but approximate) differential equation from the difference equation by taking the limit $\lambda, \tau \rightarrow 0$.

It must be noticed here that the difference equation is not Lorentz invariant, because the quantization of space-time is done referring to a special coordinate system. Lorentz invariance is to be required only of the original differential equations and not of the difference equations. Our interest at present lies mainly in academic problems, such as whether or not it is possible to formulate a consistent theory on discrete space-time, how the divergence difficulty (which was inevitable in the case of continuous space-time) is removed, and so on.

In Sec. II we reformulate classical dynamics on discrete time, where space is still continuous. We take a linear oscillator as an example and see how the quantization of space-time works. There are many ways of quantizing space-time, some of which may lead to unfavorable solutions. In order to avoid such problems we start from first principles and construct a consistent theory, that is, we see that the favorable theory is reproduced by Lagrangian formalism. Section III is devoted to classical field theory, where both space and time are quantized. The field satisfies a difference equation, which is solved in a special case. The canonical momentum conjugate to the field and the Hamiltonian are obtained. In Sec. IV the field is quantized canonically. The propagator of the field is obtained. In the case of an interacting field we have no consistent theory. However, we discuss the problems met there in some detail, especially the divergence problem.

II. CLASSICAL DYNAMICS

In this section we reformulate classical dynamics on discrete time, where space is still continuous. As classical dynamics has a firm foundation, the new formulation should not give any visible change. Under these conditions we construct the theory from first principles. We see that the Lagrangian formalism is also possible in this case.

A. A heuristic example

We begin with a linear harmonic oscillator,

$$\frac{d^2x}{dt^2}=-\omega^2 x\ , \tag{2.1}$$

as the original equation. By simple-minded quantization of space-time we obtain the following difference equation:

$$\Delta_\tau{}^2 x(t)=-\omega^2 x(t)\ , \tag{2.2}$$

where

$$\Delta_\tau x(t)\equiv\frac{1}{\tau}[x(t+\tau)-x(t)]\ . \tag{2.3}$$

Equation (2.2) is easily solved by standard methods.[4] The solution is given by

$$x(t)=c[1+(\omega\tau)^2]^{t/\tau}\cos(\theta t/\tau+\alpha),\quad \tan\theta\equiv\omega\tau \tag{2.4}$$

which diverges as t goes to infinity. If we accept it seriously and estimate the length of time that yields an observable change in the amplitude, then we have at least $t>10^{23}\,\text{sec}\simeq 10^{16}\,\text{yr}$ for $\omega\sim 1\ \text{sec}^{-1}$, because we know that λ should be smaller than 10^{-13} cm, hence $\tau<10^{-23}$ sec. This length is far larger than the age of the Universe so there is no problem in practice.

In order to calculate the energy we rewrite the expression

$$E=\frac{1}{2}\left[\frac{dx}{dt}\right]^2+\tfrac{1}{2}\omega^2x^2 \tag{2.5}$$

in the difference form

$$E=\tfrac{1}{2}[\Delta_\tau x(t)]^2+\tfrac{1}{2}\omega^2[x(t)]^2\ . \tag{2.6}$$

Substituting the solution (2.4) for $x(t)$ in Eq. (2.6) we obtain

$$E=\tfrac{1}{2}c^2\omega^2[1+(\omega\tau)^2]^{t/\tau}\ , \tag{2.7}$$

which diverges again for $t\to\infty$.

From a certain standpoint such a divergence is undesirable, though it causes no trouble in practice. In fact, the divergence results from the simple-minded quantization of space-time. If we choose the following difference equation instead of Eq. (2.2),

$$\Delta_\tau{}^2 x(t-\tau)=-\omega^2 x(t)\ , \tag{2.8}$$

we have the solution

$$x(t)=a\cos(\theta t/\tau+\alpha)\ ,\qquad \sin\theta/2\equiv\omega\tau/2 \tag{2.9}$$

which does not diverge for $t\to\infty$. For the energy the expression

$$E=\tfrac{1}{2}[\Delta_\tau x(t)]^2+\tfrac{1}{2}\omega^2 x(t)x(t+\tau) \tag{2.10}$$

instead of Eq. (2.6) gives the constant energy:

$$E=\tfrac{1}{2}a^2\omega^2\left[1-\frac{\omega^2\tau^2}{4}\right]. \tag{2.11}$$

From Eq. (2.9) we see $\omega\leqq 2/\tau$ and $0\leqq\theta\leqq 2\pi$. For one given ω there exist two θ's (say, θ_1 and θ_2) that satisfy $\theta_1+\theta_2=2\pi$.

B. Lagrangian formalism

As is seen in Sec. II A we have a wide choice in quantizing space-time. Therefore, it seems necessary to start with a first principle and to get a unique equation. In this section we formulate classical dynamics on discrete time according to the principle of least action. Hereafter we use the time unit $\tau=1$ and hence $\Delta_\tau=\Delta_1\equiv\Delta$ for simplicity.

Now we assume the Lagrangian is a function of $x(t)$ and $\Delta x(t)$, that is, a function of $x(t)$ and $x(t+1)$:

$$L(t)\equiv L[x(t),x(t+1)]\ . \tag{2.12}$$

If we require that the action integral (or sum)

$$S\equiv\sum_{t'}L(t') \tag{2.13}$$

be stationary for arbitrary variation at t,

$$x(t')\to x(t')+\epsilon\delta_{t',t}\ , \tag{2.14}$$

we have the equation of motion

$$\frac{\partial}{\partial x(t)}[L(t)+L(t-1)]=0\ , \tag{2.15}$$

which should be satisfied by $x(t)$.

Example: Linear harmonic oscillator.

Assume the Lagrangian

$$L(t)=\tfrac{1}{2}[\Delta x(t)]^2-\tfrac{1}{2}\omega^2x^2(t)\ .$$

Then we see Eq. (2.15) yields

$$\Delta^2 x(t-1)=-\omega^2 x(t)\ ,$$

that coincides with Eq. (2.8).

The momentum canonically conjugate to $x(t)$ is defined by

$$p(t) \equiv \frac{\partial L(t)}{\partial \Delta x(t)} , \tag{2.16}$$

as is in the case of continuous time. We should notice here the differentiation with respect to $\Delta x(t) = x(t+1) - x(t)$. We mean that $x(t+1) + x(t)$ is fixed constant at the time of differentiation.[5] For example, we see $\partial x(t)/\partial \Delta x(t) = -\frac{1}{2}$. The Hamiltonian is defined by

$$H \equiv p(t)\Delta x(t) - L(t) . \tag{2.17}$$

Example: Linear harmonic oscillator.

In this case we have as the conjugate momentum

$$p(t) = \Delta x(t) + (\omega^2/2)x(t) ,$$

the second term of which may seem to be rather superfluous, if we compare it with the momentum in the case of continuous time. The Hamiltonian is then

$$H = \tfrac{1}{2}[\Delta x(t)]^2 + (\omega^2/2)x(t)x(t+1) ,$$

that agrees with Eq. (2.10). Using the equation of motion, $\Delta H = 0$ is easily verified.

III. CLASSICAL FIELD THEORY

In this section we quantize both space and time and formulate the theory of classical field on discrete space-time. We treat here the theory only on the one-dimensional space for simplicity, but it is easy to extend it to the three-dimensional space. We use the space and time units of $\lambda = \tau = 1$ hereafter.

A. Field equation and its solution

Let $\phi(t,x)$ be a scalar field and assume the Lagrangian density is a function of $\phi(t,x)$ and

$$\dot{\Delta}\phi(t,x) \equiv \phi(t+1,x) - \phi(t,x) , \tag{3.1}$$

$$\Delta'\phi(t,x) \equiv \phi(t,x+1) - \phi(t,x) , \tag{3.2}$$

then it is written as

$$L(t,x) \equiv L[\phi(t,x), \phi(t+1,x), \phi(t,x+1)] . \tag{3.3}$$

The requirement that the action sum

$$S \equiv \sum_{t',x'} L(t',x') \tag{3.4}$$

be stationary for arbitrary variations at a space-time point (t,x):

$$\phi(t',x') \rightarrow \phi(t',x') + \epsilon\delta_{t',t}\delta_{x',x} \tag{3.5}$$

leads to the field equation

$$\frac{\partial}{\partial\phi(t,x)}[L(t,x) + L(t-1,x) + L(t,x-1)] = 0 . \tag{3.6}$$

Example: Klein-Gordon field.

We assume the following Lagrangian density:

$$L(t,x) = \tfrac{1}{2}[\dot{\Delta}\phi(t,x)]^2 - \tfrac{1}{2}[\Delta'\phi(t,x)]^2 - \frac{m^2}{2}\phi^2(t,x) . \tag{3.7}$$

Then the field equation is

$$\dot{\Delta}^2\phi(t-1,x) - \Delta'^2\phi(t,x-1) + m^2\phi(t,x) = 0 \tag{3.8}$$

or equivalently,

$$\phi(t+1,x) + \phi(t-1,x) - \phi(t,x+1) - \phi(t,x-1) + m^2\phi(t,x) = 0 . \tag{3.9}$$

The solution of this equation is

$$\phi(t,x) = \int_{-\pi'}^{\pi'} d\theta_1 [A(\theta_1)e^{-i(\theta_0 t - \theta_1 x)} + B(\theta_1)e^{i(\theta_0 t - \theta_1 x)}] , \tag{3.10}$$

where $A(\theta_1)$ and $B(\theta_1)$ are arbitrary functions of θ_1 and

$$\theta_0 \equiv 2\sin^{-1}[\sin^2(\theta_1/2) + m^2/4]^{1/2} , \tag{3.11}$$

$$\pi' \equiv 2\sin^{-1}(1 - m^2/4)^{1/2} . \tag{3.12}$$

Of course, π' is equal to π if $m = 0$ and θ_0 varies between $2\sin^{-1}(m/2)$ and π.

B. Canonical momentum and Hamiltonian

In the case of continuous space-time the Lagrangian density is a local quantity and therefore the momentum canonically conjugate to a field ϕ is defined by the derivative of the Lagrangian density L with respect to the time derivative of ϕ: $\partial L/\partial(\partial_0\phi)$. However, the Lagrangian density in our case of discrete space-time is a nonlocal quantity as is seen from Eq. (3.3). For this reason we define the canonical momentum conjugate to $\phi(t,x)$ by

$$\pi(t,x) \equiv \partial\mathscr{L}/\partial\dot{\Delta}\phi(t,x) , \tag{3.13}$$

where $\mathscr{L}$ is the Lagrangian:

$$\mathscr{L} = \sum_{x'} L(t,x') . \tag{3.14}$$

The differentiation with respect to $\dot{\Delta}\phi(t,x) = \phi(t+1,x) - \phi(t,x)$ is the same as in the case of Eq. (2.16).

It must be noticed here that there are many Lagrangian densities that give the same Lagrangian, as is seen from the following example:

$$\sum_{x'=-\infty}^{\infty} \phi(t_1,x')\phi(t_2,x'+n) = \sum_{x'} \phi(t_1,x'+m)\phi(t_2,x'+n+m) ,$$

where m and n are arbitrary integers. Therefore, the density is not considered as a fundamental quantity. The Hamiltonian is defined by

$$\mathscr{H} \equiv \sum_{x'} \tfrac{1}{2}[\pi(t,x'), \dot{\Delta}\phi(t,x')]_+ - \mathscr{L} , \tag{3.15}$$

where it is symmetrized for the sake of later convenience.

Example: Klein-Gordon field.

The canonical momentum is

$$\pi(t,x) = \dot{\Delta}\phi(t,x) + \left(1 + \frac{m^2}{2}\right)\phi(t,x) - \tfrac{1}{2}\phi(t,x-1) - \tfrac{1}{2}\phi(t,x+1) . \tag{3.16}$$

The last three terms seem rather superfluous from the standpoint of continuous space-time, but are necessary for the consistency of the theory. Using the field equation (3.9), it is rewritten in a simple form:

$$\pi(t,x)=\tfrac{1}{2}[\phi(t+1,x)-\phi(t-1,x)] \ . \quad (3.17)$$

The Hamiltonian is

$$\begin{aligned}\mathscr{H}=\sum_{x'}\tfrac{1}{2}\{&\phi^2(t+1,x')+\phi^2(t,x')\\&+\frac{m^2}{2}[\phi(t,x'),\phi(t+1,x')]_+\\&-\tfrac{1}{2}[\phi(t,x'),\phi(t+1,x'+1)]_+\\&-\tfrac{1}{2}[\phi(t,x'+1),\phi(t+1,x')]_+\} \ . \quad (3.18)\end{aligned}$$

It is also rewritten by the field equation to give

$$\mathscr{H}=\sum_{x'}\{\tfrac{1}{2}\phi^2(t,x')-\tfrac{1}{4}[\phi(t+1,x'),\phi(t-1,x')]_+\} \ . \quad (3.19)$$

$\dot{\Delta}\mathscr{H}=0$ is easily verified.

IV. QUANTUM FIELD THEORY

In this section we restrict ourselves to the Klein-Gordon field and quantize it canonically. In the case of the interacting field the consistent theory is not yet obtained. The difficulties and problems met there are pointed out. The divergence problem inevitable in the case of continuous space-time can be avoided in the massive case.

A. Quantization of the field

The canonical momentum conjugate to $\phi(t,x)$ is given by Eq. (3.16), that is,

$$\begin{aligned}\pi(t,x)=\phi(t+1,x)&+\frac{m^2}{2}\phi(t,x)\\&-\tfrac{1}{2}\phi(t,x-1)-\tfrac{1}{2}\phi(t,x+1) \ . \quad (4.1)\end{aligned}$$

For these quantities we assume the following equal-time commutation relations:

$$[\phi(t,x),\pi(t,x')]_-=i\delta_{x,x'} \ , \quad (4.2)$$

$$[\phi(t,x),\phi(t,x')]_-=[\pi(t,x),\pi(t,x')]_-=0 \ . \quad (4.3)$$

Substituting Eq. (4.1) for $\pi(t,x')$ in Eq. (4.2), we have

$$[\phi(t,x),\phi(t+1,x')]_-=i\delta_{x,x'} \ . \quad (4.4)$$

It might be interesting here to see if the Heisenberg equations hold in our case. For that purpose we examine the commutation relations of Hamiltonian $\mathscr{H}$ with $\phi(t,x)$ and $\pi(t,x)$:

$$\begin{aligned}[\phi(t,x),\mathscr{H}]_-&=\frac{i}{2}[\phi(t+1,x)-\phi(t-1,x)]\\&=i\pi(t,x) \ , \quad (4.5)\end{aligned}$$

$$[\pi(t,x),\mathscr{H}]_-=\frac{i}{2}[\pi(t+1,x)-\pi(t-1,x)] \ . \quad (4.6)$$

However, for a general operator $F(t,x)$ which includes powers of $\phi(t,x)$ and $\pi(t,x)$ we see

$$[F(t,x),\mathscr{H}]_-\neq\frac{i}{2}[F(t+1,x)-F(t-1,x)] \ . \quad (4.7)$$

Therefore, we can say that the Heisenberg equations do not hold for this Hamiltonian.

In order to represent the commutation relations (4.3) and (4.4) in terms of creation and annihilation operators, we expand $\phi(t,x)$ in the following form as in Eq. (3.10):

$$\begin{aligned}\phi(t,x)=\frac{1}{\sqrt{2\pi}}\int_{-\pi'}^{\pi'}\frac{d\theta_1}{(2\sin\theta_0)^{1/2}}[&a(\theta_1)e^{-i(\theta_0 t-\theta_1 x)}\\&+a^\dagger(\theta_1)e^{i(\theta_0 t-\theta_1 x)}] \ , \quad (4.8)\end{aligned}$$

where θ_0 is restricted in $0\leqq\theta_0\leqq\pi$. The commutation rules (4.3) and (4.4) lead to

$$[a(\theta_1),a^\dagger(\theta_1')]_-=\delta(\theta_1-\theta_1') \ , \quad (4.9)$$

$$[a(\theta_1),a(\theta_1')]_-=[a^\dagger(\theta_1),a^\dagger(\theta_1')]_-=0 \ . \quad (4.10)$$

However, it should be noticed that the original commutation rules (4.3) and (4.4) cannot be obtained from the commutation rules (4.9) and (4.10), because we take the domain of integration in Eq. (4.8) as $[-\pi',\pi']$ instead of $[-\pi,\pi]$. As is seen from Eq. (3.11), θ_0 that corresponds to θ_1 in the interval $\pi'<|\theta_1|\leqq\pi$ is not real (pure imaginary). Therefore, the state with such a θ_1 is unphysical. It is uncertain whether the restriction on the domain of θ_1 might bring any difficulty into the theory.

The Hamiltonian (3.18) or (3.19) is reduced to

$$\mathscr{H}=\int_{-\pi'}^{\pi'}d\theta_1\sin\theta_0 a^\dagger(\theta_1)a(\theta_1) \ , \quad (4.11)$$

where we neglect the zero-point energy. If we, therefore, define the vacuum $|0\rangle$ by

$$a(\theta_1)|0\rangle=0 \ , \quad (4.12)$$

then $a^\dagger(\theta_1)$ and $a(\theta_1)$ are the creation and annihilation operators of the states with energy eigenvalue $\sin\theta_0$, respectively.

Now we define the causal propagator by

$$D_F(t-t',x-x';m^2)\equiv\langle 0|T\phi(t,x)\phi(t',x')|0\rangle \ . \quad (4.13)$$

After simple calculation we see

$$\begin{aligned}D_F(t,x;m^2)&=\frac{i}{(2\pi)^2}\int_{-\pi'}^{\pi'}d\theta_1\int_{-\pi}^{\pi}d\phi\frac{e^{-i(\phi t-\theta_1 x)}}{4\sin^2(\phi/2)-4\sin^2(\theta_1/2)-m^2+i\epsilon}\\&=\frac{1}{4\pi}\int_{-\pi'}^{\pi'}\frac{d\theta_1}{\sin\theta_0}[\theta(t)e^{-i(\theta_0 t-\theta_1 x)}+\theta(-t)e^{i(\theta_0 t-\theta_1 x)}] \ , \quad (4.14)\end{aligned}$$

where $\theta(t)$ is the usual step function:

$$\theta(t)=1 \text{ for } t>0 ,\quad =\tfrac{1}{2} \text{ for } t=0 ,\quad =0 \text{ for } t<0 . \tag{4.15}$$

Clearly, D_F satisfies the equation

$$\dot{\Delta}^2 D_F(t-1,x;m^2)-\Delta'^2 D_F(t,\dot{x}-1;m^2)+m^2 D_F(t,x;m^2)=-i\delta_{t,0}\delta_{x,0} . \tag{4.16}$$

B. Interacting field and transition matrix

In this section we consider the Klein-Gordon field with self-interaction. There are many ways to rewrite the interaction in the continuous space-time to the one in the discrete space-time. For example, a nonlocal interaction in the discrete space-time,

$$g\phi(t+m\tau,x+n\lambda)\phi(t+m'\tau,x+n'\lambda)\phi(t+m''\tau,x+n''\lambda) ,$$

tends to a local interaction $g\phi^3(t,x)$ in the continuous space-time, as the space-time units τ and λ go to zero. However, if we assume such a nonlocal interaction, we should not only change the quantization through the change of canonical momentum conjugate to ϕ but also deal with the difficulty of superlight velocity. Therefore, we assume the local interaction also in our case of discrete space-time.

Unlike the case of continuous space-time the interaction Hamiltonian is not equal to the negative interaction Lagrangian in the case of discrete space-time:

$$\mathscr{H}_{\text{int}} \neq -\mathscr{L}_{\text{int}} . \tag{4.17}$$

In fact, the interaction Lagrangian

$$\mathscr{L}_{\text{int}}=\sum_x g\phi^3(t,x) , \tag{4.18}$$

changes the momentum canonically conjugate to $\phi(t,x)$ from $\pi(t,x)$ to $\tilde{\pi}(t,x)$:

$$\tilde{\pi}(t,x)=\pi(t,x)-\tfrac{3}{2}g\phi^2(t,x) . \tag{4.19}$$

The newly added term does not affect the quantization (4.4), but changes the Hamiltonian (3.15) to the new one:

$$\mathscr{H}+\mathscr{H}_{\text{int}}=\sum_x \tfrac{1}{2}[\tilde{\pi}(t,x),\dot{\Delta}\phi(t,x)]_+ -(\mathscr{L}+\mathscr{L}_{\text{int}}) , \tag{4.20}$$

where $\mathscr{H}_{\text{int}}$, is given by

$$\mathscr{H}_{\text{int}}=\sum_x g\{\tfrac{1}{2}\phi^3(t,x) -\tfrac{3}{4}[\phi^2(t,x),\phi(t+1,x)]_+\} , \tag{4.21}$$

which is not equal to $-\mathscr{L}_{\text{int}}$. The second term of Eq. (4.21) seems to show a nonlocality over time.

Now it appears very difficult to establish the consistent S-matrix theory on the discrete space-time. As in the case of continuous space-time we assume for a time that the S matrix

$$S\equiv T\exp\left[-i\sum_t \mathscr{H}_{\text{int}}\right] \tag{4.22}$$

is interpreted as usual. Then we see the eigenvalue of free Hamiltonian $\mathscr{H}$ is not conserved between the initial and final states. However, the eigenvalue of the new operator

$$\Theta_\mu\equiv \int_{-\pi'}^{\pi'} d\theta_1\theta_\mu a^\dagger(\theta_1)a(\theta_1) \quad (\mu=1,0) \tag{4.23}$$

is conserved. The commutator of $\phi(t,x)$ and Θ_0 gives the "formal" derivative of $\phi(t,x)$ with respect to t:

$$[\phi(t,x),\Theta_0]_-=i\frac{\partial}{\partial t}\phi(t,x) . \tag{4.24}$$

Therefore, if we suppose the Hamiltonian is Θ_0 instead of $\mathscr{H}$, we have the Heisenberg equation

$$[F(t,x),\Theta_0]_-=i\frac{\partial}{\partial t}F(t,x) . \tag{4.25}$$

On the other hand, we have the following field equation in the case of the interacting field:

$$\dot{\Delta}^2\phi(t-1,x)-\Delta'^2\phi(t,x-1)+m^2\phi(t,x)=3g\phi^2(t,x) . \tag{4.26}$$

The equation is also written in the integral form:

$$\phi(t,x)=\phi_0(t,x)+3ig\sum_{t',x'} D_F(t-t',x-x';m^2)\phi^2(t',x') , \tag{4.27}$$

where $\phi_0(t,x)$ satisfies Eq. (3.8). By iteration Eq. (4.27) is solved in a power series of the coupling constant g. Therefore, the multipoint function

$$\langle 0\,|\,\phi(t_1,x_1)\phi(t_2,x_2)\cdots\phi(t_n,x_n)\,|\,0\rangle \tag{4.28}$$

would be calculable perturbatively in principle. Hence, the transition matrix element for a certain process would be obtained in a power series of coupling constant, though the convergence of the series and the unitarity of S matrix remain unproved.

Whichever method we use, S matrix (4.22) or the multipoint function (4.28), to evaluate the transition matrix, we must consider the Feynman diagrams, some of which correspond to the divergent integral in the case of continuous space-time. The highest divergence comes from the tadpole diagram and is given by the value of the propagator at the origin $t=x=0$. In our case of discrete space-time, however, it is finite if $m\neq 0$:

$$D_F(0,0;m^2)=\frac{1}{4\pi}\int_{-\pi'}^{\pi'}\frac{d\theta_1}{\sin\theta_0} =\frac{1}{\pi}\int_m^2[(p_0{}^2-m^2)(4-p_0{}^2)\times(4+m^2-p_0{}^2)]^{-1/2}dp_0 . \tag{4.29}$$

For $m\to 0$ there are two domains relating to the divergence; one is near $p_0=0$ and the other $p_0=2$. The former apparently corresponds to the infrared divergence, which can be removed by the well-known procedure. The latter is similar to the former, if p_0 and $2-p_0$ are exchanged. This fact seems to suggest the possibility that the latter divergence be removed by a similar procedure.

Finally, we must mention the relation of our theory to the lattice gauge theory.[6] Although the original ideas are different from each other in the two theories, there are some similarities. For example, the space-time is divided into a lattice. However, in the lattice gauge theory the space-time is Euclidean, the lattice spacing is to be led to zero in the last step and the Feynman path-integral method is used for the quantization. The underlying thought of our theory is akin rather to the work of Tati.[7]

ACKNOWLEDGMENT

The present work may be said to be a result of the symposium on "space-time description of elementary particles" held once a year under the advocacy of the late Professor H. Yukawa, though it appeared fairly late. However, one may not expect hasty results of the symposium on such a fundamental problem. The author felt, in fact, stimulated from it and feels happy even now that he could participate in it. He also thanks the members of our particle group in Fukui for discussion and he is very grateful to the organizers of Nihon University Meeting held 25—27 July 1983 for giving him an occasion to talk on this problem.

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[4]The standard textbooks on difference equations. See, for example, L. Brand, *Differential and Difference Equations* (Wiley, New York, 1966).

[5]The author is indebted to Dr. M. Horibe for pointing out this differentiation.

[6]There are many references on the lattice gauge theory, but we refer to only two early works: K. Wilson, Phys. Rev. D **10**, 2445 (1974); J. Kogut and L. Susskind, *ibid.* **11**, 395 (1975).

[7]T. Tati, Prog. Theor. Phys. Suppl. **29**, 1 (1964); Prog. Theor. Phys. **37**, 754 (1967).