# Entropy content and information flow in systems with limited energy

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Physical systems of finite size and limited total energy E have limited entropy content S (alternatively, limited information-storing capacity). We demonstrate the validity of our previously conjectured bound on the specific entropy S/E in numerous examples taken from quantum mechanics (number of energy levels up to given energy), free-field systems (entropy of miscellaneous radiations for given energy), and strongly interacting particles (number of many-hadron states up to given energy). In the quantum-mechanical examples we have compared the bound directly with the logarithm of the number of levels for the harmonic oscillator, the rigid rotator, and a particle in an arbitrary potential well. For many-particle systems such as radiations, there is no closed formula for the number of configurations associated with a specified one-particle spectrum. To overcome this barrier we use an efficient numerical algorithm to calculate the number of configurations up to given energy from the spectrum. In all our examples of systems of scalar, electromagnetic, and neutrino quanta contained in spaces of various shapes, the numerical results are in harmony with the bound on S/E. This conclusion is buttressed by an approximate analytical estimate of the peak S/Ewhich leaves little doubt as to the general applicability of the bound for systems of free quanta. We consider a gas of hadrons confined to a cavity as an example of a system of strongly interacting particles. Our numerical algorithm applied to the Hagedorn mass spectrum for hadrons confirms that the number of many-hadron states up to a given energy is consistent with the bound. Finally, we show that a rather general one-channel communication system has an information-carrying capacity which cannot exceed a bound akin to that on S/E. It is argued that a complete many-channel system is similarly limited.

## I. INTRODUCTION

The entropy content S of a physical system is an important quantity in several contexts. For example, the entropy associated with microscopic degrees of freedom relates to thermal properties (heat capacity of a cup of coffee). Similarly, the maximum entropy associated with measurable macroscopic degrees of freedom quantifies the amount of information that may be coded in, and then read out of the system (computer magnetic tape). Entropy flow rate  $\dot{S}$  is also important. Thus, the maximum entropy flow rate associated with macroscopic degrees of freedom quantifies the maximal rate of information transfer (telephone). Needless to say, a specific determination of S or  $\dot{S}$  requires detailed computation starting, say, from a particular density operator for a quantum system.

If one is willing to forego precision in exchange for being spared consideration of details, then several possibilities arise. One proposal is that specific entropy (entropy to *total* energy ratio S/E) is always bounded from above in terms of the radius of the sphere that circumscribes the (complete) system.<sup>1</sup> This would mean, for example, that the information capacity of a device of arbitrary construction and logic is limited in a predictable way by its *linear* size. In a similar vein, the entropy flow rate in a onechannel system may be limited by the energy available to form the message,<sup>2</sup> by its mean energy,<sup>3</sup> or by the energy flow associated with it.<sup>4</sup> The origin of these bounds is diverse. The proposed one on S/E was suggested by an argument in black-hole physics, and received further support from an investigation of canonical distributions of quantum fields.<sup>1</sup> The associated bound on  $\dot{S}$  is an immediate extrapolation of it to traveling systems.<sup>3</sup> Bremermann's bound<sup>2</sup> on  $\dot{S}$  is a marriage of Shannon's channel capacity theorem<sup>5</sup> with considerations about the role of quantum noise. Pendry's bound<sup>4</sup> on  $\dot{S}$  is based on a calculation of the entropy carried by traveling modes populated by a canonical distribution.

The possibilities for bounds like these are manifold, essentially because there exist many inequivalent definitions of entropy: canonical, microcanonical, .... Also a complicating factor is the related fact that the term "energy" could be given various meanings: precisely defined energy, mean energy (Pendry's and our bounds), maximum available energy (Bremermann's bound),.... The best policy in the face of such ambiguity would seem to be to first define precisely what one means by energy, and then determine S or  $\dot{S}$  from that probability distribution (or density operator) which maximizes it subject to the given constraints on the energy.

For example, if the *mean* energy  $\langle E \rangle$  is fixed, the distribution is a canonical one with some "temperature." Effectively, Pendry's calculation starts from such a viewpoint. He eliminates temperature from consideration by relating the *square* of entropy flow to the energy flow. This succeeds because of the one-dimensional nature of the flow; a different relation would be needed otherwise. Thus, if it could be generalized to a three-dimensional standing system, Pendry's approach would give a bound on S related to  $\langle E \rangle^{3/4}$  reminiscent of bounds proposed by

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Page,<sup>6</sup> and by Unruh and Wald<sup>7</sup> which hold that the entropy of thermal radiation sets an upper bound on the entropy of any system with like mean energy and volume. Counterexamples to these bounds are known,<sup>8</sup> but it is clear from their nature that they do not compromise Pendry's specific treatment.

Interestingly, if one does not require that  $\langle E \rangle$  be fixed, but merely that  $S/\langle E \rangle$  be maximal, then the appropriate distribution is also a canonical one, but with a special temperature this time.<sup>1</sup> The maximum actually exists provided the ground state of the system has positive energy. This is the relevant case for most thermodynamic systems, despite the existence of field systems with negative ground energies.<sup>9</sup> However, the usefulness of such an approach for bounding S in a communication system is doubtful since here the zero-energy state of the "message" can exist.<sup>10</sup>

An alternative meaning for energy is the total energy available or accessible to the system. Bremermann formulated his bound on  $\dot{S}$  in these terms. When the S/Ebound was first proposed, Gibbons<sup>11</sup> suggested interpreting E as the available energy. He investigated some examples of fields contained in simple cavities or spaces. For effectively one-dimensional systems the applicability of the bound on S/E was well in evidence. For the more complicated systems, the labor involved in determining the bound with a pocket calculator was found to be prohibitive, and the question was left unanswered. Actually there exists a simple relation between the bound on  $S/\langle E \rangle$  and that on S/E, where E is the available energy: provided the zero of the energy scale is chosen in the same way, the former must bound the latter from above. The proof is a slight modification of one given earlier.<sup>1</sup> However, this is not very useful for the communication problem since one wants the energy of the message to start from zero, and under such conditions there is no maximum to  $S/\langle E \rangle$ .<sup>1</sup>

Our main purpose here is to investigate extensively the applicability of the bound on S/E which we state as

$$S/E < 2\pi R/\hbar c$$
, (1)

and that of the akin bound on the information flow rate I for one-channel communication systems,

$$I < \pi E / \hbar \ln 2 . \tag{2}$$

In (1) and (2) E is regarded as the total energy available to the system or signal, respectively. We conjecture that inequalities (1) and (2) are generally valid on the basis of a variety of evidence which we shall present.

Our starting point will be the definition of S. Classically, for a system which can be in any of a class of states with probability  $P_i$  for the *i*th state,

$$S = -\sum_{i} P_{i} \ln P_{i} \tag{3}$$

with  $\sum P_i = 1$ . In the quantum theory, for a system whose state is described by the density operator  $\rho$  with  $Tr(\rho)=1$ ,

$$S = -\operatorname{Tr}(\rho \ln \rho) . \tag{4}$$

Suppose we want to maximize S/E for allotted energy E. In the classical case, if N(E) denotes the number of states with energy no larger than E, then evidently S is maximized if  $P_i = 1/N(E)$  for states with energy up to E, and  $P_i = 0$  otherwise, in which case  $S = \ln N(E)$ . Then, without regard to E

$$\max(S/E) = \max[\ln N(E)/E], \qquad (5)$$

where the maximization in the right-hand side is over all allowed E. Exactly the same result applies in the quantum case. This is best seen by using the energy representation for  $\rho$ . One can thus hope to evaluate the truth of (1) when one has a grasp of the shape of the cumulative density of states N(E) for generic classical or quantum systems. In the following sections we explore this approach for various types of physical systems.

In Sec. II we show that (1) is necessarily satisfied by simple quantum-mechanical systems (vibrators and rotators) regardless of the parameters chosen for them. This has immediate relevance to the issue of storage of information in atomic or molecular degrees of freedom (futuristic data bank). In Sec. III we study a large variety of examples of noninteracting many-particle systems (bosons or fermions) confined to spaces of various shapes. These are paradigms of blackbody radiation at very low temperatures, or of the computation process seen at its most elementary level. By a complicated algorithm that calculates S(E) we verify that there are no exceptions to (1) among our examples. In Sec. IV we uncover a simple way to estimate max(S/E) for such systems when the oneparticle spectrum is known. Most interesting, for threedimensional systems, whose aspect ratios are not too extreme, we find that knowledge of the first one-particle level suffices to estimate max(S/E), and to show, with some confidence, that bound (1) is respected in general.

In Sec. V we extend the concepts developed so far to interacting many-particle systems, such as hadrons. The empirical Hagedorn spectrum of hadrons is shown to be in harmony with bound (1). Communication systems are the subject of Sec. VI. One-channel communication lines are discussed in some generality, and using techniques from Sec. II, we show that they must respect bound (2), or equivalently, Bremermann's rule. Finally, we discuss our conclusions in Sec. VII.

#### **II. ONE-PARTICLE SYSTEMS**

An atom or a molecule could, in principle, be turned into a short-term information-storage device. The information coding would exploit the multiplicity of available atomic or molecular states. Because these states usually differ in energy, it is relevant to ask what is the maximum information which may be coded for given available energy. Suppose we apply (1) taking care to include in E all the energies, i.e., rest energies as well as excitation energies. Of couse, in real atoms and molecules most of the energy is rest energy, and so (1) predicts, for typical atomic (molecular) dimensions and masses, that the limit is some 10<sup>6</sup> bits. This certainly exceeds the logarithm of the number of atomic (molecular) states below ionization (dissociation) in known atoms and molecules, so bound (1) is easily satisfied. (The seemingly discrepant case of hydrogen with its infinity of levels is easily accounted for by remembering that the highly excited states correspond to dimensions large by atomic standards.) But it is interesting to consider hypothetical atomiclike systems whose rest masses could be adjusted at will. Would not reduction of such masses eventually bring (1) into conflict with the actual value of max[ $\ln N(E)/E$ ]?

To elucidate this question we shall now consider the cumulative number of states N(E) for one-particle quantum-mechanical systems described by Schrödinger's equation. Our examples are meant to capture the essential features of the electronic, rotational, and vibrational degrees of freedom we meet in atoms and molecules. We want to see whether the peak value of  $\ln N(E)/E$  is indeed bounded by  $2\pi R/\hbar c$  as predicted by (5) and (1). As expected, the inclusion in E of the rest energy of the particle, however small, is essential for the bound to be obeyed, so we choose the zero of the energy scale accordingly.

### A. Particle in one-dimensional potential well

Our first example concerns a particle of mass m in a one-dimensional potential well. This is relevant to electronic states (radial Schrödinger equation with electron mass m), or to vibrational states (m is then a nuclear mass). Let us assume that its motion is constrained to a range of size R on either side of an appropriately chosen point regardless of its energy E. A simple way to count the number of states N up to and including E is to use the WKB formula<sup>12</sup>

$$\int \left[2m(E-V)\right]^{1/2} dx = 2\pi (N+\frac{1}{2})\hbar, \qquad (6)$$

where V(x) is the potential and the integral ranges over x for which E > V. Evidently, only the whole part of N given by (6) is to be used. For the moment we ignore the inherent inaccuracy of the WKB formula for low-lying states. Evidently, the range of x is less than 2R. Further,  $E - V \le e$  where e is E measured with respect to the bottom of the potential well. Thus,

$$N < R(2m e)^{1/2} / \pi \hbar$$
 (7)

It is also clear that  $E = e + mc^2$ . With the notation  $e_* = e/mc^2$  and  $R_* = Rmc/\hbar$  we have

$$\ln N(E)/E < YR/2\hbar c , \qquad (8)$$

$$Y = \ln(2e_{\star}R_{\star}^{2}/\pi^{2})/R_{\star}(1+e_{\star}).$$
(9)

It is clear that within the Schrödinger theory we can only consider the case  $e_* < 1$  (nonrelativistic particle). Let us now maximize Y with respect to  $e_*$ . The maximum occurs at the  $e_*$  determined by

$$2R_*^2 e_*^2 = \pi^2 e_* \exp(1 + 1/e_*), \qquad (10)$$

and amounts to  $(e_*R_*)^{-1}$ . Because  $e_* < 1$  the right-hand side of (10) is never smaller than 72.93 and so Y < 0.1656. Therefore,

$$\ln N(E)/E < 0.0828 R/\hbar c$$
 (11)

for all e. Thus, by (5) S/E always obeys the proposed bound (1) regardless of the choice of m.

#### B. Rigid rotator

Consider now a two-dimensional system, a rigid rotator with moment of inertia I and mass m is confined within a sphere of radius R. This can serve to model the rotational levels of a molecule (m is molecular mass). The rotational energy levels are given by  $e=j(j+1)\hbar^2/2I$  with the levels labeled by j (j=0,1,2,...) being 2j+1 degenerate. The total energy is  $mc^2+e$ . Obviously, N(E) is just the sum of 2j+1 from j=0 to the largest j for which  $mc^2+e$  does not yet exceed E. Denoting this by  $j_*$  we find  $N(E)=(j_*+1)^2$ . Now, we are interested in the peak value of  $\ln N(E)/E$ ; this obviously occurs for an E which is just a rotational level [if E is increased slightly, the factor E depresses the ratio while N(E) does not grow unless the next level has been reached]. Thus, with the notation  $I_*=I/mR^2$  and  $R_*=Rmc/\hbar$  we may put

$$\ln N(E)/E = 2XR /\hbar c , \qquad (12)$$

$$X = \ln(j_* + 1) [R_* + j_*(j_* + 1)/2I_*R_*]^{-1}.$$
(13)

As a function of  $j_*$ , X peaks at the  $j_*$  determined by

$$(2j_{*}+1)(j_{*}+1)\ln(j_{*}+1) - j_{*}(j_{*}+1) = 2I_{*}R_{*}^{2} \quad (14)$$

and

$$\max X = 2I_* R_* (j_* + 1)^{-1} (2j_* + 1)^{-1} . \tag{15}$$

Of course, if (14) does not give integral  $j_*$ , then the peak in X cannot be quite reached, and (14) actually gives us an upper bound on  $I_*R_*$  for a specific  $j_*$ . But if  $I_*$  and  $R_*$ are so adjusted that the full peak can be reached and  $j_*=0, 1, 2, 3, 4, 5, 6,...$ , then  $I_*R_*=0, 1.08, 5.24,13.4,$ 26.2, 44.1, and 67.7, respectively, with the increasing trend continuing indefinitely. Because the radius of gyration cannot exceed R,  $I_* < 1$  so we get upper bounds on  $I_*R_*$  itself to substitute in (15). In this way we find that

$$\ln N(E)/E < 0.346R/\hbar c \tag{16}$$

for our rigid rotator. Thus, S/E obeys the bound (1) regardless of the parameters of the rotator.

#### C. Three-dimensional harmonic oscillator

Consider next a three-dimensional isotropic harmonic oscillator of rest mass m and frequency  $\omega$ . This can serve to model the rotational-vibrational levels of a complicated polyatomic molecule (m would then be a molecular mass). Its energy levels are

$$e = (n_1 + n_2 + n_3 + 3/2)\hbar\omega , \qquad (17)$$

where  $n_i = 0, 1, 2, ...$  Again, the total energy is  $mc^2 + e$ . N(E) is evidently the number of ways in which the  $n_i$  can be added in such a way that the total energy does not exceed E. Again, the peak  $\ln N(E)/E$  is reached when E exactly corresponds to some energy level. If F(n) denotes the number of ways in which three labeled non-negative integers can be added to give the integer n, then the peak value of  $\ln N(E)/E$  is given by

$$\ln F(n)R_{*}^{-1}[1+(n+\frac{3}{2})y]^{-1}R/\hbar c$$
(18)

for some integer n > 0. Apart from the usual notation  $R_*$  we have also used  $y = \hbar \omega / mc^2$ .

The effective radius of the oscillator is not predetermined: it can be no smaller than the amplitude of the oscillator. The quantum virial theorem<sup>12</sup> assures us that the expectation of the square of the vector amplitude is just  $e/m\omega^2$ . We can thus take  $R^2$  as a small multiple  $b^2$  of this state-dependent quantity. In view of (17) we have

$$R_* = b \left[ \left( n + \frac{3}{2} \right) / y \right]^{1/2} . \tag{19}$$

As a function of y, (18) peaks at  $y = (n + \frac{3}{2})^{-1}$ , i.e., where the oscillator's energy just equals the rest energy. Of course this point is already outside the nonrelativistic domain. However, it should be clear that the formal peak value obtained with this y does bound any  $\ln N(E)/E$ realizable by the nonrelativistic oscillator. Thus,

$$\ln N(E)/E < \ln F(n) [2(n+\frac{3}{2})]^{-1} R / \hbar c$$
(20)

since b < 1. Now,  $F(n) < (n+1)^3$  so that the ratio in the right-hand side of (20) eventually decreases with n. For  $n=0, 1, 2, 3, 4, 5, 6, \ldots$  we find  $F(n)=1, 3, 6, 10, 15, 21, 28, \ldots$  so that the peak is reached for n=2:

$$\ln N(E)/E < 0.256 R/\hbar c$$
 (21)

Again we find the bound on S/E is satisfied for all values of the parameters m and  $\omega$ .

The range of applicability of our example probably transcends the pure harmonic oscillator. Any spherically symmetric potential well resembles a harmonic potential near the bottom. Since the peak in  $\ln N(E)/E$  is reached at low excitation, it is likely that some anharmonicity of the potential does not change (21).

If there is a moral to our three examples, it is that, when rest energy is included in E, the number of states accessible to a quantum-mechanical system with energy limited to E is less than  $\exp(2\pi ER/\hbar c)$ . The inclusion of rest energy in E is crucial. Without it any bound such as (1) can be surpassed by adjusting parameters of the problem, i.e., making the moment of inertia of the rigid rotator large. However, the rest energy can be small without upsetting our result. Since our examples can be tailored to electronic, vibrational, and rotational levels, we can state with some confidence that the information that could be coded in an atom or molecule is indeed bounded by (1). For real atoms and molecules the maximum must fall considerably below (1). In fact, if we consider only electronic levels (case A) for which the electron mass is the relevant one, (11) limits the information to a few tens of bits.

## III. MANY-PARTICLE SYSTEMS: NUMERICAL EXPERIMENTS

There are at least two reasons to study many-particle systems in the context of the present paper. First, a collection of many identical particles is a paradigm of blackbody radiation, and if any system might be suspect of violating bound (1), blackbody radiation would be a prime suspect. After all it has rather large entropy content for given energy. The second reason concerns the computational process. In considering ways to optimize computers, a useful reference would be a computing machine, itself composed of elementary quanta, in which information is coded in the occupation numbers of the various modes, and in which the elementary operations consist of shifting quanta from one mode to another. It is difficult to believe that any foreseeable computer composed of macroscopic components could be more energetically efficient or faster at storing, retrieving, or processing information. Thus, it is interesting to assess the information capacity of an assembly of quanta, or, equivalently, the maximal entropy for given available energy.

Consider a collection of identical noninteracting particles confined to some volume. These may be described as quanta of some field. The stationary states are described in terms of the one-particle energy spectrum. Owing to the boundary conditions on the confining surface this will be a discrete spectrum  $e_1, e_2, \ldots$  with degeneracies  $g_1, g_2, \ldots$ . A many-particle state is specified by occupation numbers  $n_1, n_2, \ldots$  for the various one-particle energy levels. In view of the lack of interactions, the total many-particle energy is  $\sum n_i e_i$ .

When energy E is available, the number of accessible states N(E) is just the number of distinct ways in which the levels may be populated in harmony with the appropriate statistics, and with  $\sum n_i e_i \leq E$ . Evidently N(0)=1, since the vacuum state is a legitimate state (we choose our zero of energy at the vacuum energy in this section and in Sec. IV). The number of ways in which a set of occupation numbers may be realized is

$$D\{n_i\} = \prod_i (n_i + g_i - 1)! / n_i! (g_i - 1)!$$
(22)

for bosons, and

$$D\{n_i\} = \prod_i g_i! / (g_i - n_i)! n_i!$$
(23)

for fermions (with  $n_i < g_i$ ). These formulas take into account the identity of particles and, when appropriate, the Pauli principle. Precisely at  $E = \sum n_i e_i$ , N(E) undergoes an increment equal to  $D\{n_i\}$  and stays at its new value as E increases until E reaches the next higher many-particle energy level. Thus, N(E) is a sum of step functions.

The usual microcanonical entropy is just  $\ln D$ ; it is obviously very discontinuous, being a "comb" function with peaks at the many-particle levels. Following Gibbons we shall define entropy as  $S(E) = \ln N(E)$ , a somewhat better-behaved function of E. Clearly,  $S(E) > \ln D$  so that any statement about the maximum of S(E)/E carries over to  $\ln D/E$ .

Given a generic one-particle spectrum, there exists no general analytic expression for N(E). It is possible to make some statements about its behavior for large E, but when the quantity of interest is the peak value of S(E)/E, one is interested in N(E) for intermediate E as well. Thus, we have adopted an "empirical" approach in this section.

(i) Define a spectrum of interest.

(ii) Populate its levels according to some pattern which assures inclusion of all many-particle states up to some energy ceiling.

(iii) Bin the many-particle states by energy and thus

form an approximation to N(E) which improves with the fineness of the bins.

Owing to the labor involved, we implemented this program on a microcomputer. A variety of spectra for scalar, vector, and spinor fields in boxes of various shapes, and in spaces with various dimensions and topologies were tried. The strategy for populating the levels was the following. First, a single quantum was successively promoted through the levels in the sense of increasing  $e_i$  until this exceeded the ceiling. At each step the number of states was calculated with (22) or (23) and these numbers were binned by energy. Then a second quantum was added at the lowest level, while the first was returned to the lowest available level (if fermions and  $g_1 = 1$  this was  $e_2$ ). Then the first quantum was successively promoted level by level until the ceiling was exceeded at which point the second quantum was promoted one level up, and the first was returned to the lowest available state not below that of the second. Then the promotion of the first quantum began again, and the pattern was repeated. Numbers of states were calculated from (22) or (23) at each promotion and binned. When the two-quantum system already exceeded the ceiling at the start of a promotion series, a third quantum was added at the lowest level, the second was put in the lowest available level, and the first into the lowest available level above that. This pattern was repeated until already at the addition of a new quantum, the ceiling was exceeded. This populating pattern assures inclusion of all states up to the ceiling. At this stage a count of the number of states in bins up to a given E gives a good approximation to N(E). In this way N(E) was obtained for E up to the stated ceiling.

Not surprisingly, the computing time required to calculate N(E) grows very rapidly with the ceiling energy. In

most cases the ceiling was set at some seven times the energy of the first level. For some examples (notably those involving one-dimensional and two-dimensional spaces) we went up to 20 times the first level. The number of bins used was usually 200 which provided adequate accuracy.

Only massless fields were considered since for given available energy a nonvanishing rest mass reduces the number of accessible states. Thus, the N(E) we obtain is actually the upper envelope of N(E) for massive fields, and our conclusions about a bound on S(E)/E apply as well to massive fields (masses are included in E as in Sec. II). We considered the following fields.

(a) The electromagnetic (EM) field without sources obeying perfect conductor boundary conditions on walls of the box, or appropriate periodicity conditions for closed spaces of non-Euclidean topology.

(b) The conformal scalar (CS) field without sources obeying Dirichlet boundary conditions on walls, or periodicity conditions for closed spaces. The conformal scalar equation coincides with the usual scalar equation in flat spacetime. Otherwise<sup>13</sup>

$$\Box \phi - \frac{1}{4} [(q-2)/(q-1)] R_c \phi = 0 , \qquad (24)$$

where  $R_c$  is the scalar curvature and q+1 is the dimensionality of the spacetime. We have insisted on (24) for consistency; our other field equations are conformally invariant.

(c) The neutrino field (SP) obeying the two-spinor Weyl equation and periodicity conditions for a closed space, or a special boundary condition<sup>1</sup> or a spherical box.

We considered a variety of spaces. First we considered all three fields confined to a spherical box. The one-

TABLE I. Energy levels for conformal scalar (CS), electromagnetic (EM), and neutrino (SP) fields in various spaces.

Field	Space	Levels $e_i^{a,b}$	<i>g</i> i	Ranges
CS	Sphere	ħcj <sub>nl</sub> /R	2 <i>l</i> +1	$n=1,2,\ldots, l=0,1,\ldots$
SP	Sphere	ňcj <sub>nl</sub> /R	2(2l+1)	$n = 1, 2, \ldots l = 0, 1, \ldots$
EM	Sphere	$\hbar c(j_{nl} \text{ or } j'_{nl})/R$	2l+1	$n, l = 1, 2, \ldots$
CS	Line	n <b>h</b> cπ/d	1	$n=1,2,\ldots$
CS	Square	$\hbar c \pi (n^2 + m^2)^{1/2}/d$	1	n,m=1,2,
CS	<i>S</i> 1	nňc /2R	1	$n=1,2,\ldots$
CS	<i>S</i> 2	$(l+1/2)\hbar c/R$	2l + 1	<i>l</i> =0,1,
CS	<i>S</i> 3	nňc /R	$n^2$	$n = 0, 1, \ldots$
ЕМ	<i>S</i> 3	$(n+1)\hbar c/R$	2n(n+1)	$n=1,2,\ldots$
SP	<i>S</i> 3	$(n+1/2)\hbar c/R$	2n(n+1)	$n=1,2,\ldots$

<sup>a</sup>R denotes radius and d denotes linear size.

 $j_l$  and  $j'_l$  denote the *n*th root of the *l*th spherical Bessel function and its derivative, respectively.

particle spectra have been worked out in Ref. 1, for example. Likewise, we considered all three fields in an Einstein universe, a static three-sphere (S3). The CS and SP levels are given by Al'taie and Dowker,<sup>14</sup> and the EM levels by Mashhoon.<sup>15</sup> To investigate the effects of varying numbers of dimensions we compared a CS confined to a circle (S1) and to the surface of a sphere (S2) with that in S3 using periodic conditions throughout. The curvature term appears in the last two examples. For S2, where  $R_c = 2/R^2$  (R is sphere's radius) and q = 2, this term changes the usual l(l+1) eigenvalue to  $(l+\frac{1}{2})^2$ . Likewise, we compared a CS confined to a one-dimensional line and to a two-dimensional square with that in a cube (see below). The eigenvalues for all the above cases are summarized in Table I. Finally, we considered both CS and EM in three-dimensional rectangular boxes with a variety of aspect ratios to check for the effect of shape. For a box with sides a, b, and c, the energy levels are

$$e = (i^2/a^2 + i^2/b^2 + k^2/c^2)^{1/2} \pi \hbar .$$
<sup>(25)</sup>

For CS, i, j, k=1, 2, 3, ... and g=1. For EM, levels with one of i, j, or k vanishing are allowed and they have g=1. Levels with  $ijk \neq 0$  all have g=2. There are no levels with two vanishing quantum numbers. We did not consider SP in rectangular boxes because of the legendary difficulty in formulating appropriate boundary conditions.

The number of distinct levels required in the actual computations varied from 20 in the one-dimensional cases to several hundreds for the rectangular boxes.

In all cases we found that S/E as a function of E starts from zero at E = 0, rises rapidly, and with some oscillation to what is often an extended plateau, and then decreases slowly at larger E's. The three curves reproduced in Fig. 1 (50 bins were used to produce them) are, to a large extent, typical of the behavior for three-dimensional systems. On general grounds we know that the decrease of S/E must continue since when the number of quanta in the system is large (large  $E/e_1$ ), thermodynamics is applicable and predicts  $S \propto E^{3/4}$  (Stefan-Boltzmann formulas). In Fig. 2 we see two examples of the S/E curve for lower-dimensional systems. Here the oscillations can be stronger, and the decrease at large E steeper. This last fits in with the thermodynamic prediction that  $S \propto E^{2/3}$ 



FIG. 1. The entropy-to-entropy ratio as a function of energy (in units of the energy of the first one-particle level) for (A) a scalar field in a unit cube, (B) an electromagnetic field in a box of dimensions  $1 \times \frac{2}{3} \times \frac{1}{5}$ , and (C) a neutrino field in a unit sphere. Units are such that  $\hbar = c = 1$ .



FIG. 2. The entropy-to-entropy ratio as a function of energy (in units of the first one-particle level) for (A) a conformal scalar field confined to the surface of a unit sphere, and (B) a scalar field confined to a unit one-dimensional space (Dirichlet boundary condition). We take  $\hbar = c = 1$ .

for two-dimensional systems, and  $S \propto E^{1/2}$  for onedimensional systems.

Since we are mainly interested in  $\max(S/E)$ , we do not reproduce the curves for all systems, but merely list  $\max(S/E)$  for each in column 3 of Table II. These values were obtained by scanning the results from numerical experiments using 200 bins. It may be seen by comparing with the last column in Table II that bound (1) is satisfied without exception.

## IV. MANY-PARTICLE SYSTEMS: ANALYTIC APPROACH

As mentioned earlier, there is no general analytic expression for N(E) or even for  $\max(S/E)$  given a generic system. But it is also evident that detailed computations such as those in Sec. III are not the perfect answer when one wishes to understand large classes of systems. We need a more theoretic approach for that. Here we develop an analytic approximation for  $\max(S/E)$  which allows us to do just that.

Recall the definition of the partition function  $Z(\beta)$  for a system: the sum of  $\exp(-\beta E)$  over all many-particle energy eigenvalues E. Since N(E) increases by unity at each eigenvalue, we may replace

$$\sum_{E} \to \int_{0_{-}}^{\infty} dN(E) , \qquad (26)$$

and performing an integration by parts, we have

$$Z = \beta \int_{0_{-}}^{\infty} \exp(-\beta E) N(E) dE . \qquad (27)$$

The boundary terms from the integration by parts are absent because  $N(0_{-})=0$ , and because it is assumed implicitly that N(E) does not grow as fast as exponentially for large E (thus, black holes and hadron gas—see Sec. V—are excluded from this treatment).

A significant way to rewrite (26) is

$$Z = \beta \int \exp\{-E[\beta - \ln N(E)/E]\} dE .$$
 (28)

The useful thing here is the explicit appearance of S(E)/E as we have defined it. Now in Sec. III we found that  $\ln N(E)/E$  quickly rises from zero to a broad plateau from which it drops only slowly at rather large E. Since at large E the integrand is already small, this means that

TABLE II. Numerically computed  $\max(S/E)$ , analytic approximation to it, and the universal bound  $2\pi R$  ( $\hbar$ , c, and radius or longest linear dimension taken as unity).

Field	Space	$\max(S/E)^{a}$	$\zeta(4)^{1/4}$ b	$2\pi R$
CS	Sphere	0.446	0.452	6.283
EM	Sphere	0.711	0.718	6.283
SP	Sphere	0.526	0.537	6.283
CS	Line	0.216	0.325	6.283
CS	Square	0.220	0.256	4.443
CS	<i>S</i> 1	0.162	0.193	6.283
CS	S 2	1.39	2.025	6.283
CS	<b>S</b> 3	1.02	1.124	6.283
EM	<i>S</i> 3	0.977	0.944	6.283
SP	S3	1.130	1.140	6.283
CS	$1 \times 1 \times 1$	0.264	0.269	5.441
EM	$1 \times 1 \times 1$	0.390	0.384	5.441
CS	1×0.95×0.9	0.250	0.248	5.174
EM	1×0.95×0.9	0.365	0.366	5.174
CS	1×0.5×0.5	0.164	0.162	3.848
EM	1×0.5×0.5	0.243	0.239	3.848
CS	1×1×0.25	0.154	0.147	4.512
EM	1×1×0.25	0.245	0.270	4.512
CS	1×0.66×0.2	0.122	0.115	3.828
EM	1×0.66×0.2	0.197	0.213	3.828
CS	1×0.25×0.25	0.0998	0.0968	3.332
EM	$1 \times 0.25 \times 0.25$	0.149	0.144	3.332
CS	$1 \times 1 \times 0.1$	0.0984	0.0931	4,454
EM	$1 \times 1 \times 0.1$	0.218	0.257	4.454
CS	1×0.1×0.1	0.0507	0.0480	3.173
EM	1×0.1×0.1	0.0768	0.0723	3.173

<sup>a</sup>From the numerical algorithm.

<sup>b</sup>From Eq. (37).

one can approximate well the integral by using  $\max(S/E)$ in lieu of  $\ln N(E)/E$ . Then carrying out the integral explicitly and solving for  $\max(S/E)$  gives

$$\max(S/E) \ge \beta \{1 - \exp[-\ln Z(\beta)]\}.$$
<sup>(29)</sup>

Here the  $\geq$  reminds us that rigorously there is an inequality, but that the equality is close by. And obviously the equality is closest when the right-hand side is maximized over  $\beta$ .

All this is general. For a system of many noninteracting particles, there exists a well-known alternative expression for  $\ln Z$ ,<sup>16</sup>

$$\ln Z = \sum_{i} \mp g_{i} \ln[1 \mp \exp(-\beta e_{i})] , \qquad (30)$$

where the sum is over one-particle energy levels, and upper (lower) signs correspond to bosons (fermions). If one has the one-particle spectrum on hand,  $\ln Z$  is easily calculated to good accuracy from (30) (only a few seconds of microcomputer time needed). By trial and error one can find the  $\beta$  for which the right-hand side of (29) is largest. Then a good approximation to  $\max(S/E)$  is obtained by taking the equality in (29). Thus, with rather limited computation one can circumvent the onerous algorithm of Sec. III which can take anywhere from 20 min to several hours of microcomputer time to find  $\max(S/E)$ .

The above strategy is still one based on concrete com-

putations, and cannot be used for more than one system at a time. To get a more flexible approach, let n(e) be the number of one-particle energy levels not exceeding e. Obviously n(0)=0 (we exclude systems with zero modes—for the rationale see Sec. VII). One should distinguish n(e) from N(E). They are the same only for oneparticle systems like those discussed in Sec. II. For a many-particle system N(E) is a complicated functional of n(e), and therein lies the subtlety of our subject. We may now replace the sum in (30) by an integral over dn(e). An integration by parts allows us to rewrite (30) as

$$\ln Z = \int n(e) [\exp(\beta e) \mp 1]^{-1} de .$$
 (31)

At this point we introduce the " $\zeta$  function" for the one-particle spectrum

$$\zeta(p) = \sum_{i} g_i e_i^{-p} \,. \tag{32}$$

For a system in q space dimensions, the  $\zeta$  function converges for  $p \ge q$ . It is evident that<sup>11</sup>

 $n(e) < \zeta(p)e^{+p} \tag{33}$ 

because in  $\zeta(p)$  there are *n* terms, each of the form  $e_i^{-p}$  with  $e_i < e_i$ , as well as other positive terms. With the help of (33) we can integrate (31) to get

$$\ln \mathbf{Z}(\beta) < \zeta(p) \beta^{-p} \zeta_R(p+1) \Gamma(p+1) , \qquad (34)$$

where  $\zeta_R$  is the Riemann  $\zeta$  function and  $\Gamma$  is the factorial function.

Now the idea is to approximate  $\ln Z$  by a fixed fraction k of the power of  $\beta$  in the right-hand side of (34). Recall that we only need  $\ln Z$  at the  $\beta$  for which the right-hand side of (29) is maximized, so the approximation is not necessarily as crude as it would seem. Further, we are free to choose p to optimize the approximation. In this connection we recall that for a system in q space dimensions, n(e) rises as  $e^{q}$  for large e (volume of phase space). Thus, if we wish to employ a single inequality like (33), we must choose  $p \ge 3$  for it to be valid for onedimensional, two-dimensional, and three-dimensional systems. The choice p = 3 is not very convenient because for three-dimensional systems,  $\zeta(3)$  converges only marginally:  $\xi(p)$  diverges for p < 3. This means that to get a good approximation to  $\zeta(3)$ , one must perform the sum in (32) over many levels. One may not know enough about the system to be able to do this. As p is raised, fewer and fewer levels are needed to get  $\zeta(p)$  to reasonable accuracy. However, as p is raised the power law in (33) looks less and less like n(e) at high e. Thus, as a compromise value we choose p = 4.

Replacing now  $\ln Z(\beta)$  in (29) by k times the right-hand side of (34) and maximizing the right-hand side of (29) over  $\beta$ , we find

 $\max(S/E) \simeq [24k\zeta_R(5)\zeta(4)/x]^{1/4} [1 - \exp(-x)], \quad (35)$ 

where the (positive) x is determined by

$$\exp(x) - 1 - 4x = 0 . \tag{36}$$

This last gives x=2.336. At this point we choose k=0.141 with a view to doing away with the numerical

coefficient in (35). This cavalier approach, which does yield a very simple expression, is mainly justified by the results. We get the estimate

$$\max(S/E) \cong \zeta(4)^{1/4} . \tag{37}$$

In Table II we have tabulated this estimate of  $\max(S/E)$  for the various systems considered in Sec. III (the  $\zeta$  function was calculated using all levels available in the numerical calculation of S/E). It may be seen that the agreement with the "experimental" values is very good, typically better than 10%. The larger discrepancies, of order 50%, occur only for one-dimensional and two-dimensional systems. Thus, for three-dimensional systems the rule (37) seems to be quite adequate when a not very accurate estimate of  $\max(S/E)$  is desired.

When the one-particle spectrum is poorly known, so that it is not possible to explicitly compute  $\zeta(4)$ , a crude estimate of it may be obtained by simply replacing it by its first term,  $g_1e_1^{-4}$  (the first level can usually be determined easily by variational methods). The approximation is not as bad as it sounds because  $\zeta(4)$  appears only to the power  $\frac{1}{4}$  in (37). The new approximation should be best for systems whose first level is well isolated from those above it. For such a system the first level does indeed make a major contribution to  $\zeta(4)$ . Numerical checks for the systems in Table II show that the simplified estimate of  $\max(S/E)$  usually underestimates the actual value, but almost never by more than a factor of 2. Somewhat larger discrepancies occur precisely for boxes of large aspect ratio for which there are many energy levels just above the first. Thus, for systems with moderate aspect ratios the simplified estimate provides a "rough and ready" idea of the largest possible S/E.

Our simplified estimate for max(S/E) also provides a way to deal with large classes of systems. We have shown earlier<sup>1</sup> that for a scalar field in a three-dimensional box of arbitrary shape (Dirichlet boundary conditions),  $e_1 > \pi \hbar c / R$ , where R is the circumscribing radius of the box. Similarly, for the electromagnetic field (perfect conductor boundary),  $e_1 > 2.082\hbar c/R$ . Insofar as our proposed boundary condition for the neutrino field<sup>1</sup> makes sense,  $e_1 > \pi \hbar c / R$  for it. All the examples in Table II (boxes with some degree of symmetry) ascribe  $g_1 = 1$  to the scalar and electromagnetic fields. For boxes with lower symmetry, this rule should continue to hold. Our single example for the neutrino field in a box has  $g_1 = 2$ and this may continue to be true for lower symmetry. Combining these limitations with our simplified estimate we find

$$\max(S/E) < 0.784 R/\hbar c$$
 (38)

According to our findings, the actual values of  $\max(S/E)$  are unlikely to exceed bound (38) by more than a factor of 2. Indeed, every exact result for  $\max(S/E)$  in Table II concurs with (38). It thus seems safe to conclude that systems of scalar, electromagnetic, or neutrino quanta in boxes of arbitrary shapes (but with not too large aspect ratios) will respect the more generous blanket bound (1). Extending this conclusion to systems with high aspect ratios is an important problem for the future.

## V. STRONGLY INTERACTING PARTICLES: HADRON GAS

Sections III and IV have left two important questions unanswered. Is the bound on S/E still valid in the presence of interactions? Is it still valid when many species of particles are available? It is reasonable to expect that weak interactions between particles should not affect the conclusions reached earlier, but that outcome would seem uncertain in the face of strong interactions? In order to check this point we turn to the example of a hadron gas, where (almost by definition) the interactions are strong. Now hadrons come in a myriad species, so the example is also well tailored to elucidation of the influence of multiplicity of particle species. It has often been argued<sup>6,7</sup> that multiplicity of particle species will lead to a violation of the bound. However, it has been shown<sup>8</sup> that such violation, if it occurs at all, demands very large multiplicity (maybe  $10^9$  species). Much of the polemics are academic: they refer to massless noninteracting particles of which there are only a few species in nature. Hadrons with their rich mass spectrum invite a new look at this issue.

Consider, then, a box sufficiently ample to contain hadrons. Energy E is allocated to it and may be materialized in any permissible way into hadrons. What is the maximum S/E of the box contents? For simplicity we here ignore the contribution to the number of quantum states from motion of the hadrons. After all, we have shown in Secs. III and IV that even for massless particles the number of states associated with motion in the box falls well short of what is required to violate the bound. For our (very) massive hadrons this statement should be all the more correct. But is not the rationale just stated good for noninteracting particles, but of doubtful relevance to the hadron case? To answer this quibble we shall argue, in the spirit of the bootstrap idea,<sup>17</sup> that the influence of hadronic interactions is approximately accounted for by considering simultaneously the full complement of hadronic species. In this way we sidestep the thorny issue of interactions, while at the same time focusing on the multiplicity of species, which interests us in its own right. Thus, by number of quantum states (and entropy) we mean here the contributions from the variety of species (the entropy first dealt with by Gibbs in connection with the famous paradox bearing his name).

In principle the determination of the run of S/E with energy could be accomplished by the method of Sec. III. One would list the hadronic mass levels and their degeneracies, and would populate them with an algorithm differing from that previously described only in that it would treat some levels as bosonic with formula (22), and some as fermionic with formula (23), (mesons vs baryons). In reality the hadron mass spectrum is so populous that it would be almost hopeless, and certainly pointless, to enter into such detail. After all, one only wants S(E) which is not very sensitive to the fine details of the many-particle spectrum, let alone the one-particle spectrum. Thus, we simplify our treatment in two senses. First, we treat all mass levels as bosonic. Clearly, this can only have the effect of overestimating S, and that is permissible when one desires an upper bound on S/E. Second, we make use of a smoothed density of hadron mass levels instead of using a precise list of resonances.

In practice we employed the semiempirical Hagedorn density of levels<sup>17</sup>

$$\mu(e)d \ e = 26\ 300(2.5 \times 10^4 + e^2)^{-5/4} \exp(e/160)d \ e \ . \tag{39}$$

Here all energies e are meant to be expressed in MeV. The formula includes spin and isospin multiplicity as well as hadron-antihadron duplicity. This density is known to bound the empirical hadron spectrum, and has also been supported by theoretical arguments.<sup>17,18</sup>

By means of (39) we constructed an artificial discrete spectrum in the range 0 to 1500 MeV by partitioning the energy into intervals of widths varying from 100 MeV at low energies to 10 MeV at high energies. The center of each interval was regarded as a discrete energy level, and assigned a degeneracy factor equal to the number of hadron states which reside in the interval according to (39). Then the many-hadron states with energies up to E were counted with the help of the algorithm of Sec. III (bosonic version), and the run of S/E with energy was determined, with S(E) being the logarithm of the number of hadron combinations with energy not greater than E. We stress again that kinetic energies were not included in E.

The curve S/E vs E rises with some oscillations up to a peak at 0.007 MeV<sup>-1</sup>, and for E > 700 MeV becomes very flat at a value 0.0056 MeV<sup>-1</sup>. There is an almost imperceptible decrease after that (the contrast with the curves in Figs. 1 and 2 in this respect results from the exponential rise of the Hagedorn spectrum). Is the peak S/E found here consistent with bound (1)? The box containing the hadrons should be large enough to fit in the typical hadron, which is definitely an extended particle. Thus the box (assumed spherical for simplicity) should be no smaller than 1 fm in radius. To take it smaller would mean hadrons could not remain intact, and would have to dissolve into quarks changing entirely the terms of our problem. For  $R > 1 \times 10^{-13}$  cm,

$$2\pi R/\hbar c > 0.0318 \text{ MeV}^{-1}$$
, (40)

so that the bound (1) is obeyed with more than a factor of 4 to spare. We conclude that strong interactions and high multiplicity of species do not, at least for the obvious example in nature, cause a violation of the conjectured bound on S/E.

One could also consider an enclosure too small to contain hadrons. The energy would then be materialized as quarks, leptons, and various gauge particles. According to contemporary physics there are only some  $10^2$  relevant species (if the number of generations is 3–4), most of them massive. Further, for a sufficiently small enclosure the quarks can be regarded as weakly interacting (asymptotic freedom). The S/E of such a system can be bounded by the method of Sec. III.

First, we regard the particles as massless—this can only overestimate S/E. Next, we replace each spinor particle by two scalar ones, and each vector one by three. For nearly free massive particles this should be a reasonable approximation if all that is desired is an entropy. Also by supplanting spinors by scalars and so sidestepping Pauli's principle, one is again exaggerating S/E. As the number of effective scalars we adopt  $10^3$  which obviously leads to a further overestimate of S/E. This quantity is computed with the algorithm of Sec. III by the simple device of taking each one-particle level as  $10^3$ -fold degenerate. The conclusion is that for the realistic system

 $S/E < 2.13R/\hbar c$  . (41)

Thus, the system respects bound (1).

#### **VI. INFORMATION FLOW**

In this section we consider the validity of the bound (2) on the flow of entropy or information. As mentioned in Sec. I, this bound is merely a reinterpretation of that proposed in Ref. 3. We shall first be considering a onechannel communication line, essentially a material "pipe" which is supposed to guide signals coded in some field to their destination. We want to compute the maximum flow  $\hat{S}$  for allotted energy E of the signal.

In order not to complicate the analysis, assume the pipe is straight and of constant cross section, at least over the length scale of interest. We need not assume the cross section to be circular; in fact, it could be multiply connected, i.e., coaxial cable, or two parallel wires. The pipe may be hollow and conducting, or solid dielectric. Whether a scalar or electromagnetic field is to be guided, each field component will satisfy a wave equation

$$\nabla^2 f - v^{-2} \partial^2 f / \partial t^2 = 0 , \qquad (42)$$

where v denotes either c for the scalar or electromagnetic field in a hollow pipe, or c/(index of refraction) for the electromagnetic field in a dielectric pipe. For simplicity we ignore dispersion in the dielectric (v assumed independent of frequency). The symmetry along the pipe (coordinate z) allows us to concentrate on solutions of the form

$$f = g(x, y)e^{ikz - i\omega t}, \qquad (43)$$

where x and y are coordinates in the cross section of the pipe. Denoting by  $\Delta$  the two-dimensional Laplacian in x and y, we find that

$$\Delta g = (k^2 - \omega^2 / v^2) g = -bg .$$
 (44)

In (44) we are confronted with an eigenvalue problem for the operator  $\Delta$ . The eigenvalues *b* determine the dispersion relation for waves traveling along the pipe. We may assume  $b \ge 0$ . To have negative *b* would mean that the group velocity

$$v_{g} = vk(b+k^{2})^{-1/2} \tag{45}$$

can exceed c for long wavelengths. If b > 0, the frequencies of traveling (hence, information-bearing) waves have a threshold at  $vb^{1/2}$ . On dimensional grounds we expect the lowest  $b^{1/2}$ , if nonzero, to be of order of the reciprocal transversal size of the pipe. Thus, for a thin pipe the smallest energy invested in the quanta could be quite large, and for given E we would get only a few possible states; this would suppress  $\dot{S}$ . We thus see that the highest  $\dot{S}$  are expected for a pipe which admits the eigenvalue b = 0. For such there is no threshold. An example of such a pipe is the coaxial cable.<sup>19</sup> Henceforth, we deal

only with such pipes which, for given allotted E, can carry information the fastest.

The one-quantum states out of which the signals will be built are thus specified entirely by k. In the usual manner we discretize k by invoking periodic boundary conditions for the field (large period L). The allowed k are integer multiples of  $2\pi/L$ , and there are two such modes for each k (sine and cosine). Thus, we have  $e_i = 2\pi\hbar cL^{-1}i$  and  $g_i = 2$  for i = 1, 2, 3, ... With the algorithm of Sec. III we have established that

$$\max(S/E) = 0.162L/\hbar c$$
, (46)

where, as always, S stands for the natural logarithm of the number of many-quantum states which can be populated with allotted energy E (Bose-Einstein statistics were assumed).

Because the longest wavelengths involved are L, we must regard L as the size of our signal. The signal thus takes at least a time L/v in sweeping by a fixed point since by (45) the signaling speed never exceeds v. We may thus infer  $\dot{S}$  by dividing by L/v. Further dividing by ln2 to convert from entropy units to bits we have

$$I < 0.234 E / \hbar$$
, (47)

where I, information flow, is expressed in bits/s. Bound (47) is still a factor 15 below our proposed universal bound.<sup>3</sup> The latter, Eq. (2), as well as Bremermann's rule,<sup>2</sup> thus seem to be obeyed in the realm of one-channel communication systems, at least those for which our straight pipes with constant cross section are a good model.

What happens when several channels are available? It would seem that an opportunity would open to surpass the bound since by allocating the allotted energy in different ways among the channels, one can increase the number of available states. Actually the number of channels required to do this is large. A calculation along the lines of those in Sec. III shows that as many as  $2 \times 10^8$ may be required (in essence one considers a pipe whose every mode is N degenerate, with N the number of communication channels). But this does not mean that it is possible to design a *complete* communication system which violates bound (2).

Having many channels available is only half the solution. One still has to solve the problem of integrating the signals from different channels into a coherent whole. To complete the communication process all signals must reach a central unit in the "receiver" where they can be decoded in concert. Conceptually this calls for an accessory channel that connects sequentially into this or that primary channel, and conveys its information to the decoding device. It can be seen that the accessory channel will be the bottleneck of the system, for it too must be subject to bound (2). If it has available the same E which was available to the primary channels, the full system cannot communicate information faster than a onechannel system.

It could be argued that by boosting the energy available to the signal at the final stage, one can fully match the higher communication capacity of the primary channels. This is true but does not alter the fact that at some stage in its trip, the signal ties up as large an E as required by the bound. In fact, when originally described,<sup>3</sup> the bound was stipulated to apply with E being the energy of the *received* signal

One could try to dispense with the bottleneck channel by connecting the primary channels directly to the central unit and scheduling the bursts of information in the different channels so that they would arrive one after the other. But then the system would no longer be a manychannel system; only one channel would be active at any time, and the energy available to it, E, would limit its  $\dot{I}$  by (2). So even if E can be reused by the next channel, (2) still limits  $\dot{I}$  for the ensemble of channels.

The many-channel system can convey information faster than allowed by (2) if the number of channels is  $> 2 \times 10^8$ , and if the information, upon arriving, is left in unintegrated format. For example, a large bundle of our pipes could presumably convey the picture of an object, and form its likeness upon a screen at a rate faster than (2) if each pixel of the image is produced by one or a few channels. However, if what is desired is the further step of bringing the image into a "brain" to evoke recognition—"bird" or "flower"— then one will run into the bottleneck channel, and bound (2) should limit the rate of flow of information for the full process.

We conclude that the full communication process to its end as integrated information at a central unit cannot transcend the stated bound (2), even if many channels are involved.

### VII. CONCLUSIONS AND DISCUSSION

One conclusion supported by several examples (Sec. II) is that a nonrelativistic quantum-mechanical system with limited size R has a limited number of energy levels (including degeneracy) not exceeding the exponential of a number of order of the ratio of R to the Compton length of the system. This conclusion is in harmony with bound (1) when E is interpreted there as rest energy.

For systems of identical particles (quantum fields) confined to spaces of finite effective radius R, one may conclude, on the basis of many numerical examples (Sec. III) and a crude analytic argument (Sec. IV), that the microcanonical entropy is indeed limited by bound (1), with Ebeing the total energy available to the particles (including rest masses, if any). Vacuum energy plays no role in this formulation; thus the result is not subject to the problem pointed out by Page,<sup>6</sup> Unwin,<sup>9</sup> and Deutsch<sup>10</sup> for the earlier formulation of the S/E bound.<sup>1</sup> There are some systems which approach bound (1) to within a factor of 5 (Table II).

One seeming loophole in the results of Secs. III and VI hinges on their being based on the assumption that the cavities enclosing the fields are empty. In a cavity filled with dielectric all formulas relating to S/E of photons would be changed by the replacement  $c \rightarrow c/(\text{index of re-fraction})$ . If the index is large, the peak S/E is increased proportionately. For example, from Table II we might infer that S/E of photons in a sphere filled with dielectric of index 10 slightly exceeds bound (1). However, this

would be a hasty conclusion. The electromagnetic modes in a dielectric are a property, not only of the electromagnetic field, but also of the collective oscillations of the dielectric (the photons are "dressed"). Hence, in the spirit of Secs. II and III we must include the dielectric rest energy in E. Now, the particles comprising the dielectric are confined within effective radius R, so their Compton lengths must be smaller. Hence, each contributes rest energy at least as large as  $\hbar c/R$ . When these energies are included in E, max(S/E) is much reduced below the naive estimate, since in all cases considered S/E peaks at energies a few times  $\hbar c/R$ . It is of importance that unless there are many dielectric particles (and S/E is strongly suppressed by the large E), the (in dielectric) one-photon spectrum loses all but its lower levels because the medium is not smooth as seen by short-wavelength excitations. This has the effect of suppressing S. We believe these effects succeed in keeping S/E (as interpreted) below bound (1), but have not carried out extensive checks.

Another obvious loophole in Secs. III and IV is that the possibility of zero (eigenfrequency) modes is not considered. It would seem that when such a mode is present, a violation of bound (1) is possible because there are several zero-energy many-particle states, so S/E can diverge. Actually the problem goes well beyond the province of the bound on S/E: when a zero mode exists, microcanonical entropy is infinite for any energy. This is so because for any many-particle state one can construct an infinity of states with like energy by just adding quanta to the zero mode one at a time. Thus, N(E) is always infinite.

But our intuition revolts against such a conclusion. It must be that the entropy so defined is not physically relevant, that it must be "renormalized." And the obvious way to do this is to exclude from N(E) all states which have some quanta in the zero mode. This purely pragmatic approach may be justified theoretically in light of the observation that zero modes (for example that of the conformal scalar field in S3-see Table I) are generally spatially homogeneous. We may then think of the quanta in the zero mode as a "condensate" akin to the classical part of a quantum field that arises in the theory of symmetry breaking.<sup>20</sup> When such a field is second-quantized, creation and annihilation operators are associated only with the nonzero modes. The homogeneous static part (the vacuum value) is left classical. Adopting such a philosophy here means that N(E) will ignore quanta in the zero mode. This justifies the above *ad hoc* procedure; the loophole is thus closed. In effect we treated the conformal scalar field in S3 just in this way in Sec. III.

In principle very large numbers of particle species  $(>10^9)$  can help to violate bound (1).<sup>8</sup> However, in nature large numbers of species occur only among the strongly interacting hadrons. The evidence marshalled in Sec. V supports the conclusion that interactions compensate for the presence of many species, and bound (1) is still respected.

The close relation between entropy and information allows us to draw the further conclusion (Sec. VI) that a one-channel communication system has a limited information-transfer capacity in terms of the energy available to the signal. The limit deduced supports our proposed bound on  $\dot{I}$ —inequality (2) as well as the very similar Bremermann's rule.<sup>2</sup> It also seems likely, as mentioned in Sec. VII, that a many-channel system is limited by the same bound.

#### ACKNOWLEDGMENTS

Gary Gibbons was the first to promote the "microcanonical" approach to testing the S/E bound. His early attempt encouraged me to try extensive numerical experiments. Rolf Landauer's criticism inspired the approach of Sec. VI. This research was initiated in the congenial atmosphere of the Aspen Center of Physics.

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