Static spin-polarized cylinder in the Einstein-Cartan theory of gravitation

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An exact solution of the Einstein-Cartan field equations is obtained, which represents the gravitational field of a static perfect fluid cylinder with a net spin polarization along its axis of symmetry. It is found that the gravitational field is stationary, with the cylinder's spin giving rise to inertial-frame dragging.

I. INTRODUCTION

There has been a considerable growth of interest in the Einstein-Cartan (-Sciama-Kibble) (EC) theory of gravitation[1–5] in recent years, as a result of a most attractive set of features that characterize this theory. Among these the ones mostly cited in the relevant literature seem to be the following.

(i) The EC theory is a natural, as well as moderate, extension of general relativity (GR). This is reflected in the fact that on the one hand it retains GR's model of spacetime as a differentiable manifold with a metric connection, the metric field being reducible at each point of the manifold to a Minkowski metric, while on the other hand its field equations are identical with Einstein's, as long as the gravitational sources carry no intrinsic or spin angular momentum. It differs from GR in not demanding that the connection, when expressed in a coordinate basis, be symmetric, letting, instead, this asymmetry be determined by the sources' spin density.

(ii) The EC theory is an improvement of GR because it provides a geometric framework for the description of spin angular momentum, a physical attribute of elementary particles as fundamental as their mass.

(iii) It is a gauge theory of the Lorentz group (or of the Poincaré group, or of the group of affine transformations, depending on the author), and gauge field theories have been very successful in recent years. This aspect of the EC theory derives from the very method its field equations were obtained by Sciama[2] and Kibble,[3] and has been extensively analyzed and emphasized by Hehl,[5] Trautman,[4] and their collaborators.

A new theory, however, or an extension of a successful one, starts gaining real strength when, besides being aesthetically satisfying, it gives birth to new predictions, insights, and unified descriptions of physically related phenomena. It is a problem deriving from this category of expectations that we consider in this paper.

Specifically, since the EC theory allows for the intrinsic angular momentum of a mass distribution to effect the gravitational field produced by the latter, it is natural to expect that spin can replace orbital angular momentum in giving rise to characteristic phenomena associated with this physical quantity as predicted by GR. It is well known, for example, that the orbital angular momentum of a source gives rise to the "dragging of inertial frames",[6] by producing magneticlike components in the gravitational field around the source. This effect was first exhibited in an approximate solution of Einstein's field equations obtained by Lense and Thirring,[7] which represents the gravitational field of a rigidly rotating sphere. It was also found to be present in van Stockum's[8] later exact solution describing the field of a rotating infinite cylinder of dust.

An encouraging result in this direction was first obtained by Arkuzewski *et al.*,[9] who showed that, in the linear approximation, the EC theory reproduces the Lense-Thirring effect in the case of a static sphere with nonvanishing spin angular momentum. About the same time, however, Prasanna's[10] exact solution for a static spin-polarized cylinder in the EC theory was published, in which this characteristic effect was lacking.

It is the issue of the static spin-polarized cylinder in the context of the EC theory that we take up again in this paper. In the following sections it will be shown that the use of matching and junction conditions appropriate to the EC theory, combined with a proper choice of stress-energy tensor, leads to the exact solution obtained by the author recently,[11] in which a static but spin-polarized cylinder produces essentially the same gravitational field as van Stockum's rotating cylinder of dust. This restores the equivalence of spin to orbital angular momentum as far as effects external to the gravitating source are concerned.

The structure of the paper is as follows. In Sec. II, an outline of the EC theory is given, in order to set the notation and introduce the geometric and physical quantities appearing in the rest of the paper. Section III contains a description of the specific forms of the sources assumed in the derivation of the interior solution presented in Sec. IV. Section V is devoted to the solution describing the exterior region and its matching to the solution given in the previous section. Lastly, in Sec. VI, the distinctive features of the exact solution obtained are discussed and a (thought) experiment is proposed for testing its validity.

II. THE EC THEORY OF SPACETIME

The geometric model of spacetime employed by the EC theory consists of a four-dimensional differentiable manifold with a linear connection compatible with a metric of signature $(-,+,+,+)$. In terms of a set of vector fields $\{e_\alpha\}$, $\alpha=0,1,2,3$, which span the tangent space at each point of the spacetime manifold, the components $\Gamma^\alpha{}_{\beta\gamma}$ of the connection are determined by the covariant derivative formula (see Ref. 12 for details on notation)

$$\nabla_\alpha e_\beta = \Gamma^\gamma{}_{\beta\alpha} e_\gamma \ , \tag{2.1}$$

while the compatibility of the connection with the metric g is expressed by the condition

$$g_{\alpha\beta,\gamma} = e_\gamma(g_{\alpha\beta}) = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} \ , \tag{2.2}$$

where

$$g_{\alpha\beta} = g(e_\alpha, e_\beta) \ , \quad \Gamma_{\alpha\beta\gamma} = g_{\alpha\delta}\Gamma^\delta{}_{\beta\gamma} \ . \tag{2.3}$$

The torsion of such a metric compatible connection is defined as the tensor field with components

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma} - C_{\beta\gamma}{}^\alpha \ , \tag{2.4}$$

where the quantities $C_{\beta\gamma}{}^\alpha$ are determined by the commutators of the basis vector fields, when written in the form

$$[e_\alpha, e_\beta] = C_{\alpha\beta}{}^\gamma e_\gamma \ . \tag{2.5}$$

What distinguishes the EC theory from classical GR is that in the former theory the torsion of spacetime is not allowed to vanish identically. Instead, it is coupled to the intrinsic or spin angular momentum density of matter. The latter quantity is represented by a tensor field with components $S^\alpha{}_{\beta\gamma}$, and the spin-torsion relation is expressed by the equation

$$T^\alpha{}_{\beta\gamma} + 2\delta^\alpha_{[\beta} T_{\gamma]} = \kappa S^\alpha{}_{\beta\gamma} \ , \tag{2.6}$$

where $T_\alpha \equiv T^\gamma{}_{\alpha\gamma}$, κ is the GR gravitational constant, and square brackets denote antisymmetrization. Since Eq. (2.6) is an algebraic one, the name of field equations is reserved for the set

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa t_{\alpha\beta} \ , \tag{2.7}$$

where $R = R^\alpha{}_\alpha$, $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$, $R^\alpha{}_{\beta\gamma\delta}$ being the curvature tensor, and $t_{\alpha\beta}$ is the canonical or dynamical stress-energy tensor, which, in general, is asymmetric. The antisymmetric part of Eq. (2.7), however, is a disguised form of the conservation of angular momentum equation

$$2R_{[\alpha\beta]} = 2\kappa t_{[\alpha\beta]} = (\nabla_\gamma + T_\gamma) S^\gamma{}_{\beta\alpha} \ . \tag{2.8}$$

It can, thus, be subtracted, leaving the symmetric part

$$R_{(\alpha\beta)} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa t_{(\alpha\beta)} \tag{2.9}$$

as the field equations proper.

III. THE FORM OF THE SOURCE TENSOR FIELDS

In the solution to be presented in the following sections, the source of the gravitational field will be considered to consist of a perfect fluid of the Weyssenhoff type, for which the spin-density tensor takes the form

$$S^\alpha{}_{\beta\gamma} = u^\alpha S_{\beta\gamma} \ , \quad u^\alpha S_{\alpha\beta} = 0 \ , \tag{3.1}$$

where $S_{\alpha\beta} = S_{[\alpha\beta]}$, and u^α is the fluid particles' velocity four-vector, with $u^\alpha u_\alpha = -1$.

As for the form of the canonical stress-energy tensor, use will be made of recent results by Ray and Smalley,[13] who have constructed a stress-energy tensor appropriate to a perfect fluid in the EC theory, on the basis of a Lagrangian variational principle. The Ray-Smalley tensor takes into account the contribution of spin to the specific internal energy, ϵ, of the fluid, the thermodynamics of the latter being expressed by the formula

$$d\epsilon = T\,ds - pd(1/\rho) + \tfrac{1}{2} w_{\alpha\beta} ds^{\alpha\beta} \ , \tag{3.2}$$

where T is the temperature, s the specific entropy, p the pressure, ρ the mass density, $s_{\alpha\beta} = \rho^{-1} S_{\alpha\beta}$, and $w_{\alpha\beta}$ the angular velocity of the fluid. The last quantity can best be described in terms of an orthonormal frame $\{e_\alpha\}$, in which the four-velocity of a fluid particle is given by $u^\alpha = \delta^\alpha_0$ and its spin by $s_{\alpha\beta} = 2\lambda(x)\delta^1_{[\alpha}\delta^2_{\beta]}$. In such a frame,

$$w_{\alpha\beta} = g(e_\alpha, \nabla_0 e_\beta) = \Gamma_{\alpha\beta 0} \ . \tag{3.3}$$

The Ray-Smalley tensor for the stress-energy of the fluid can now be written in terms of quantities already defined as

$$t^{\mathrm{RS}}_{(\alpha\beta)} = \rho(1+\epsilon+p/\rho) u_\alpha u_\beta + p g_{\alpha\beta} + u_{(\alpha} S_{\beta)_\gamma} \dot{u}^\gamma - w^\gamma{}_{(\alpha} S_{\beta)_\gamma} + u_{(\alpha} S^\gamma_{\beta)} w_{\gamma\delta} u^\delta \ , \tag{3.4}$$

where the overdot denotes covariant derivative with respect to u^α. In what follows the source term $t_{(\alpha\beta)}$ of Eqs. (2.9) will be identified with $t^{\mathrm{RS}}_{(\alpha\beta)}$, and the units will be chosen such that $\kappa = 1 = c$, the speed of light.

IV. THE INTERIOR SOLUTION

Following Prasanna,[10] we will now consider a spin-polarized medium occupying the $0 \le r \le R$ region of a coordinate system in which the metric reads

$$ds^2 = -e^{2\nu} dt^2 + r^2 e^{-2\nu} d\varphi^2 + e^{2(\mu-\nu)}(dr^2 + dz^2) \ , \tag{4.1}$$

where μ, ν are functions only of r, and (r, φ) represent polar coordinates. It will be assumed that all particles of this medium have their spin aligned along the axis of cylindrical symmetry (the z axis) and their velocity along the hypersurface orthogonal timelike killing vector ∂_t. It will be further assumed that the tensor fields $u^\alpha, S_{\alpha\beta}$ have vanishing Lie derivatives along all three Killing vector fields $\partial_t, \partial_\phi, \partial_z$. In terms of the orthonormal basis $(e_0 = e^{-\nu}\partial_t, e_1 = e^{\nu-\mu}\partial_r, e_2 = r^{-1}\partial_\phi, e_3 = e^{\nu-\mu}\partial_z)$, the above assumptions are expressed as

$$u^\alpha = \delta^\alpha_0 \ , \quad S_{\alpha\beta} = 4S(r)\delta^1_{[\alpha}\delta^2_{\beta]} \ , \tag{4.2}$$

where condition (3.1) was taken into consideration. When the same condition is used in Eq. (2.6), it follows that

$$T_\alpha = 0 \ , \quad T^\alpha{}_{\beta\gamma} = S^\alpha{}_{\beta\gamma} = u^\alpha S_{\beta\gamma} \ , \tag{4.3}$$

which, combined with (4.2), shows that $T^0_{12} = -T^0_{21} = 2S$ are the only nonvanishing components of the torsion tensor.

Given that the geometry of the EC spacetime is completely determined by the metric and torsion tensor fields, the assumptions laid out above permit us to characterize the spacetime region under study as cylindrically symmetric and static.

The connection coefficients, curvature tensor, and all other geometric quantities for the metric (4.1) and spin tensor (4.2) are given in Ref. 10, from where we quote the results pertinent to later calculations. Thus, the nonvanishing components of the angular velocity are given by

$$w_{01}=-w_{10}=-\nu' e^{\nu-\mu} ,$$
$$w_{12}=-w_{21}=-S , \tag{4.4}$$

where a prime denotes differentiation with respect to r,

$$\dot{u}^{\alpha}=\delta_1^{\alpha}\nu' e^{\nu-\mu} \tag{4.5}$$

and

$$\dot{S}_{\alpha\beta}=-4\delta^0_{[\alpha}\delta^2_{\beta]}S\nu' e^{\nu-\mu} . \tag{4.6}$$

Having assumed that $S_{12}=-S_{21}$ are the only nonvanishing components of the spin tensor and that on the level of thermodynamics $w_{\alpha\beta}=(\partial\epsilon/\partial s^{\alpha\beta})$, according to Eq. (3.2), it is consistent to put $w_{01}=-w_{10}=0$. From Eqs. (4.4) and (4.6) it is clear that this is equivalent to assuming $\dot{S}_{\alpha\beta}=0$, i.e., that spin is conserved. Either one of these equations then implies that $\nu=C_1$, a constant, which we require to vanish in order for the metric (4.1) to reduce to that of Minkowski on the z axis ("elementary flatness"). According to Eq. (4.5), $\dot{u}^{\alpha}$ also vanishes, and, then, use can be made of Eqs. (4.2) and (4.4) in order to write Eq. (3.4) as

$$t_{(\alpha\beta)}=\mathrm{diag}(m,p_1,p_2,p_3) , \tag{4.7}$$

where

$$m=\rho(1+\epsilon) , \quad p_1=p_2=p-2S^2 , \quad p_3=p . \tag{4.8}$$

The vanishing of the right-hand side of Eq. (4.6), on the other hand, leads via Eqs. (2.8) and (4.3) to

$$t_{[\alpha\beta]}=0 , \tag{4.9}$$

while the field equations (2.9) become

$$\begin{aligned} &-\mu'' e^{-2\mu}+S^2=t_{00}=m ,\\ &(\mu'/r)e^{-2\mu}+S^2=t_{11}=p-2S^2 ,\\ &\mu'' e^{-2\mu}+S^2=t_{22}=p-2S^2 ,\\ &-(\mu'/r)e^{-2\mu}+S^2=t_{33}=p ,\\ &-(S'+\mu' S)e^{-\mu}=t_{02}=0 . \end{aligned} \tag{4.10}$$

The last of these equations demands that

$$S=S_0 e^{-\mu} , \tag{4.11}$$

where S_0 is an integration constant, which renders the system of the remaining equations very easy to solve. The result reads

$$m=p=2S^2 , \quad 2\mu=-(S_0 r)^2+C_2 , \tag{4.12}$$

where C_2 is another integration constant. The condition of elementary flatness discussed earlier makes C_2 vanish, and so we can write the final form of the interior metric as

$$ds^2=-dt^2+r^2 d\phi^2+e^{-S_0^2 r^2}(dr^2+dz^2) . \tag{4.13}$$

V. THE EXTERIOR SOLUTION AND MATCHING

The EC field equations for vacuum reduce to those of GR, namely,

$$R_{\alpha\beta}=0 . \tag{5.1}$$

A well-known static cylindrically symmetric solution of Eqs. (5.1) is the metric of Levi-Civita,[14]

$$ds^2=-r^{2l}dt^2+r^{2(1-l)}d\phi^2+L^2 r^{-2(1-l)}(dr^2+dz^2) , \tag{5.2}$$

where l,L are constants. It is, therefore, tempting to consider (5.2) as the metric representing the gravitational field produced by the static cylinder under study, and use the well-known Lichnerowicz[15] junction conditions in order to match this solution to the interior metric. This is the approach followed by Prasanna.[10] Arkuzewski *et al.*[16] have shown, however, that the matching and junction conditions appropriate to the EC theory are not those of Lichnerowicz, but a new set, which in our case read

(i) the fluid particles move along the $r=R$ hypersurface, separating the cylinder from the vacuum,

(ii) the component of stress normal to this hypersurface vanishes,

(iii) the metric functions are continuous at $r=R$, and

(iv)

$$\partial_r g_{\alpha\beta}\,|_{\,r=R+0}=\partial_r g_{\alpha\beta}\,|_{\,r=R-0}+2K_{r(\alpha\beta)} , \tag{5.3}$$

where $\alpha,\beta\neq r$.

The first two of the above conditions are satisfied by our interior solution, since $u^{\alpha}=\delta_0^{\alpha}$ and t_{11} vanishes as a result of Eqs. (4.7) and (4.12). The last two conditions, however, cannot be simultaneously satisfied by the Levi-Civita metric in the form (5.2), because Eq. (5.3) demands that

$$\partial_r g_{t\phi}\,|_{\,r=R+0}=2S_0 R . \tag{5.4}$$

This difficulty, along with the fact that it was the Lewis[17] stationary cylindrically symmetric solution that van Stockum was able to match to his solution for the interior of a rotating cylinder of dust, led us to consider the Lewis metric as a candidate for the description of the gravitational field outside a spin-polarized cylinder. The Lewis metric is given by

$$ds^2=-r(\alpha_1{}^2 e^{\psi}-\gamma_1{}^2 e^{-\psi})dt^2+r(-\alpha_2{}^2 e^{\psi}+\gamma_2{}^2 e^{-\psi})d\phi^2+2r(-\alpha_1\alpha_2 e^{\psi}+\gamma_1\gamma_2 e^{-\psi})dt\,d\phi+e^{2\bar{\mu}}(dr^2+dz^2) , \tag{5.5}$$

where

$$\psi=-k\ln(r/r_0) , \quad 2\bar{\mu}=\tfrac{1}{2}(k^2-1)\ln r+D \tag{5.6}$$

and $k, r_0, D, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ are arbitrary constants, the last four of which must satisfy the condition

$$\alpha_1\gamma_2 - \alpha_2\gamma_1 = 1 . \tag{5.7}$$

All these constants are specified in terms of S_0 and R, by making the metric (5.5) agree with (4.13) at $r = R$, and satisfy Eq. (5.4) there. The algebra being simple but tedious, we give only the final result, which makes Eq. (5.5) read

$$\begin{aligned} ds^2 = &-(A^2x^{1-k} - B^2x^{1+k})dt^2 + R^2(-B^2x^{1-k} + A^2x^{1+k})d\phi^2 \\ &-2RAB(x^{1-k} - x^{1+k})dt\,d\phi + x^{(k^2-1)/2}e^{-S_0{}^2R^2}(dr^2 + dz^2) , \end{aligned} \tag{5.8}$$

where

$$x = r/R , \quad A^2 = (1+k)/2k , \quad B^2 = (1-k)/2k , \quad k = [1-(2S_0R)^2]^{1/2} . \tag{5.9}$$

The expression (5.8) for the metric is given for $2S_0R < 1$, so that, according to (5.9), k is real and positive. The metric form for $2S_0R = 1$ can be obtained as the $k = 0$ limit of (5.8), while for $2S_0R > 1$ and k pure imaginary a solution still obtains, but lacks any physical interest, since it leads to the $z =$constant surfaces having finite extent, as discussed in detail in the last section of van Stockum's paper.[8]

VI. DISCUSSIONS AND CONCLUSIONS

The gravitational field of the static spin-polarized perfect fluid cylinder described in the previous sections is determined by the cylinder's spin density and size only. This is reflected in the fact that the expressions (4.13) and (5.8) for the metric field in the interior and exterior regions, respectively, involve only $2S_0$, the spin density on the cylinder's axis of symmetry, and R, the cylinder's coordinate radius. Thus, for $S_0R = 0$ we obtain the flat spacetime of Minkowski.

The dynamical role of spin in the physical problem studied, on the other hand, is twofold. First, inside the cylinder it acts as an anisotropic negative pressure which prevents the cylinder from collapsing under the mutual attraction of its elements. This can be made explicit by transferring all terms involving the spin density on the right-hand side of the field equations (4.10), thus obtaining the field equations of GR with an "effective" stress-energy tensor describing a "pressure" of the sort just mentioned. Second, the cylinder's spin is responsible for the presence of the $t-\phi$ cross terms in the metric field outside. These magneticlike terms reflect the "dragging of inertial frames," usually produced by mass distributions with orbital angular momentum described in the context of GR. To our knowledge, the solution obtained above is the first exact solution of the EC field equations, exhibiting the double effect of spin described in this paragraph.

Of course, inertial-frame dragging is an effect associated with stationary spacetimes and not simply with time-space components in the metric tensor. Thus, our reference to this effect in connection with the metric (5.8) might seem not well founded, since the gravitational field outside the cylinder is static, a fact that can be made explicit via a linear transformation of the $t-\phi$ coordinates which brings the metric (5.8) to the manifestly static Levi-Civita form (5.2). This is also the case with van Stockum's solution and, till very recently, it was considered paradoxical that a rotating source produces a static gravitational field. The paradox was resolved by Stachel,[18] who noted that the transformation that gauges away the off-diagonal terms in the exterior metric is not a proper one for the entire spacetime manifold, due to the periodic nature of the ϕ coordinate.

In contrast with van Stockum's solution where the interior region is stationary, in our case the gravitational field is static even inside the cylinder. However, one still cannot find a transformation that eliminates the $t-\phi$ components of the metric globally. Equivalently, no time-like Killing vector can be found which is hypersurface orthogonal everywhere. Thus, although static by regions, our solution is globally stationary.

Stachel[18] has also shown that globally stationary but locally static spacetimes provide a gravitational analog of the well-known Aharonov-Bohm effect.[19] In terms of this analogy, the static gravitational field produced by van Stockum's rotating cylinder of dust corresponds to the electrostatic field outside an infinite charged rotating cylinder. The Aharonov-Bohm effect demonstrates the nonlocal electromagnetic effects of the rotating charge. Similarly, one can in principle observe the global gravitational effects of the cylinder's rotating mass via an optical experiment. Specifically, the cylinder's orbital angular momentum is expected to bring about a shift in the interference pattern produced by reuniting two components of a coherent beam of light after passing them around opposite sides of the cylinder.

The solution described in this paper, on the other hand, provides the gravitational analog to the case of the infinitely long magnet. Thus, an optical experiment of the type performed by Chambers, in which the Aharonov-Bohm effect was observed to be produced by a magnetized iron whisker, can in principle prove the global gravitational effect of the spin angular momentum of the source.

In connection with the analogy with classical electromagnetism, it should be observed that the fact that the metric of our solution is static in the interior region of the cylinder corresponds to the fact that inside an infinitely long magnet the magnetic field $\vec{\mathrm{H}}$ vanishes. This is not the case with the rotating charged cylinder, and this is reflected in the fact that inside van Stockum's cylinder spacetime is stationary.

The analogy established above brings forth the issue of the relation between the EC theory and Einstein's GR. According to Adamowicz,[20] "the relation between the EC theory and GR is similar to that between Maxwell's theory of continuous media and classical microscopic electrodynamics; torsion appears as a by-product of the process of averaging of GR, exactly as one introduces the polarization and magnetization vectors in the process of averaging of microscopic Maxwell's equations." Hehl *et al.*,[5] however, disagree with this conclusion of Adamowicz and claim that in the context of EC theory spin is meant to be the spin of elementary particles, therefore torsion exists already on an elementary level. Actually, the analog of classical macroscopic electromagnetism seems to favor the argument of Hehl *et al.* Because, even though one can model the effects of a piece of magnetic material in terms of a continuous distribution of infinitesimal loops of electric current (Ampère's model), the fact remains that the magnetic moments of its constituent elementary particles have their origin in the particles' (quantum-mechanical) spin and thus exist already on an elementary level.

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