Theory of the relativistic spinning particle: Hamiltonian formulation and world-line invariance

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The theory of the classical relativistic spinning particle, called the pure gyroscope, is reformulated in the language of Dirac's constraint Hamiltonian dynamics. So formulated, the particle trajectory in Minkowski space is shown to satisfy the condition of world-line invariance, in the form of covariant predictivity.

I. INTRODUCTION

In 1926 Frenkel[1] introduced a model of a classical relativistic point particle, which has come to be known as the pure gyroscope.[2] This model proved to be interesting for several reasons. (a) It was the first Lagrangian formulation for a spinning particle at the classical, relativistic level. (b) It allowed the extension of the correspondence principle to the domain of relativistic quantum theory. The pure gyroscope was shown, via Ehrenfest's theorem, to be the classical limit for a class of first-order relativistic wave equations,[3] e.g., the Majorana equation.[4] (c) Because of the complex gyrations of the pure gyroscope within a small region of space,[2] it was considered as a classical model for an elementary particle.[5,6] This did not meet with success, but the analysis, coupled with the extended correspondence principle, shed some light on the kinematic content of certain relativistic wave equations.[3,4,7] The truly significant attempt to upgrade the structure of the pure gyroscope to that of a relativistic rotator model for an elementary particle is due to Corben.[8,9] Other models based on "reinvention" of the pure gyroscope analysis also occur.[10]

In this paper we again re-examine the pure gyroscope for the long-range purpose of constructing an elementary-particle model with interacting pure gyroscopes as constituents. To be most useful the theory should rest on sound Hamiltonian ground. To this end we here reformulate the theory of the free pure gyroscope as a problem in Dirac's constraint Hamiltonian dynamics,[11] and then check to see if so formulated the theory passes the test of world-line invariance.[12]

In Sec. II we briefly review those features of the pure-gyroscope kinematics which are relevant to the subsequent discussion. In Sec. III we present the constraint Hamiltonian formulation for this model. In Sec. IV we give an explicit proof of the invariance of the pure-gyroscope world line by showing that exactly the same world line will be calculated by all inertial observers. The method of proof, called functional reparametrization, requires the introduction of a free, spinless spectator particle to represent an inertial observer,[13,14] and corresponds to a covariant predictivity requirement.

II. FREE GYROSCOPE KINEMATICS

A spinning particle, described covariantly, is one whose angular momentum tensor is given by

$$M_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu+s_{\mu\nu}\ , \tag{1}$$

where x_μ is the instantaneous position four-vector,[15] p_μ is the four-momentum conjugate to x_μ, and $s_{\mu\nu}$ is the antisymmetric spin angular momentum tensor. The pure gyroscope (PG) is that spinning particle whose spin tensor is subject to the constraints[2]

$$s_{\mu\nu}v_\nu=0\ , \tag{2}$$

where $v_\mu=\dot{x}_\mu$, the instantaneous velocity four-vector, is the derivative of x_μ with respect to σ, the PG proper time, so that $v_\mu v_\mu=-1$. There are three linearly independent constraint equations in (2) and they force the s_{i4} to vanish in the frame in which $\vec{v}=0$, the intrinsic rest frame (IRF). Therefore, at rest, the PG is a pure magnetic dipole.[16]

Since the particle is free, $\dot{M}_{\mu\nu}=0=\dot{p}_\mu$. Therefore, if $s_{\mu\nu}$ is not required to be separately conserved, then

$$\dot{s}_{\mu\nu}=p_\mu v_\nu-p_\nu v_\mu \tag{3}$$

and in the most general case the velocity and momentum four-vectors are not collinear. As a consequence of (2) the spin tensor obeys the identity[17]

$$s_{\mu\nu}s_{\nu\rho}s_{\rho\sigma}=-s_0{}^2 s_{\mu\sigma}\ , \tag{4}$$

where the constant $s_{\mu\nu}s_{\mu\nu}=2s_0{}^2$, and s_0 is the spin magnitude in the IRF. We further define the intrinsic rest mass

$$m=-v_\mu p_\mu \tag{5}$$

which is conserved as a consequence of (2) and (3). It also follows from (3), (4), and (5) that

$$mv_\mu=p_\mu+s_{\mu\nu}s_{\nu\sigma}p_\sigma\ /s_0{}^2\ . \tag{6}$$

Multiplying through by p_μ and using (5) yields the identity

$$\phi=p_\mu p_\mu+p_\mu s_{\mu\nu}s_{\nu\sigma}p_\sigma\ /s_0{}^2+m^2=0\ . \tag{7}$$

Successive derivatives of (6) yield

$$s_0{}^2\ddot{v}_\mu = mp_\mu - M^2 v_\mu \ , \tag{8}$$

where $M^2 = -p_\mu p_\mu$. Equation (8) admits the solution

$$x_\mu(\sigma) = A_\mu(\sigma_0)\sin\omega_1\sigma + B_\mu(\sigma_0)\cos\omega_1\sigma + (m/M^2)p_\mu\sigma \tag{9a}$$

which gives the world line of the PG, parametrized by its own proper time, and

$$v_\mu(\sigma) = \omega_1[A_\mu(\sigma_0)\cos\omega_1\sigma - B_\mu(\sigma_0)\sin\omega_1\sigma] + mp_\mu/M^2 \ , \tag{9b}$$

where $\omega_1 = M/s_0$ and the amplitudes are given in terms of the initial conditions by

$$B_\mu(\sigma_0) = x_\mu(\sigma_0)\cos\omega_1\sigma_0 - [v_\mu(\sigma_0)\sin\omega_1\sigma_0]/\omega_1 - (mp_\mu/M^2)[\sigma_0\cos\omega_1\sigma_0 - (\sin\omega_1\sigma_0)/\omega_1] \ , \tag{10a}$$

$$A_\mu(\sigma_0) = x_\mu(\sigma_0)\sin\omega_1\sigma_0 + [v_\mu(\sigma_0)\cos\omega_1\sigma_0]/\omega_1 - (mp_\mu/M^2)[\sigma_0\sin\omega_1\sigma_0 + (\cos\omega_1\sigma_0)/\omega_1] \tag{10b}$$

and satisfy

$$p_\mu A_\mu(\sigma_0) = p_\mu B_\mu(\sigma_0) = 0 \tag{11}$$

by virtue of (5). Equations (9) describe a particle executing a helical trajectory in Minkowski space. This oscillatory motion is the classical *Zitterbewegung*. As viewed from the momentum rest frame (MRF), defined by $\vec{p} = 0$, these oscillations are pure spacelike and take place in a spatial region with dimensions of the classical Compton wavelength.[2]

From the covariant generalization of the Pryce center of mass (c.m.),[18,19]

$$X_\mu = x_\mu + s_{\mu\nu}p_\nu/p_\sigma p_\sigma \ , \tag{12}$$

it follows that

$$\dot{X}_\mu = mp_\mu/M^2 = \langle v_\mu(\sigma)\rangle \ , \tag{13}$$

where $\langle v_\mu(\sigma)\rangle$ is the average velocity of the particle along its world line. Therefore,

$$X_\mu(\sigma) = (mp_\mu/M^2)\sigma = \langle x_\mu(\sigma)\rangle \tag{14}$$

where $\langle x_\mu(\sigma)\rangle$ is the average position of the particle in Minkowski space. Further, using (12), the angular momentum tensor (1) may be expressed as

$$M_{\mu\nu} = X_\mu p_\nu - X_\nu p_\mu + S_{\mu\nu} \ , \tag{15}$$

where

$$S_{\mu\nu} = s_{\mu\nu} + (s_{\mu\sigma}p_\sigma p_\nu - s_{\nu\sigma}p_\sigma p_\mu)/M^2 \tag{16}$$

for which $\dot{S}_{\mu\nu} = 0$ and $S_{\mu\nu}p_\nu = 0$. Therefore, $S_{\mu\nu}S_{\mu\nu} = 2S^2$, where S is the spin magnitude in the MRF. The relation $MS = ms_0$ then obtains[4,20] and is the principal reason the PG fails as a candidate for an elementary-particle model.

III. HAMILTONIAN FORMULATION

There are two bases for the usual Hamiltonian formulation of the PG:[19] the postulated set of fundamental Poisson brackets (PB)

$$(x_\mu, x_\nu) = 0 = (p_\mu, p_\nu), \quad (x_\mu, p_\nu) = \delta_{\mu\nu} \ , \tag{17a}$$

$$(x_\mu, s_{\rho\sigma}) = 0 = (p_\mu, s_{\rho\sigma}) \ , \tag{17b}$$

$$(s_{\mu\nu}, s_{\rho\sigma}) = s_{\mu\rho}\delta_{\nu\sigma} + s_{\nu\sigma}\delta_{\mu\rho} - s_{\mu\sigma}\delta_{\nu\rho} - s_{\nu\rho}\delta_{\mu\sigma} \tag{17c}$$

and the postulated proper-time Hamiltonian

$$H = \phi/2m = 0 \tag{18}$$

with ϕ given by (7). Equations (17) ensure that

$$(M_{\mu\nu}, M_{\rho\sigma}) = M_{\mu\rho}\delta_{\nu\sigma} + M_{\nu\rho}\delta_{\mu\sigma} - M_{\mu\sigma}\delta_{\nu\rho} - M_{\nu\rho}\delta_{\mu\sigma} \ , \tag{19a}$$

$$(M_{\mu\nu}, p_\rho) = p_\nu\delta_{\mu\rho} - p_\mu\delta_{\nu\rho} \tag{19b}$$

which is the requisite Poincaré group realization,[21] and give as well

$$(M_{\mu\nu}, x_\rho) = x_\nu\delta_{\mu\rho} - x_\mu\delta_{\nu\rho} \ , \tag{20a}$$

$$(M_{\mu\nu}, X_\rho) = X_\nu\delta_{\mu\rho} - X_\mu\delta_{\nu\rho} \ , \tag{20b}$$

$$(X_\mu, p_\nu) = \delta_{\mu\nu} \ , \tag{20c}$$

$$(X_\mu, X_\nu) = S_{\mu\nu}/M^2 \ . \tag{20d}$$

Equation (20d) expresses the essential nonlocality of the covariant c.m.[19] Accepting (18) as the Hamiltonian, the PB equations of motion are

$$(x_\mu, H) = [p_\mu + (s_{\mu\sigma}s_{\sigma\nu}p_\nu)/s_0{}^2]/m = v_\mu \ , \tag{21a}$$

$$(s_{\mu\nu}, H) = p_\mu v_\nu - p_\nu v_\mu = \dot{s}_{\mu\nu} \ , \tag{21b}$$

which, together with the constraints (2), suffice to reproduce the analysis of Sec. II.

To put the Hamiltonian analysis on a firm footing we must treat *all* the constraints according to the rules of Dirac's formalism.[11] In this scheme we start by postulating the constraints

$$\phi = p_\mu p_\mu + (p_\mu s_{\mu\sigma}s_{\sigma\nu}p_\nu)/s_0{}^2 + m^2 \approx 0 \ , \tag{22}$$

$$\theta_\mu = s_{\mu\nu}[p_\nu + (s_{\nu\sigma}s_{\sigma\rho}p_\rho)/s_0{}^2] \approx 0 \ , \tag{23}$$

and the fundamental PB (17). We then obtain the following PB among the constraints;

$$(\phi, \phi) = 0 \ , \tag{24a}$$

$$(\phi, \theta_\mu) = 0 \ , \tag{24b}$$

$$(\theta_\mu, \theta_\nu) = s_{\mu\nu} + (s_{\mu\sigma}s_{\sigma\rho}s_{\rho\nu})/s_0{}^2 \approx 0 \ . \tag{24c}$$

Thus, the postulated constraints are all first class. There are no second-class constraints so the total Hamiltonian is

$$h = \lambda\phi + \lambda_\mu\theta_\mu \tag{25}$$

and the Dirac brackets coincide with the PB. Further, since θ_μ has vanishing PB with $x_\nu, p_\nu, s_{\nu\sigma}$, we may set $\lambda_\mu = 0$, and take

$$H=\lambda\phi \tag{26}$$

as the total Hamiltonian. The undetermined coefficient, λ, represents the gauge freedom still remaining.

From (26) we obtain

$$\dot{x}_\mu=(x_\mu,H)=2\lambda[p_\mu+(s_{\mu\nu}s_{\nu\sigma}p_\sigma)/s_0{}^2]=v_\mu \tag{27}$$

and (21b). The dot now means differentiation with respect to an as-yet-unspecified temporal parameter τ. From (27) we obtain

$$v_\mu v_\mu=-4m^2\lambda^2\ , \tag{28a}$$

$$v_\mu p_\mu=-2m^2\lambda\ , \tag{28b}$$

so that

$$x_\mu(\tau)=(2m^2\lambda/M^2)p_\mu\tau\ , \tag{29a}$$

$$\dot{x}_\mu\dot{x}_\mu=-4m^4\lambda^2/M^2=(m^2/M^2)v_\mu v_\mu\ . \tag{29b}$$

The gauge-invariant equations of motion now lead to the generalization of (9) and (10):

$$x_\mu(\tau)=A_\mu(\tau_0)\sin\omega\tau+B_\mu(\tau_0)\cos\omega\tau+(2m^2\lambda/M^2)p_\mu\tau\ , \tag{30a}$$

$$v_\mu(\tau)=\omega[A_\mu(\tau_0)\cos\omega\tau-B_\mu(\tau_0)\sin\omega\tau]+(2m^2\lambda/M^2)p_\mu\ , \tag{30b}$$

where $\omega=2m\lambda\omega_1$ and

$$A_\mu(\tau_0)=x_\mu(\tau_0)\sin\omega\tau_0+[v_\mu(\tau_0)\cos\omega\tau_0]/\omega \\ -(2m^2\lambda p_\mu/M^2)[\tau_0\sin\omega\tau_0+(\cos\omega\tau_0)/\omega]\ , \tag{31a}$$

$$B_\mu(\tau_0)=x_\mu(\tau_0)\cos\omega\tau_0-[v_\mu(\tau_0)\sin\omega\tau_0]/\omega \\ -(2m^2\lambda p_\mu/M^2)[\tau_0\cos\omega\tau_0-(\sin\omega\tau_0)/\omega]\ . \tag{31b}$$

The choice $\lambda=\lambda_1=(2m)^{-1}$ gives $v_\mu v_\mu=-1$, and the kinematic description of Sec. II is recovered with $\tau=\sigma$ and $\omega=\omega_1$. There is another interesting gauge choice that has meaning within the isolated PG system: $\lambda=\lambda_2=M/2m^2$ gives $\dot{X}_\mu\dot{X}_\mu=-1$. The temporal parameter is then Σ, the proper time of the c.m. The world line (30a) is then explicitly

$$x_\mu(\Sigma)=A_\mu(\Sigma_0)\sin\omega_2\Sigma+B_\mu(\Sigma_0)\cos\omega_2\Sigma+(p_\mu/M)\Sigma \tag{32}$$

with $A_\mu(\Sigma_0)$ and $B_\mu(\Sigma_0)$ gotten from (31) with $\tau_0=\Sigma_0$, $\omega=\omega_2=(M/m)\omega_1$, and $\lambda=\lambda_2$. It is important to note that (32) and (9a), in fact, describe the same world line. This is easily shown since from (29b) we have[22]

$$M/m=\sigma/\Sigma=\lambda_2/\lambda_1=\omega_2/\omega_1\ . \tag{33}$$

Therefore,

$$x_\mu[\Sigma(\sigma)]=A_\mu(\Sigma_0)\sin\omega_1\sigma+B_\mu(\Sigma_0)\cos\omega_1\sigma+(mp_\mu/M^2)\sigma \tag{34}$$

and if $A_\mu(\Sigma_0)=A_\mu(\sigma_0), B_\mu(\Sigma_0)=B_\mu(\sigma_0)$, where $\Sigma_0=\Sigma(\sigma_0)$, (34) is reduced to (9a) for all σ. Specifically with $\sigma_0=0$ and thus $\Sigma_0=0$, we obtain

$$x_\mu(\Sigma_0)=x_\mu(\sigma_0),\ \ v_\mu(\Sigma_0)=(M/m)v_\mu(\sigma_0)\ . \tag{35}$$

It is important to understand the sense in which the simple variable change (33) corresponds to world-line reparametrization. The relation

$$\hat{p}_\mu X_\mu=-\Sigma\ , \tag{36}$$

where $\hat{p}_\mu=p_\mu/M$, assigns the parameter Σ to X_4 in the MRF, the c.m. proper time. By the construction (12), $p_\mu(x_\mu-X_\mu)=0$ and therefore,

$$\hat{p}_\mu x_\mu=-\Sigma \tag{37}$$

is an identity (not a constraint) which simultaneously assigns the parameter Σ to x_4 in the MRF. If the dot in (29b) means differentiation with respect to this parameter, then the gauge is fixed and the PG world line is parametrized by the c.m. proper time, according to (32). If on the left-hand side of (37) we use $x_\mu(\sigma)$, as given by (9a), then the right-hand side of (37) becomes $\Sigma(\sigma)$, the functional form of the reparametrization scheme which assigns a value of Σ for each value of σ. The result is (33).

The gauge freedom in the theory permits the PG world line to be parametrized by its own rest clock, or by a clock at rest with respect to the c.m. The functional relationship between the two parameters then allows the reparametrization, from one gauge to the other, after the respective world lines have been constructed. An explicit relationship, such as (35), between the two sets of initial conditions, ensures world-line invariance (WLI) under this functional reparametrization.

IV. WORLD-LINE INVARIANCE

It is the purpose of this section to show that the world line of the PG is the same for all inertial observers. The method used is the demonstration of WLI under functional reparametrization. To do this we define an inertial observer as a free, spinless particle, whose world line is a geodesic in Minkowski space.[13,14] This free particle will be the spectator whose proper clock is used to parametrize the world line of the PG.

The spectator is described by the Hamiltonian

$$H_s=\lambda_s(k_\mu k_\mu+1)\approx 0\ , \tag{38}$$

where k_μ is the spectator four-momentum divided by the spectator rest mass m_s. If z_μ denotes the instantaneous position four-vector of the spectator, then $(z_\mu,k_\nu)=\delta_{\mu\nu}/m_s$, so that

$$\dot{z}_\mu=(z_\mu,H_s)=2\lambda_s k_\mu/m_s\ . \tag{39}$$

The gauge choice, $\lambda_s=m_s/2$, fixes the clock which parametrizes the spectator world line to read his own proper time. Call this temporal parameter τ. The spectator world line is then given by

$$z_\mu(\tau)=k_\mu\tau\ . \tag{40}$$

We now define a two-particle system, consisting of the PG plus the spectator. The Hamiltonian for this system consists of the separately vanishing parts, (26) and (38):

$$H=\lambda\phi+(m_s/2)(k_\mu k_\mu+1)\approx 0\ . \tag{41}$$

Since this is a two-particle system, the Poincaré generators are

$$p_\mu=p_\mu+m_s k_\mu\ , \tag{42a}$$

$$M_{\mu\nu}=(x_\mu p_\nu-x_\nu p_\mu+s_{\mu\nu})+(z_\mu k_\nu-z_\nu k_\mu)m_s\ . \tag{42b}$$

The group realization is preserved since the position and momentum of the spectator have zero PB with the position, momentum, and spin tensor of the PG.

The world line of the PG is given by (30a), but we must yet fix the gauge of the PG so that the parameter τ in that expression is, in fact, the spectator proper time. To do this we might impose the gauge constraint[13]

$$k_\mu x_\mu=-\tau \tag{43}$$

which also gives

$$k_\mu(x_\mu-z_\mu)=0\ . \tag{44}$$

Equation (43) seems to represent the obvious generalization of the scheme discussed in the last section; in the spectator rest frame (SRF), defined by $\vec{k}=0$, the parameter τ is assigned to x_4, and the separation between the PG and the spectator is pure spacelike. There is, however, a problem with this scheme. By the construction (12), the separation between the PG and its c.m. is pure spacelike in the MRF. Therefore, unless $k_\mu\sim\hat{p}_\mu$, there are two distinct inertial frames in which the PG oscillations *(Zitterbewegung)* are pure spacelike, and this cannot be. This difficulty is also seen when we try to test for WLI. We must start with the PG world line given by (9a), and then use (43) in the form $k_\mu x_\mu(\sigma)=-\tau(\sigma)$ to find the functional form to be used for τ in (30a). Explicitly, this gives

$$\tau(\sigma)=A_k(\sigma_0)\sin\omega_1\sigma+B_k(\sigma_0)\cos\omega_1\sigma+(mp_k/M^2)\sigma\ , \tag{45}$$

where

$$A_k(\sigma_0)=-k_\mu A_\mu(\sigma_0),\quad B_k(\sigma_0)=-k_\mu B_\mu(\sigma_0),$$
$$p_k=-k_\mu p_\mu\ .$$

It is not difficult to see that with the expression (45) used for τ in (30a), there is no exact relationship between the two sets of A's and B's such that $x_\mu[\tau(\sigma)]=x_\mu(\sigma)$ for all σ, unless A_k and $B_k=0$, but this holds only if $k_\mu\sim\hat{p}_\mu$.

The way around this difficulty is to relax the requirement that in the SRF the separation between the instantaneous position of the PG and the spectator be pure spacelike, and require that this condition be satisfied only on the average. Thus, (43) is replaced by the constraint

$$k_\mu X_\mu=-\tau \tag{46}$$

and similarly, X_μ replaces x_μ in (44). Recall that $X_\mu(\Sigma)=(p_\mu/M)\Sigma$, thus $\tau(\Sigma)=(p_k/M)\Sigma$, but $\Sigma=(m/M)\sigma$ and, therefore, (45) is replaced by

$$\tau(\sigma)=(mp_k/M^2)\sigma\ . \tag{47}$$

The substitution of (47) into (30a) then gives

$$\begin{aligned}x_\mu[\tau(\sigma)]&=A_\mu(\tau_0)\sin(\omega mp_k/M^2)\sigma\\&\quad+B_\mu(\tau_0)\cos(\omega mp_k/M^2)\sigma\\&\quad+(2\lambda m^3p_k/M^4)p_\mu\sigma\ .\end{aligned} \tag{48}$$

Since $\omega=2m\lambda\omega_1$, the choice forced by (46),

$$\lambda=M^2/2m^2p_k \tag{49}$$

reduces (48) to (9a) with

$$A_\mu(\tau_0)=A_\mu(\sigma_0),\quad B_\mu(\tau_0)=B_\mu(\sigma_0)\ ,$$

where $\tau_0=(mp_k/M^2)\sigma_0$. Thus, WLI under functional reparametrization is proved, since the spectator geodesic is arbitrary. It may appear that the relaxed gauge constraint (46) has introduced the possibility of unwanted timelike oscillations. Such is not the case, however, since the PG oscillations are pure spacelike in the MRF regardless of gauge constraints.

V. CONCLUSION

We have recast the theory of a relativistic spinning particle, the pure gyroscope, in the language of Dirac's constraint Hamiltonian dynamics. There are no second-class constraints and the total Hamiltonian is, aside from a gauge parameter, the one previously considered. Through the introduction of a spectator particle (inertial observer), we have used this gauge freedom to demonstrate world-line invariance under functional reparametrization. This invariance ensures that all inertial observers will construct the same world line for the pure gyroscope, and amounts to covariant predictivity.

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