Effects of multiple Higgs bosons on tree unitarity

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(Received 11 June 1984)

The abundance of new Higgs bosons in many extensions of the electroweak theory motivate an investigation of their effect on partial-wave scattering amplitudes. For SU(2)×U(1), tree unitarity is shown to be valid for W and Z amplitudes provided three quartic couplings from the Higgs potential are small ($|f| < 16\pi$). The mass spectrum of the Higgs boson is almost unconstrained but for the simple requirement that at least one neutral scalar must have a mass $M < 1$ TeV. Similar results are obtained in a general broken gauge theory for the scattering of arbitrary combinations of scalar and vector bosons.

I. INTRODUCTION AND SUMMARY

In order to determine the effects of a very large Higgs-boson mass, some years ago Lee, Quigg, and Thacker investigated the minimal SU(2)×U(1) model with a single Higgs doublet.[1,2] Any such theory satisfies exact unitarity and thus has partial-wave amplitudes that satisfy $|a_J| \le 1$. Unitarity holds for all values of the masses and couplings. If, however, the theory is to be weakly coupled, then one expects that the amplitudes computed in lowest-order perturbation theory (i.e., tree approximation) should satisfy $|a_J| \le 1$. This tree-unitarity property will only be satisfied for certain ranges of the masses and couplings and therefore provides a quantitative distinction between weak coupling and strong coupling. At high energy, the amplitudes that pose the greatest threat to tree unitarity are those involving longitudinally polarized gauge bosons Z_L and W_L or the Higgs boson H. Lee, Quigg, and Thacker find, for example, in the tree approximation that

$$a_0(Z_LZ_L \to Z_LZ_L) \underset{s \gg M^2}{\to} -3\sqrt{2}G_FM^2/16\pi \,, \qquad (1.1)$$

where the Higgs-boson mass M is presumed larger than M_Z. Requiring $|a_0| \le 1$ thus bounds the Higgs-boson mass:

$$M^2 < 8\pi\sqrt{2}/3G_F \,. \qquad (1.2a)$$

In the minimal model, bounding M^2 is equivalent to bounding the quartic coupling in the potential

$$V = -\tfrac{1}{2}\mu^2\Phi^\dagger\Phi + \tfrac{1}{6}f(\Phi^\dagger\Phi)^2 \,. \qquad (1.3)$$

Since $M^2 = fv^2/3$ and $G_F = 1/\sqrt{2}v^2$, where v is the vacuum expectation value, the bound (1.2a) is equivalent to

$$f < 16\pi \,. \qquad (1.2b)$$

If tree unitarity (1.2) is not satisfied, the bosonic sector of the theory becomes strongly coupled. Considerable work has gone into investigating the properties of such a strongly coupled theory,[3] including resonance formation[4] and copious particle production.[5]

The above analysis is specific to the minimal model with one Higgs doublet. There are now many SU(2)×U(1) models that employ many Higgs multiplets in an assortment of representations (singlets, doublets, triplets, etc.). The motivations for this proliferation are often quite reasonable: to solve the strong CP problem, to predict fermion masses and mixing angles, to make parity a spontaneously broken symmetry, to implement supersymmetry or grand unification. Once there are many scalar fields there is no longer a simple relation between the various masses M and the various quartic couplings f, so that it is rather difficult to anticipate how (1.2a) and (1.2b) might generalize. The purpose of this paper is to find the general conditions on scalar masses and couplings necessary for a nonminimal theory to satisfy tree unitarity.

The investigation is begun in Sec. II. The scattering of longitudinally polarized W's and Z's are computed in the center of mass. Terms of order α at all energies are omitted since they pose no threat to tree unitarity. Completely arbitrary numbers and types of Higgs bosons ϕ_n with masses M_n can be exchanged in the s, t, and u channels. Individual diagrams will often grow with energy like E^2 or E^4; however these terms cancel between diagrams and leave amplitudes that are asymptotically constant at high energy but proportional to various Higgs-boson masses ${M_n}^2$.[6] A typical example is[7]

$$T(Z_LZ_L \to Z_LZ_L) = \frac{-1}{4{M_Z}^4}\sum_n {M_n}^2{A_n}^2\left[\frac{s}{s-{M_n}^2}+\frac{t}{t-{M_n}^2}+\frac{u}{u-{M_n}^2}\right], \qquad (1.4)$$

where $s \gg {M_Z}^2$ and ${M_n}^2 \gg {M_z}^2$. The coupling $ZZ\phi_n$ is

$$A_n = 2g^2(T_3T_3v)_n/\cos^2\theta_W \,.$$

The asymptotic values ($s \to \infty$) of this amplitude is constant in energy and angle with a $J=0$ projection

$$a_0(Z_LZ_L \to Z_LZ_L) \underset{s \gg {M_n}^2}{\to} -3\left[\sum_n {M_n}^2{A_n}^2\right]\Big/ 64\pi {M_Z}^4 \,. \qquad (1.5)$$

This reduces to (1.1) in the minimal Higgs model. Naively, each A_n is of order gM_Z so that (1.5) appears to be of order $G_F\sum_n M_n^2$. This would mean that any superheavy Higgs boson would violate tree unitarity.

Remarkably, it turns out that making a Higgs-boson mass M_n larger is automatically compensated by the $ZZ\phi_n$ coupling A_n becoming smaller. This is guaranteed by a sum rule

$$3\sum_n M_n^2 A_n^2 = 4M_Z^4 f_{zzzz} , \quad (1.6)$$

where f_{zzzz} is a particular quartic coupling from the Higgs potential. (In fact, it is just the self-coupling of the would-be Goldstone boson that becomes the longitudinal degree of freedom of the Z gauge boson.) Because of this sum rule

$$a_0(Z_L Z_L \to Z_L Z_L) \underset{s \gg M_n^2}{\longrightarrow} -f_{zzzz}/16\pi . \quad (1.7)$$

A necessary condition for tree unitarity is thus

$$f_{zzzz} < 16\pi . \quad (1.8)$$

This is also shown to be sufficient to guarantee tree unitarity for nonasymptotic energies and for all other partial waves a_J.

The necessary and sufficient condition (1.8) for tree unitarity is the appropriate generalization of (1.2b). It is also interesting to ask what happened to the mass condition (1.2a). The answer is that from the sum rule (1.6) one can derive an inequality

$$M_{\min}^2 < f_{zzzz}/3\sqrt{2}G_F \quad (1.9)$$

for the mass of the lightest, neutral scalar boson.[8] This bound holds whether or not tree unitarity applies. If tree unitarity does apply, then because of (1.8) we have

$$M_{\min}^2 < 8\pi\sqrt{2}/3G_F . \quad (1.10)$$

Thus tree unitarity constrains only one scalar mass. Furthermore, satisfying (1.10) does not guarantee tree unitarity.

The analysis of $Z_L Z_L \to W_L^- W_L^+$ and $W_L^- W_L^+ \to W_L^- W_L^+$ performed in Sec. II gives exactly analogous results. Basically the complexities of the Higgs structure are concealed by the theory. The point is emphasized further if one evaluates the three amplitudes in the "low-energy" regime $M_Z^2 \ll s \ll M_n^2$. Amplitude (1.4) is negligibly small (i.e., order α) for these energies, but the remaining two are of order $G_F s$. These yield the isoscalar and isotensor amplitudes

$$a_0^{(I=0)}(s) \underset{s \ll M_n^2}{\longrightarrow} G_F s/8\pi\sqrt{2} ,$$
$$a_0^{(I=2)}(s) \underset{s \ll M_n^2}{\longrightarrow} -G_F s/16\pi\sqrt{2} . \quad (1.11)$$

Because of various identities, all details of the Higgs structure again vanish so that (1.11) is precisely the same result as obtained by Lee, Quigg, and Thacker[1] in the minimal model. Because the isoscalar amplitude (1.11) is attractive and grows with energy, it suggests that in the strong coupling limit a bound state $I=0$ scalar boson will arise regardless of the number of elementary scalars.

Section III examines the scatterings in which the external particles are various mixtures of gauge bosons and scalars. This is most easily done in a general broken gauge theory, not specifically SU(2)×U(1). There are five classes of amplitudes: $\phi\phi\to\phi\phi$, $A_L\phi\to\phi\phi$, $A_L A_L\to\phi\phi$, $A_L A_L\to A_L\phi$, and $A_L A_L\to A_L A_L$. Because of the connection between longitudinally polarized vectors and the would-be Goldstone bosons, the last four amplitudes turn out to be just projections of the fundamental amplitude

$$T(\phi_1\phi_2\to\phi_3\phi_4) = -f_{1234} - \sum_n \left[\frac{e_{12n}e_{34n}}{s-M_n^2} + \frac{e_{13n}e_{24n}}{t-M_n^2} + \frac{e_{14n}e_{23n}}{u-M_n^2} \right] . \quad (1.12)$$

At infinite energy this is pure $J=0$ and satisfies tree unitarity if $|f_{1234}| < 16\pi$. At finite energy the amplitude is controlled by the cubic couplings e_{ijk}. Unfortunately, there is no elegant sum rule for these couplings. It is possible to prove that

$$e_{ijk}^2 < \tfrac{16}{5}\bar{f}\max(M_i^2, M_j^2, M_k^2) , \quad (1.13)$$

where $\bar{f}$ is the maximum quartic coupling in the theory. From this one can show that (1.12) will satisfy tree unitarity in the relativistic regime

$$s \gtrsim 5\max(M_1^2, M_2^2, M_3^2, M_4^2)$$

provided

$$\bar{f} < 5\pi/N , \quad (1.14)$$

where N is the total number of scalar-exchange diagrams allowed by the quantum numbers. If tree unitarity is satisfied for (1.12) then it is automatically satisfied for the four other classes of scattering involving various numbers of longitudinal gauge bosons. Section III lacks the elegance of Sec. II, but the fundamental result is the same: Small quartic couplings guarantee unitarity for boson-boson scattering. No additional restrictions on masses or cubic couplings are necessary.

The appendixes contain results that may be useful in other applications. Appendix A derives various sum rules of which (1.6) is a very special case. Appendix B proves in an arbitrary gauge the well known theorem relating amplitudes for longitudinally polarized gauge bosons to those for would-be Goldstone bosons. Appendix C proves the bound (1.13) on cubic couplings.

II. W AND Z SCATTERING IN SU(2)×U(1) WITH ARBITRARY HIGGS MULTIPLETS

In order to investigate an SU(2)×U(1) theory with many Higgs multiplets of various types, it is convenient to assemble all the real spinless field into a single multiplet $\phi_j(x)$. The Hermitian matrices T_1, T_2, T_3, and Y that act on the ϕ_j are antisymmetric and pure imaginary. Vacuum expectation values $v_k = \langle \phi_k(x) \rangle_0$ acquired via spontaneous symmetry breaking produce vector-meson masses

$$M_Z{}^2 = g^2 v_j [(T_3)^2]_{jk} v_k / \cos^2\theta_W \; ,$$
$$M_W{}^2 = \tfrac{1}{2} g^2 v_j [T(T+1) - (T_3)^2]_{jk} v_k \tag{2.1}$$

and keep the photon massless ($Q_{jk} v_k = 0$). Without loss of generality we will work in the basis in which the scalar mass matrix is diagonal

$$\partial^2 V / \partial\phi_n \partial\phi_k \mid_{\phi=v} = M_n{}^2 \delta_{nk} \; .$$

The shift $\phi_n(x) = v_n + \phi_n'(x)$ produces a trilinear coupling of ϕ_n' to two vectors given by

$$\mathscr{L}_{\text{cubic}} = \tfrac{1}{2} Z_\mu Z^\mu A_n \phi_n' + W_\mu^- W^{\mu +} B_n \phi_n' + (W_\mu^+ Z^\mu C_n \phi_n' + \text{H.c.}) + (\tfrac{1}{2} W_\mu^+ W^{\mu +} D_n \phi_n' + \text{H.c.}) \; ,$$

$$A_n = \frac{2g^2}{\cos^2\theta_W} [(T_3)^2 v]_n \; ,$$

$$B_n = g^2 \{ [T(T+1) - (T_3)^2] v \}_n \; , \tag{2.2}$$

$$C_n = \frac{g^2 \sqrt{2}}{\cos\theta_W} (T_- T_3 v)_n - \frac{g^2 M_Z{}^2 \cos^2\theta_W}{\sqrt{2} M_W{}^2} (T_- v)_n \; ,$$

$$D_n = g^2 (T_- T_- v)_n \; .$$

Note that A_n and B_n are real but C_n and D_n are complex because T_- is complex; A_n and B_n are orthogonal to the neutral Goldstone-boson direction $(T_3 v)_n$; C_n is orthogonal to the charged Goldstone-boson direction, $C_n (T_+ v)_n = 0$. Combining (2.1) and (2.2) implies that $A_n v_n = 2M_Z{}^2$, $B_n v_n = 2M_W{}^2$, $C_n v_n = 0$, and $D_n v_n = 0$.

All amplitudes are computed in the center of mass at $s \gg M_Z{}^2$. The partial-wave decomposition of the scattering amplitude is

$$T(s, \cos\theta) = 16\pi \sum_{J=0}^{\infty} (2J+1) a_J(s) P_J(\cos\theta) \tag{2.3}$$

normalized so that the elastic cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \mid T \mid^2 \; .$$

A. Computation of amplitudes

1. $Z_L Z_L \to Z_L Z_L$

The simplest tree-approximation amplitude for scattering longitudinally polarized gauge bosons is $Z_L Z_L \to Z_L Z_L$. No exchanges of gauge bosons are possible. Only the three Higgs-boson-exchange diagrams shown in Fig. 1 contribute. The E^2 growth of each diagram cancels between the three because $s + t + u = 4M_Z{}^2$ and the result is

$$T(Z_L Z_L \to Z_L Z_L) = -\sum_n \frac{M_n{}^2 A_n{}^2}{4 M_Z{}^4} \left[\frac{s}{s - M_n{}^2} + \frac{t}{t - M_n{}^2} + \frac{u}{u - M_n{}^2} \right] . \tag{2.4}$$

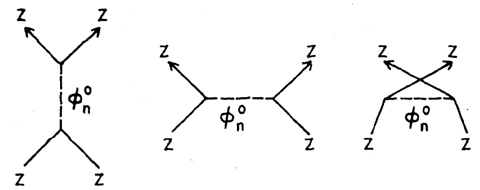

FIG. 1. The three types of diagrams that contribute to $Z_L Z_L \to Z_L Z_L$ in (2.4).

2. $Z_L Z_L \to W_L^- W_L^+$

For $Z_L Z_L \to W_L^- W_L^+$ the $W^\pm$ exchange graph and the contact graph shown in Fig. 2 all grow like E^4. After cancellations among these three graphs the remainder grows like E^2:

$$T_{\text{vector}} = \frac{g^2 \cos^2\theta_W}{4 M_W{}^4} M_Z{}^2 s \; .$$

Exchanges of neutral scalars in the s channel and charged scalars in the t and u channels give

$$T_{\text{scalar}} = \frac{-1}{4 M_Z{}^2 M_W{}^2} \sum_n \left[A_n B_n \frac{s^2}{s - M_n{}^2} + \mid C_n \mid^2 \left[\frac{t^2}{t - M_n{}^2} + \frac{u^2}{u - M_n{}^2} \right] \right] .$$

The E^2 growth of this amplitude is canceled by the vector exchange because of the identity

$$\sum_n (A_n B_n - \mid C_n \mid^2) = g^2 \cos^2\theta_W M_Z{}^4 / M_W{}^2 \; . \tag{2.5}$$

The final result is

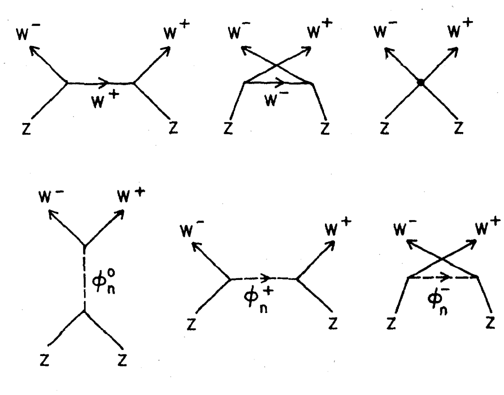

FIG. 2. The diagrams that contribute to $Z_L Z_L \to W_L^- W_L^+$. The high-energy growth of the $W^\pm$-exchange and contact graphs cancels against that of the scalar-exchange graphs to give (2.6).

$$T(Z_LZ_L\to W_L^-W_L^+)=-\sum_n \frac{M_n^2}{4M_Z^2M_W^2}\left[A_nB_n\frac{s}{s-M_n^2}+|C_n|^2\left[\frac{t}{t-M_n^2}+\frac{u}{u-M_n^2}\right]\right]. \tag{2.6}$$

3. $W_L^-W_L^+\to W_L^-W_L^+$

The computation on $W_L^-W_L^+\to W_L^-W_L^+$ is similar to, but slightly more complicated than, the previous case. The two Z-exchange graphs, the two γ-exchange graphs, and the contact graph shown in Fig. 3 all grow like E^4 at high energy but cancellations among them eliminate this and leave a remainder that grows like E^2:

$$T_{\text{vector}}=\frac{g^2}{4M_W^4}(3M_Z^2\cos^2\theta_W-4M_W^2)u\ .$$

Exchanges of neutral scalars in the s and t channels and of charge 2 scalars in the u channel give

$$T_{\text{scalar}}=\frac{-1}{4M_W^4}\sum_n\left[B_n^2\left[\frac{s^2}{s-M_n^2}+\frac{t^2}{t-M_n^2}\right]+|D_n|^2\frac{u^2}{u-M_n^2}\right].$$

The E^2 growth of the vector and scalar contributions cancel because of the identity

$$\sum_n(-B_n^2+|D_n|^2)=g^2(3M_Z^2\cos^2\theta_W-4M_W^2)\ . \tag{2.7}$$

The physical amplitude is then

$$T(W_L^-W_L^+\to W_L^-W_L^+)=-\sum_n\frac{M_n^2}{4M_W^4}\left[B_n^2\left[\frac{s}{s-M_n^2}+\frac{t}{t-M_n^2}\right]+|D_n|^2\frac{u}{u-M_n^2}\right]. \tag{2.8}$$

B. Asymptotic values

1. *Low energy*

At energies below all Higgs-boson masses ($M_Z^2\ll s\ll M_n^2$) the amplitudes (2.4), (2.6), and (2.8) have the limits

$$T(Z_L\to Z_LZ_L)\underset{s\ll M_n^2}{\longrightarrow}\frac{1}{M_Z^2}\sum_n A_n^2\ ,$$

$$T(Z_LZ_L\to W_L^-W_L^+)\underset{s\ll M_n^2}{\longrightarrow}\frac{1}{4M_Z^2M_W^2}\sum_n(A_nB_n-|C_n|^2)s\ ,$$

$$T(W_L^-W_L^+\to W_L^-W_L^+)\underset{s\ll M_n^2}{\longrightarrow}-\frac{1}{4M_W^4}\sum_n(B_n^2-|D_n|^2)u.$$

Using $\sum_n A_nv_n=2M_Z^2$ and the Schwartz inequality, one can show that the first of these is less than $2\pi\alpha/\rho^2$ (where $\rho=M_W/M_Z\cos\theta_W$) and is thus negligible since terms of order α have already been dropped. Because of the identities (2.5) and (2.7) the remaining two amplitudes can be written

$$T(Z_LZ_L\to W_L^-W_L^+)\underset{s\ll M_n^2}{\longrightarrow}2\sqrt{2}G_Fs/\rho^2\ ,$$
$$T(W_L^-W_L^+\to W_L^-W_L^+)\underset{s\ll M_n^2}{\longrightarrow}\sqrt{2}G_Fs(1+\cos\theta)(4-3/\rho^2)\ . \tag{2.9}$$

Experimentally $\rho\approx1$, so that all details of the Higgs structure disappear and the results are identical with those of the minimal model.[1] The $J=0$ projections of (2.9) in the isospin channels $I=0,2$ are given in (1.11).

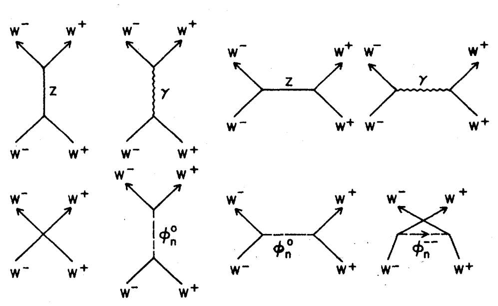

FIG. 3. The diagrams for $W_L^-W_L^+\to W_L^-W_L^+$. The high-energy growth to the Z-exchange, γ-exchange, and contact graphs cancels against that of the scalar-exchange graphs to give (2.8).

2. *Infinite energy*

At infinite energy the three amplitudes (2.4), (2.6), and (2.8) all approach constants:

$$T(Z_LZ_L\to Z_LZ_L)\underset{s\to\infty}{\longrightarrow}-3\sum_n M_n^2A_n^2/4M_Z^4\ , \tag{2.10a}$$

$$T(Z_LZ_L\to W_L^-W_L^+)\underset{s\to\infty}{\longrightarrow}-\sum_n M_n^2(A_nB_n+2|C_n|^2)/4M_Z^2M_W^2\ , \tag{2.10b}$$

$$T(W_L^-W_L^+\to W_L^-W_L^+)\underset{s\to\infty}{\longrightarrow}-\sum_n M_n^2(2B_n^2+|D_n|^2)/4M_W^4\ . \tag{2.10c}$$

Since these asymptotic values are independent of angle, the $J=0$ partial wave amplitudes are

$$a_0(\infty)=T(\infty)/16\pi$$

and the $J>0$ amplitudes all vanish at $s=\infty$. The question of whether $|a_0|$ is less than unity then rests on how large the sums are in (2.10). Naively, one would estimate that the couplings (2.2) are of order g^2v so that each term in (2.10) would be of order $M_n{}^2/v^2 \sim M_n{}^2G_F$.

Since many theories (e.g., grand unified models) have enormous scalar masses M_n, it would appear that $|a_0|$ will easily exceed unity. Remarkably, this is not the case because whenever a particular M_n is very large, the corresponding couplings A_n, B_n, C_n, and D_n are automatically small, contrary to the naive estimate. The proof of this relies on a sum rule derived in Appendix A. The scalar potential V is invariant, before spontaneous symmetry breaking, under infinitesimal transformations

$$\phi_n(x)\to\phi_n(x)+\epsilon^\alpha\theta^\alpha_{nj}\phi_j(x) ,$$

where θ^α is the antisymmetric matrix representation of any symmetry (e.g., T_+, T_-, T_3, or Y). Although the symmetries are spontaneously broken, the Higgs-boson masses will automatically obey

$$\sum_n M_n{}^2[(\theta^\alpha\theta^\beta v)_n(\theta^\gamma\theta^\delta v)_n+(\theta^\alpha\theta^\gamma v)_n(\theta^\beta\theta^\delta v)_n +(\theta^\alpha\theta^\delta v)_n(\theta^\beta\theta^\gamma v)_n] =f_{ijkl}(\theta^\alpha v)_i(\theta^\beta v)_j(\theta^\gamma v)_k(\theta^\delta v)_l , \quad (2.11)$$

where f is the quartic Higgs coupling in v:

$$V=\tfrac{1}{24}f_{ijkl}\phi_i\phi_j\phi_k\phi_l+\cdots . \quad (2.12)$$

To apply this to (2.10) it is useful to define two unit vectors

$$z_i=g(T_3v)_i/M_z\cos\theta_W,\quad z_i^*z_i=1 ,$$
$$w_i=g(T_-v)_i/M_W\sqrt{2},\quad w_i^*w_i=1 . \quad (2.13)$$

These are Goldstone-boson eigenvectors of the scalar mass matrix. Of course, because of the Higgs mechanism there are no scalar field excitations in these directions; instead the gauge bosons Z and W propagate as massive, three-component particles. The self-couplings of the would-be Goldstone bosons are

$$f_{zzzz}=f_{ijkl}z_iz_jz_kz_l ,$$
$$f_{zzw\bar{w}}=f_{ijkl}z_iz_jw_kw_l^* . \quad (2.14)$$
$$f_{w\bar{w}w\bar{w}}=f_{ijkl}w_iw_j^*w_kw_l^* .$$

A specific application of (2.11) results from choosing

$$\theta^\alpha=\theta^\beta=\theta^\gamma=\theta^\delta=g\sqrt{2}T_3/\cos\theta_w$$

and yields

$$3\sum_n M_n{}^2A_n{}^2=4M_Z{}^4f_{zzzz} . \quad (2.15a)$$

Choosing $\theta^\alpha=\theta^\beta=g\sqrt{2}\,T_3/\cos\theta_W$ but $\theta^\gamma=gT_+$ and $\theta^\delta=gT_-$ produces

$$\sum_n M_n{}^2(A_nB_n+2\,|\,C_n\,|^2)=4M_Z{}^2M_W{}^2f_{zzw\bar{w}} . \quad (2.15b)$$

Similarly, if $\theta^\alpha=\theta^\delta=gT_-$ and $\theta^\beta=\theta^\gamma=gT_+$ then

$$\sum_n M_n{}^2(2B_n{}^+\,|\,D_n\,|^2)=4M_W{}^4f_{w\bar{w}w\bar{w}} . \quad (2.15c)$$

These sum rules guarantee that whenever any of the Higgs masses M_n become large, the corresponding couplings to gauge bosons (A_n, B_n, C_n, and D_n) automatically become small. The asymptotic values (2.10) are

$$T(Z_LZ_L\to Z_LZ_L)\underset{s\to\infty}{\to} -f_{zzzz} , \quad (2.16a)$$

$$T(Z_LZ_L\to W_L^-W_L^+)\underset{s\to\infty}{\to} -f_{zzw\bar{w}} , \quad (2.16b)$$

$$T(W_L^-W_L^+\to W_L^-W_L^+)\underset{s\to\infty}{\to} -f_{w\bar{w}w\bar{w}} . \quad (2.16c)$$

Tree unitarity is satisfied (i.e., $|a_0|<1$) provided the three Higgs couplings are small (i.e., $|f|<16\pi$). Thus, for $s=\infty$, the gauge bosons are weakly coupled if and only if the Higgs bosons are weakly coupled: the Higgs-boson masses, *per se*, are not the determining factor.

C. Tree unitarity

It is now necessary to examine $|a_J|$ for nonasymptotic energies.

1. J = 0 amplitudes

The $J=0$ partial wave projections of (2.4), (2.6), and (2.8) may be written

$$a_0(Z_LZ_L\to Z_LZ_L)=-f_{zzzz}F_1(s)/16\pi ,$$
$$a_0(Z_LZ_L\to W_L^-W_L^+)=-f_{zzw\bar{w}}F_2(s)/16\pi , \quad (2.17)$$
$$a_0(W_L^-W_L^+\to W_L^-W_L^+)=-f_{w\bar{w}w\bar{w}}F_3(s)/16\pi ,$$

where

$$F_1(s)=\frac{1}{N_1}\sum_n M_n{}^2A_n{}^2[p_n(s)+2q_n(s)] ,$$

$$F_2(s)=\frac{1}{N_2}\sum_n M_n{}^2[A_nB_np_n(s)+2\,|\,C_n\,|^2q_n(s)] , \quad (2.18)$$

$$F_3(s)=\frac{1}{N_3}\sum_n M_n{}^2[B_n{}^2p_n(s)+(B_n{}^2+|\,D_n\,|^2)q_n(s)] .$$

All energy dependence is contained in the two functions

$$p_n(s)=\frac{s}{s-M_n{}^2},\quad q_n(s)=1-\frac{M_n{}^2}{s}\ln\left[1+\frac{s}{M_n{}^2}\right] . \quad (2.19)$$

The normalization constants N_i are

$$N_1=3\sum_n M_n{}^2A_n{}^2 ,$$
$$N_2=\sum_n M_n{}^2(A_nB_n+2\,|\,C_n\,|^2) , \quad (2.20)$$
$$N_3=\sum_n M_n{}^2(2B_n{}^2+|\,D_n\,|^2) ,$$

so that it is necessary to use the sum rules (2.15) to verify (2.17). Obviously

$$F_i(s)\underset{s\to\infty}{\to} \text{ for } i=1,2,3 .$$

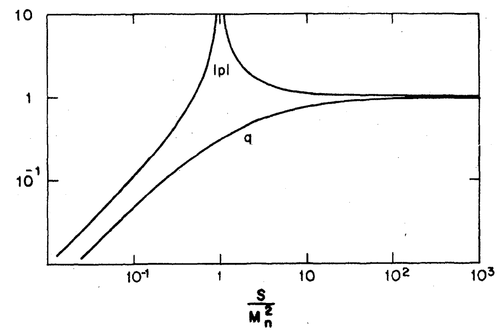

FIG. 4. The magnitude of the functions p and q that determine the energy dependence of the partial-wave amplitudes a_0 in (2.18).

The function p_n diverges at $s=M_n{}^2$, but at any nonresonant energy Fig. 4 shows that $|p_n|$ and q_n are nearly always less than 1. Thus, away from resonances $|F_i(s)| \leq 1$ or equivalently

$$|a_0| \leq |f|/16\pi \ . \tag{2.21}$$

Consequently, small Higgs-boson couplings ($|f| \leq 16\pi$) guarantee tree unitarity ($|a_0| \leq 1$) for all nonresonant energies.

At s near $M_n{}^2$ the amplitudes should include the finite decay width of the Higgs mesons. The decay of a Higgs meson into pairs of longitudinal gauge bosons restores elastic unitarity and ensures that $|a_0| \leq 1$ even at resonance.

2. $J>0$ amplitudes

For $J>0$ the partial-wave projections of (2.4), (2.6), and (2.8) are

$$a_J(Z_LZ_L \rightarrow Z_LZ_L) = \frac{f_{zzzz}}{16\pi N_1}\sum_n M_n{}^2 A_n{}^2[1+(-1)^J] \times r_{J,n}(s) \ ,$$

$$a_J(Z_LZ_L \rightarrow W_L^- W_L^+) = \frac{f_{zzw\bar{w}}}{16\pi N_2}\sum_n M_n{}^2 |C_n|^2[1+(-1)^J] \times r_{J,n}(s) \ , \tag{2.22}$$

$$a_J(W_L^- W_L^+ \rightarrow W_L^- W_L^+) = \frac{f_{w\bar{w}w\bar{w}}}{16\pi N_3}\sum_n M_n{}^2[B_n{}^2+(-1)^J|D_n|^2] r_{J,n}(s) \ ,$$

where all energy dependence is contained in

$$r_{J,n}(s) = \frac{2M_n{}^2}{s} Q_J\left[1+\frac{2M_n{}^2}{s}\right] \tag{2.23}$$

and Q_J are the associated Legendre functions.

Figure 5 shows that $r_J < 0.1$ for all $s/M_n{}^2$. Because of this, (2.22) implies that each of the amplitudes satisfies

$$|a_J| \leq (0.1)|f|/16\pi \tag{2.24}$$

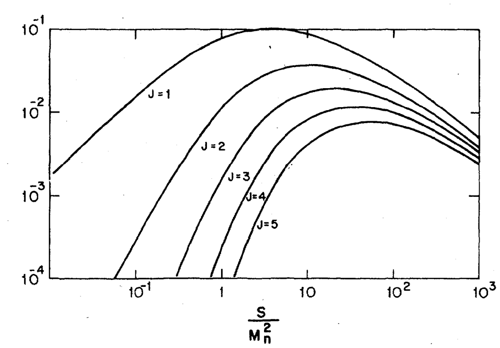

FIG. 5. The functions r_J that determine the energy dependence of the higher-partial-wave amplitudes (2.22) and (3.30).

at all energies. Thus, tree unitarity is even more easily satisfied for the higher partial waves than $J=0$.

III. TWO-BOSON SCATTERING IN ANY BROKEN GAUGE THEORY

In the previous section the external particles were always W's and Z's. One would like to consider reactions in which some (or all) of the external particles are spinless Higgs bosons. For example, can the reaction $Z_LH^0 \rightarrow H^+H^-$ satisfy tree unitarity in a theory which allows the exchange of many heavy Higgs bosons? Once one generalizes the Higgs sector there are very many allowed scatterings of W's, Z's and H's. To keep the discussion concise, it is most convenient to generalize the gauge sector as well. Consequently, we adopt the notation of Weinberg[9] for discussing a general broken gauge theory.

The Lagrangian density for the bosons is

$$\mathscr{L} = -\tfrac{1}{4}F^{\alpha}_{\mu\nu}F^{\alpha,\mu\nu} + \tfrac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i - V(\phi) \ , \tag{3.1}$$

$$F^{\alpha}_{\mu\nu} = \partial_\mu A^{\alpha}_{\nu} - \partial_\nu A^{\alpha}_{\mu} - C^{\alpha\beta\gamma}A^{\beta}_{\mu}A^{\gamma}_{\nu} \ , \tag{3.2}$$

where $C^{\alpha\beta\gamma}$ are the real, antisymmetric structure constants of the gauge group. The group need not be simple [e.g., SU(2)×U(1)] and the independent coupling constants are included in the $C^{\alpha\beta\gamma}$. The scalar fields are organized into a single real multiplet $\phi_i(x)$, which will transform reducibly under the gauge group. The gauge covariant derivative is

$$(D_\mu\phi)_i = \partial_\mu\phi_i + iA^{\alpha}_{\mu}\theta^{\alpha}_{ij}\phi_j \ , \tag{3.3}$$

where θ^α is the matrix representation (including coupling constants) of the αth group generator:

$$[\theta^\alpha, \theta^\beta] = iC^{\alpha\beta\gamma}\theta^\gamma \ . \tag{3.4}$$

These matrices are antisymmetric because the ϕ_i are real:

$$\theta^{\alpha}_{ji} + \theta^{\alpha}_{ij} = 0 \ . \tag{3.5}$$

The value of the ϕ_i which minimizes the potential is v_i. After shifting, $\phi_i(x) = v_i + \phi'_i(x)$, the potential for ϕ' is

$$V(\phi') = \tfrac{1}{2}M^2{}_{ij}\phi'_i\phi'_j + \tfrac{1}{6}e_{ijk}\phi'_i\phi'_j\phi'_k + \tfrac{1}{24}f_{ijkl}\phi'_i\phi'_j\phi'_k\phi'_l \ . \tag{3.6}$$

Note that there can be no terms linear in ϕ' because the minimum of V occurs at $\langle \phi_i' \rangle = 0$ by definition. Without loss of generality one may assume that a basis has been chosen in which the scalar-boson mass matrix and the gauge-boson mass matrix are both diagonal:

$$M^2{}_{ij} = \delta_{ij} M_j^2 \, , \qquad v_i (\theta^\alpha \theta^\beta) v_j = \delta^{\alpha\beta} \mu_\alpha^2 \, . \tag{3.7}$$

A. Computation of amplitudes

All two-boson amplitudes are calculated in the limit $s \gg \mu_\alpha^2$. Since we are interested in the effects of large Higgs masses, it is assumed that $M_n^2 \gg \mu_\alpha^2$ (for all ϕ_n and A_L^α). No assumption is made about the relative size of s and the various M_n^2. As in Sec. II the computations require combining vector and scalar exchanges in each channel. After cancellations any contributions to the amplitudes that are of order $\theta^2 \sim \alpha$ are dropped since they constitute no threat to tree unitarity.

All these amplitudes turn out to have the property that at infinite energy they approach a constant $T(\infty)$ that is independent of angle. For $s = \infty$ the partial-wave amplitudes are

$$a_J(\infty) = \delta_{J,0} T(\infty)/16\pi \tag{3.8}$$

and will satisfy tree unitarity if $|T(\infty)| < 16\pi$. The behavior at finite energy will be discussed in a later section.

1. $\phi_i \phi_j \to \phi_k \phi_l$

Six exchange graphs and one contact graph combine to give

$$T(\phi_i\phi_j \to \phi_k\phi_l) = -f_{ijkl} - \sum_n \left[\frac{e_{ijn}e_{kln}}{s - M_n^2} + \frac{e_{ikn}e_{jln}}{t - M_n^2} + \frac{e_{iln}e_{jkn}}{u - M_n^2} \right] . \tag{3.9}$$

The infinite energy limit of this amplitude,

$$T(\phi_i\phi_j \to \phi_k\phi_l) \underset{s\to\infty}{\longrightarrow} -f_{ijkl} \tag{3.10}$$

obviously satisfies tree unitarity if $|f| < 16\pi$.

2. $A_L^\alpha \phi_j \to \phi_k \phi_l$

With one external gauge boson, no contact graph is possible. The six exchange graphs give

$$T(A_L^\alpha \phi_j \to \phi_k\phi_l) = \frac{-1}{\mu_\alpha} \sum_n \left[\theta^\alpha_{jn} e_{nkl} \left[\frac{s - M_j^2}{s - M_n^2} \right] + \theta^\alpha_{kn} e_{njl} \left[\frac{t - M_k^2}{t - M_n^2} \right] + \theta_{ln}{}^\alpha e_{njk} \left[\frac{u - M_l^2}{u - M_n^2} \right] \right] . \tag{3.11}$$

At infinite energy,

$$T(A_L^\alpha \phi_j \to \phi_k\phi_l) \underset{s\to\infty}{\longrightarrow} \frac{-1}{\mu_\alpha} \sum_n \left[\theta^\alpha_{jn} e_{nkl} + \theta^\alpha_{kn} e_{njl} + \theta^\alpha_{ln} e_{njk} \right] .$$

This limit looks rather intractable because of the presence of the cubic couplings e_{nkl}. However, the identity (A3) proved in Appendix A allows this limit to be expressed as

$$T(A_L^\alpha \phi_j \to \phi_k\phi_l) \underset{s\to\infty}{\longrightarrow} -f_{ijkl} u_i^\alpha \, , \tag{3.12}$$

where u_i^α is the unit vector

$$u_i^\alpha = \frac{(\theta^\alpha v)_i}{\mu_\alpha} , \sum_i |\mu_i^\alpha|^2 = 1 \, . \tag{3.13}$$

This vector points along the direction of the would-be Goldstone boson that became the longitudinal component of A^α. Tree unitarity is again satisfied at $s = \infty$ provided $|f_{ijkl}| < 16\pi$.

3. $A_L^\alpha A_L^\beta \to \phi_k \phi_l$

The six possible exchange graphs and one contact graph sum to

$$T(A_L^\alpha A_L^\beta \to \phi_k\phi_l) = \frac{1}{\mu_\alpha \mu_\beta} \sum_n \left[h_n^{\alpha\beta} e_{nkl} \left[\frac{s}{s - M_n^2} \right] + \theta^\alpha_{kn} \theta^\beta_{nl} \rho_n(t) + \theta^\beta_{kn} \theta^\alpha_{nl} \rho_n(u) \right] , \tag{3.14}$$

$$h_n^{\alpha\beta} \equiv \tfrac{1}{2}(\{\theta^\alpha, \theta^\beta\} v)_n \, ,$$

$$\rho_n(t) \equiv [t M_n^2 - \tfrac{1}{2}(t + M_n^2)(M_k^2 + M_l^2) + M_k^2 M_l^2]/(t - M_n^2) \, . \tag{3.15}$$

The function ρ_n depends on the nth exchanged meson only through its mass M_n. In the infinite-energy limit

$$T(A_L^\alpha A_L^\beta \to \phi_k\phi_l) \underset{s\to\infty}{\longrightarrow} \frac{1}{\mu_\alpha \mu_\beta} \sum_n [h_n^{\alpha\beta} e_{nkl} + (\theta^\alpha_{kn}\theta^\beta_{nl} + \theta^\beta_{kn}\theta^\alpha_{nl})(M_n^2 - \tfrac{1}{2}M_k^2 - \tfrac{1}{2}M_l^2)] \, .$$

Because of identity (A4), this limit may be written

$$T(A_L^\alpha A_L^\beta \to \phi_k\phi_l) \underset{s\to\infty}{\longrightarrow} -f_{ijkl} u_i^\alpha u_j^\beta \, . \tag{3.16}$$

Tree unitarity is satisfied at $s = \infty$ if the quartic couplings are small ($|f_{ijkl}| < 16\pi$).

4. $A_L^\alpha A_L^\beta \to A_L^\gamma \phi_l$

Here there are six exchange diagrams that yield

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma \phi_l) = \frac{-1}{\mu_\alpha \mu_\beta \mu_\gamma} \sum_n [h_n^{\alpha\beta} \theta^\gamma_{nl} \sigma_n(s) + h_n^{\alpha\gamma} \theta^\beta_{nl} \sigma_n(t) + h_n^{\beta\gamma} \theta^\alpha_{nl} \sigma_n(u)] \, , \tag{3.17}$$

$$\sigma_n(s)=(sM_n{}^2-\tfrac{2}{3}sM_l{}^2-\tfrac{1}{3}M_n{}^2M_l{}^2)/(s-M_n{}^2)\ .$$

At infinite energy,

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma \phi_l) \underset{s\to\infty}{\to} \frac{-1}{\mu_\alpha\mu_\beta\mu_\gamma} \times \sum_n (h_n^{\alpha\beta}\theta_{nl}^\gamma + h_n^{\alpha\gamma}\theta_{nl}^\beta + h_n^{\beta\gamma}\theta_{nl}^\alpha) \times (M_n{}^2-\tfrac{2}{3}M_l{}^2)\ .$$

Because of (A5) the right-hand side is equal to the projection of the quartic coupling along the Goldstone-boson directions,

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma \phi_l) \underset{s\to\infty}{\to} -f_{ijkl}u_i^\alpha u_j^\beta u_k^\gamma\ , \tag{3.18}$$

and will satisfy tree unitarity if $|f| < 16\pi$.

5. $A_L^\alpha A_L^\beta \to A_L^\gamma A_L^\delta$

One contact graph plus six exchange graphs combine to give

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma A_L^\delta)=\frac{-1}{\mu_\alpha\mu_\beta\mu_\gamma\mu_\delta}\sum_n \left[h_n^{\alpha\beta}h_n^{\gamma d}\frac{sM_n{}^2}{s-M_n{}^2} + h_n^{\alpha\gamma}h_n^{\beta\delta}\frac{tM_n{}^2}{t-M_n{}^2} + h_n^{\alpha\delta}h_n^{\beta\gamma}\frac{uM_n{}^2}{u-M_n{}^2}\right]\ . \tag{3.19}$$

The amplitudes discussed in Sec. II [viz., (2.4), (2.6), and (2.8)] result from specializing this to the scattering of W's and Z's. At infinite energy

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma A_L^\delta) \underset{s\to\infty}{\to} \frac{-1}{\mu_\alpha\mu_\beta\mu_\gamma\mu_\delta} \times \sum_n M_n{}^2(h_n^{\alpha\beta}h_n^{\gamma\delta} + h_n^{\alpha\gamma}h_n^{\beta\delta}+h_n^{\alpha\delta}h_n^{\beta\gamma})\ .$$

Because of the identity (A6) this limit is equal to

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma A_L^\delta) \underset{s\to\infty}{\to} -f_{ijkl}u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta \tag{3.20}$$

and will satisfy tree unitarity if $|f| < 16\pi$.

B. A theorem for longitudinal gauge bosons

Before computing any partial-wave projections, it is very useful to recognize that all five of the above amplitudes are related. By using (A1)–(A3) one can show that for any s,

$$T(A_L^\alpha \phi_j \to \phi_k\phi_l)=T(\phi_i\phi_j\to\phi_k\phi_l)u_i^\alpha\ , \tag{3.21a}$$

where u^α is the unit vector (3.13) along the Goldstone-boson direction corresponding to the αth gauge boson. [At $s=\infty$ this reduces to the previous result (3.12).] Similarly one can show by using Appendix A that for any s,

$$T(A_L^\alpha A_L^\beta \to \phi_k\phi_l)=T(\phi_i\phi_j\to\phi_k\phi_l)u_i^\alpha u_j^\beta\ , \tag{3.21b}$$

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma\phi_l)=T(\phi_i\phi_j\to\phi_k\phi_l)u_i^\alpha u_j^\beta u_k^\gamma\ , \tag{3.21c}$$

$$T(A_L^\alpha A_L^\beta \to A_L^\gamma A_L^\delta)=T(\phi_i\phi_j\to\phi_k\phi_l)u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta\ . \tag{3.21d}$$

The underlying reason for these relations is a theorem due to Cornwall, Levin, and Tiktopoulos[6] which states that in any high-energy reaction

$$T(A_L^\alpha,\ldots)=T(\phi_i,\ldots)u_i^\alpha+O(\mu_\alpha/\sqrt{s})\ . \tag{3.22}$$

The proof of this given by Lee, Quigg, and Thacker,[1] for the 't Hooft–Feynman ($\xi=1$) gauge is generalized in Appendix B to an arbitrary ξ gauge.

C. Tree unitarity

It is quite tedious to analyze the partial-wave projections of the five amplitudes in Sec. III A. Fortunately, this is not necessary. All the amplitudes involving longitudinal gauge bosons can be obtained from $T(\phi_i\phi_j \to\phi_k\phi_l)$ because of (3.21). These identities apply to the amplitudes considered as functions of s, t, and u. Of course, a particular value of t will correspond to different c.m. scattering angles in two reactions that have different external masses. (For example, $\phi\phi\to\phi\phi$ vs $A_L\phi\to\phi\phi$.) This is a severe problem for nonrelativistic scattering, but once the particles are relativistic the relation between t and c.m. scattering angle becomes mass independent:

$$t\approx\tfrac{1}{2}s(-1+\cos\theta)\ .$$

Thus, for relativistic particles, relations (3.21) imply relations between the c.m. partial-wave amplitudes such as

$$a_J(A_L^\alpha\phi_j\to\phi_k\phi_l)=a_J(\phi_i\phi_j\to\phi_k\phi_l)u_i^\alpha\ , \tag{3.23}$$

corresponding to (3.21a). The restriction to relativistic scattering means only that the c.m. energy be larger than M_j,M_k,M_l. (The masses of A_L and the Goldstone mode $\phi_i u_i^\alpha$ are already negligible.) No assumption is made about the relative magnitude of the c.m. energy and the masses M_n of the exchanged particles.

It is thus only necessary to examine the partial-wave amplitudes for the fundamental process $\phi_i\phi_j\to\phi_k\phi_l$. To avoid a confusion of subscripts it is convenient to label the process $\phi_1\phi_2\to\phi_3\phi_4$.

1. $J=0$ amplitudes

The $J=0$ partial-wave transform of (3.9) is

$$a_0(\phi_1\phi_2\to\phi_3\phi_4)=\frac{-f_{1234}}{16\pi}+\sum_n R_n(s)\ ,$$

$$16\pi R_n(s)=\frac{e_{12n}e_{34n}}{s-M_n{}^2} + (e_{13n}e_{24n}+e_{14n}e_{23n})\frac{1}{s}\ln\left[1+\frac{s}{M_n{}^2}\right]\ . \tag{3.24}$$

At $s=\infty$, $R_n=0$ and the amplitude satisfies tree unitarity if $|f_{1234}|<16\pi$. For finite s it appears that small quartic couplings will not guarantee tree unitarity because of the possibility of very large cubic couplings e_{ijk}.

Remarkably, these cubic couplings are also bounded. Denote by $\bar{f}$ the maximum quartic coupling in the potential:

$$\bar{f}=\max|f_{ijkl}| \ .$$

Then the cubic couplings are bounded by $\bar{f}$ and a Higgs-boson mass:

$$e_{ijk}{}^2<(\tfrac{16}{5})\bar{f}\max(M_i{}^2,M_j{}^2,M_k{}^2) \ . \tag{3.25}$$

This result is proved in Appendix C.

To make use of this it is convenient to define M as the largest mass of the external particles:

$$M^2=\max(M_1{}^2,M_2{}^2,M_3{}^2M_4{}^2) \ . \tag{3.26}$$

Then (3.25) implies

$$\frac{5\pi}{\bar{f}}|R_n(s)|<\max(M^2,M_n{}^2)\left[\left|\frac{1}{s-M_n{}^2}\right|+\frac{2}{s}\ln\left[1+\frac{s}{M_n{}^2}\right]\right] . \tag{3.27}$$

If all four external particles have the same mass, then kinematics requires $s>4M^2$. If one particle is much heavier than the other three then the kinematic minimum is $s>M^2$. In addition, we want s large enough that we can neglect external masses and use (3.23). In the following we keep $s\geq 5M^2$.

The bound (3.27) is plotted in Fig. 6 for various values of $M_n{}^2/M^2$ between 1 and 10^{-4}. The bound falls with energy like M^2/s. At the lowest energy $s=5M^2$ it is roughly $(0.4)\ln(5M^2/M_n{}^2)$ for $M_n{}^2\ll M^2$. However, since only scalars with masses larger than M_W and M_Z have been kept, the exchanged mass M_n must be at least 100 GeV. For $M_n{}^2/M^2=10^{-4}$, the external mass M is already 10^4 GeV.

The same bound (3.27) is plotted in Fig. 7 for various values of $M_n{}^2/M^2$ between 1 and 10^4. For s below the resonance mass $M_n{}^2$ the value is 3; above the resonance it falls with energy like $M_n{}^2/s$. Inclusion of the finite width

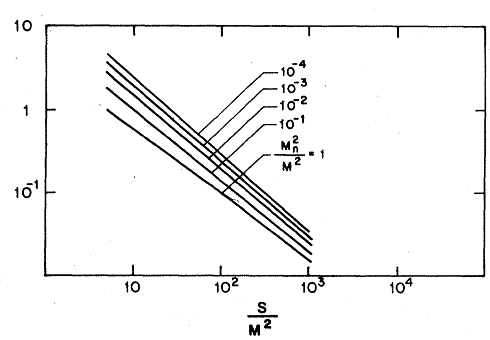

FIG. 6. The bound (3.27) in the relativistic regime $s>5M^2$ for various values of the exchanged-scalar mass $M_n{}^2\leq M^2$.

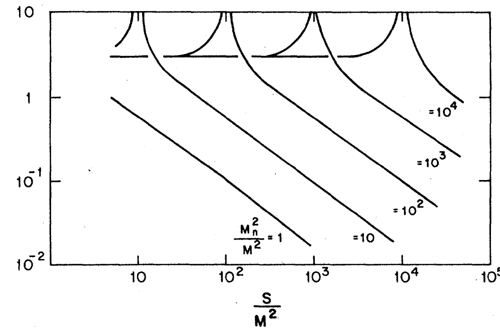

FIG. 7. The bound (3.27) in the relativistic regime $s\geq 5M^2$ for various values of the exchange-scalar mass $M_n{}^2\geq M^2$. For each value of $M_n{}^2$ the bound is constant at the value 3 at energies below the resonance pole $s=M_n{}^2$.

of the ϕ_n boson exchanged in the s channel will, of course, guarantee elastic unitarity at the resonance energies and allows us to ignore the pole structure in Fig. 7.

We can summarize the results of Figs. 6 and 7 in the crude bound

$$\frac{5\pi}{\bar{f}}|R_n(s)|<3 \ .$$

Actually this is too crude. The value 3 arises because in passing from (3.24) to (3.27) we have assumed a worst-case scenario in which a particular scalar ϕ_n has appropriate quantum numbers to be exchanged in the s, t, and u channels. Actually, the value 3 should be replaced by the number of channels (3, 2, 1, or 0) available to ϕ_n. From (3.24) a_0 is a sum over the various R_n so that

$$|a_0(\phi_1\phi_2\to\phi_3\phi_4)|<N\bar{f}/5\pi \ , \tag{3.28}$$

where N is the total number of scalar-exchange diagrams allowed by the quantum numbers. Tree unitarity is guaranteed if

$$|\bar{f}|<5\pi/N \ . \tag{3.29}$$

2. $J>0$ amplitudes

The partial wave projection of (3.9) for $J>0$ is

$$a_J(\phi_1\phi_2\to\phi_3\phi_4)=\frac{1}{16\pi}\sum_n[e_{13n}e_{24n}+(-1)^Je_{14n}e_{23n}]\times\frac{1}{M_n{}^2}r_{J,n}(s) \ ,$$
$$r_{J,n}(s)=\frac{2M_n{}^2}{s}Q_J\left[1+\frac{2M_n{}^2}{s}\right] , \tag{3.30}$$

and Q_J are the associated Legendre functions. To bound this one can use the inequality (3.25):

$$|a_J(\phi_1\phi_2\to\phi_3\phi_4)|\leq\frac{\bar{f}}{5\pi}\sum_n\frac{\max(M^2,M_n{}^2)}{M_n{}^2}2r_{J,n}(s) \ , \tag{3.31}$$

where the factor 2 comes from assuming that both t and

u channels contribute.

Again there are two cases. For small values of $M_n{}^2/M^2$ each term in (3.31) has a similar energy dependence to Fig. 6. The maximum occurs at the minimum energy $s=5M^2$ and is roughly $(0.4)\ln(5M^2/M_n{}^2)$ for $M_n{}^2 \ll M^2$ just as before.

For large values of $M_n{}^2/M^2$ each term in (3.31) is just $2r_J$ and as shown in Fig. 5, $r_J < 0.1$ at all energies. Thus, in either case the bound (3.28) generalizes to higher partial waves

$$|a_J(\phi_1\phi_2 \to \phi_3\phi_4)| < N\bar{f}/5\pi ,$$

where N is now the total number of t or u channel scalar-exchange diagrams that are allowed by the quantum numbers. From this and (3.21) it follows that all scatterings of scalar and vector bosons have small partial waves provided the quartic Higgs couplings are small. There is no additional assumption necessary about the cubic couplings or the scalar mass spectrum.

ACKNOWLEDGMENTS

This work was supported in part by a grant from the National Science Foundation.

APPENDIX A: COUPLING-CONSTANT SUM RULES

The various sum rules used in Secs. II C and III A are based on the gauge invariance of the potential V. The first four derivatives of V define the masses and couplings:

$$0=\partial V/\partial\phi_i\,|_v,\quad M^2{}_{ij}=\partial^2 V/\partial\phi_i\partial\phi_j\,|_v ,$$
$$e_{ijk}=\partial^3 V/\partial\phi_i\partial\phi_j\partial\phi_k\,|_{\phi=v},$$
$$f_{ijkl}=\partial^4 V/\partial\phi_i\partial\phi_j\partial\phi_k\partial\phi_l\,|_v .$$

After shifting the fields by v, $\phi_i=v_i+\phi_i'$, the potential expressed in terms of ϕ' may be written as (3.6). However, because the original fields ϕ_i have simpler gauge transformation properties it is more convenient to stay with them.

V is invariant under $\phi_i \to \phi_i+\epsilon^\alpha\theta^\alpha_{in}\phi_n$. Consequently, the function

$$W^\alpha=(\partial V/\partial\phi_i)\theta^\alpha_{in}\phi_n$$

has the property that $W^\alpha=0$ for any value of ϕ. Evaluating

$$\partial W^\alpha/\partial\phi_j,\quad \partial^2 W^\alpha/\partial\phi_j\partial\phi_k,\quad \partial^3 W^\alpha/\partial\phi_j\partial\phi_k\partial\phi_l$$

at the point $\phi=v$ gives

$$M^2{}_{ij}(\theta^\alpha v)_i=0 , \tag{A1}$$

$$[\theta^\alpha,M^2]_{jk}=e_{ijk}(\theta^\alpha v)_i , \tag{A2}$$

$$\theta^\alpha_{jn}e_{nkl}+\theta^\alpha_{kn}e_{njl}+\theta^\alpha_{ln}e_{njk}=f_{ijkl}(\theta^\alpha v)_i . \tag{A3}$$

Result (A1) is Goldstone's theorem that $M^2{}_{ij}$ has an eigenvector with zero eigenvalue for each symmetry of the potential that is spontaneously broken ($\theta^v\neq 0$). All three results were derived by Weinberg.[9]

From these, several other identities follow. Multiplying (A3) by $(\theta^\beta v)_j$ and symmetrizing the result in (α,β) yields

$$\sum_n[\tfrac{1}{2}(\{\theta^\alpha,\theta^\beta\}v)_n e_{nkl}+(\theta^\alpha_{kn}\theta^\beta_{nl}+\theta^\beta_{kn}\theta^\alpha_{nl})(M_n{}^2-\tfrac{1}{2}M_k{}^2-\tfrac{1}{2}M_l{}^2)]=-f_{ijkl}(\theta^\alpha v)_i(\theta^\beta v)_j \tag{A4}$$

in the basis where the scalar mass matrix is diagonal. If this is multiplied by $(\theta^\gamma v)_k$ and then symmetrized (α,β,γ) the result is

$$\sum_n[\tfrac{1}{2}(\{\theta^\alpha,\theta^\beta\}v)_n\theta^\gamma_{nl}+\tfrac{1}{2}(\{\theta^\alpha,\theta^\gamma\}v)_n\theta^\beta_{nl}+\tfrac{1}{2}(\{\theta^\beta,\theta^\gamma\}v)_n\theta^\alpha_{nl})(M_n{}^2-\tfrac{2}{3}M_l{}^2)=f_{ijkl}(\theta^\alpha v)_i(\theta^\beta v)_j(\theta^\gamma)_k . \tag{A5}$$

Multiply (A5) by $(\theta^\delta v)_k$ and symmetrizing in $(\alpha,\beta,\gamma,\delta)$ yields

$$\sum_n M_n{}^2\tfrac{1}{4}[(\{\theta^\alpha,\theta^\beta\}v)_n(\{\theta^\gamma,\theta^\delta\}v)_n+(\{\theta^\alpha,\theta^\gamma\}v)_n(\{\theta^\beta,\theta^\delta\}v)_n+(\{\theta^\alpha,\theta^\delta\}v)_n(\{\theta^\beta,\theta^\gamma\}v)_n]=f_{ijkl}(\theta^\alpha v)_i(\theta^\beta v)_j(\theta^\gamma v)_k(\theta^\delta v)_l . \tag{A6}$$

Identities (A3), (A4), (A5), and (A6) are used in obtaining the $s\to\infty$ limit of the amplitudes for $A\phi\to\phi\phi$, $AA\to\phi\phi$, $AA\to A\phi$, and $AA\to AA$ in Sec. III. Result (A6) is necessary in Sec. II for the special case of W and Z scattering.

APPENDIX B: THE LONGITUDINAL-GAUGE-BOSON THEOREM

Here is a simple proof of the theorem (3.22) for an arbitrary ξ gauge. To the Lagrangian (3.1) one must add a gauge-fixing term

$$\mathscr{L}_{\rm gf}=\tfrac{1}{2}\xi\sum_\alpha\left[\partial^\nu A^\alpha_\nu+\frac{i}{\xi}\phi_i\theta^\alpha_{ij}v_j\right]^2 .$$

The resulting vector and scalar propagators are[9]

$$D^\alpha_{\sigma\nu}(k)=\frac{g_{\sigma\nu}}{k^2-\mu_\alpha{}^2}+\frac{(1-\xi)k_\sigma k_\nu}{(k^2-\mu_\alpha{}^2)(\xi k^2-\mu_\alpha{}^2)} ,$$
$$D_{ij}(k)=\frac{\delta_{ij}}{k^2-M_j{}^2}-\sum_\beta\frac{(\theta^\beta v)_i(\theta^\beta v)_j}{k^2(\xi k^2-\mu_\beta{}^2)} . \tag{B1}$$

The T matrices computed from these are ξ independent.

The gauge-fixing term induces the constraint

$$\partial^\nu A^\alpha_\nu = \frac{-i}{\xi}\phi_i \theta^\alpha_{ij} v_j \ . \tag{B2}$$

To compute a scattering amplitude $T(A^\alpha_L,\dots)$ one must amputate the external leg of the Green's function $G(A^\alpha_\sigma,\dots)$. Since

$$(k^2-\mu_\alpha{}^2)\epsilon^\sigma_L D^\alpha_{\sigma\nu} = \epsilon_{L\nu} \ ,$$

where $\epsilon^\sigma_L(K) = (|\vec{k}|, E\hat{k})/\mu_\alpha$, the appropriate reduction formula is

$$T(A^\alpha_L,\dots) = (k^2-\mu_\alpha{}^2)\epsilon^\sigma_L G(A^\alpha_\sigma,\dots) \ . \tag{B3}$$

At high energy ($s \gg \mu_\alpha{}^2$)

$$\epsilon^\sigma_L = k^\sigma/\mu_\alpha + O(\mu_\alpha/\sqrt{s}) \ .$$

Since the vector propagator (B1) has the property

$$(\xi k^2-\mu_\alpha{}^2)k^\sigma D^\alpha_{\sigma\nu} = k_\nu \ ,$$

(B3) can be replaced at high energy by

$$T(A^\alpha_L,\dots) = (\xi k^2-\mu_\alpha{}^2)\frac{k^\sigma}{\mu_\alpha}G(A^\alpha_\sigma,\dots) \ . \tag{B4}$$

The constraint (B2) in momentum space is

$$k^\sigma A^\alpha_\sigma = \frac{1}{\xi}\phi_i \theta^\alpha_{ij} v_j \ ,$$

so that (B4) is equivalent to

$$T(A^\alpha_L,\dots) = \left[k^2 - \frac{\mu_\alpha{}^2}{\xi}\right] G(\phi_i,\dots)\frac{(\theta^\alpha v)_i}{\mu_\alpha} \ . \tag{B5}$$

Since the scalar propagator has the property

$$\left[k^2 - \frac{\mu_\alpha{}^2}{\xi}\right](\theta^\alpha v)_i D_{ij} = (\theta^\alpha v)_j \ ,$$

(B5) is precisely the reduction formula for the Goldstone-boson field:

$$T(A^\alpha_L,\dots) = T(\phi_i,\dots)\frac{(\theta^\alpha v)_i}{\mu_\alpha} \ . \tag{B6}$$

This theorem holds for each longitudinal gauge boson and thus explains the observations (3.21).

APPENDIX C: BOUNDS ON CUBIC COUPLINGS

The upper limits on the partial-wave amplitudes in Sec. III C are obtained from the bounds on the dimensionful cubic couplings to be derived here. The potential expressed in terms of the shifted fields ϕ', where $\phi_i(x) = v_i + \phi'_i(x)$, has the form

$$V = \tfrac{1}{2}M^2{}_{ij}\phi'_i\phi'_j + \tfrac{1}{6}e_{ijk}\phi'_i\phi'_j\phi'_k + \tfrac{1}{24}f_{ijkl}\phi'_i\phi'_j\phi'_k\phi'_l \ . \tag{C1}$$

The shifted field is defined by the property $\langle\phi'_i\rangle = 0$. Therefore, the minimum value of V is zero. This means that e_{ijk} cannot be too large or else V will become negative for some values of ϕ'_i.

It is convenient to let $\phi'_i = n_i r$, where $\sum_i n_i{}^2 = 1$. Then

$$V = \tfrac{1}{2}\widetilde{M}^2 r^2 + \tfrac{1}{6}\widetilde{e} r^3 + \tfrac{1}{24}\widetilde{f} r^4 \ , \tag{C2}$$

$$\widetilde{M}^2 = M^2{}_{ij}n_i n_j \ ,$$

$$\widetilde{e} = e_{ijk}n_i n_j n_k \ , \tag{C3}$$

$$\widetilde{f} = f_{ijkl}n_i n_j n_k n_l \ .$$

Expression (C2) must be positive for any $r \neq 0$. For the choice $r = -2\widetilde{e}/\widetilde{f}$,

$$V = 2\left[\widetilde{M}^2 - \frac{\widetilde{e}^2}{3\widetilde{f}}\right]\left[\frac{\widetilde{e}}{\widetilde{f}}\right]^2 . \tag{C4}$$

For this to be positive requires

$$\widetilde{e}^2 < 3\widetilde{f}\widetilde{M}^2 \ . \tag{C5}$$

From the inequality we can obtain a bound on individual couplings e_{ijk}. It is convenient to work in the basis in which $M^2{}_{ij}$ is diagonal and to focus on a particular subspace spanned by ϕ_1, ϕ_2, ϕ_3. Let $S = \{1,2,3,\}$ and define

$$\bar{f} = \max_S |f_{ijkl}| \ ,$$
$$M^2 = \max_S (M_i{}^2) \ . \tag{C6}$$

Then (C5) implies the weaker result

$$(e_{ijk}n_i n_j n_k)^2 < 3\bar{f}M^2 \ , \tag{C7}$$

where $n = (n_1,n_2,n_3)$ is an arbitrary vector satisfying

$$n_1{}^2 + n_2{}^2 + n_3{}^2 = 1 \ . \tag{C8}$$

For $n_1 = 1$, $n_2 = n_3 = 0$ this yields

$$e_{111}{}^2 < 3\bar{f}M^2 \ , \tag{C9}$$

and similarly for e_{222} and e_{333}.

To obtain bounds on mixed couplings e_{122}, e_{123}, etc., takes more work. We write our (C7) for eight different unit vectors $(\pm n_1, \pm n_2, \pm n_3)$ where all signs are chosen independently. The sum of these inequalities is

$$(e_{111}n_1{}^2 + 3e_{122}n_2{}^2 + 3e_{133}n_3{}^2)n_1{}^2 + (1\leftrightarrow 2) + (1\leftrightarrow 3) + (6e_{123}n_1n_2n_3)^2 < 3\bar{f}M^2 \ . \tag{C10}$$

Since the left side is a sum of positive terms, it implies

$$(6e_{123}n_1n_2n_3)^2 < 3\bar{f}M^2$$

for all n_i. The choice $n_1{}^2 = n_2{}^2 = n_3{}^2 = \frac{1}{3}$ yields

$$e_{123}{}^2 < \tfrac{9}{4}\bar{f}M^2 \ . \tag{C11}$$

To bound mixed couplings of the form e_{ijj} we deduce from (C10)

$$(e_{111}n_1{}^2 + 3e_{122}n_2{}^2 + 3e_{133}n_3{}^2)^2 n_1{}^2 < 3\bar{f}M^2 \ .$$

To bound e_{122}, choose $n_1{}^2 = \frac{1}{3}$, $n_2{}^2 = \frac{2}{3}$, $n_3{}^2 = 0$:

$$(e_{111} + 6e_{122})^2 < 81\bar{f}M^2 \ . \tag{C12}$$

There are three possibilities to consider. Case I: e_{111} and e_{122} have the same sign. Then (C12) implies

$$e_{122}{}^2 < \tfrac{9}{4}\bar{f}M^2 \ . \tag{C13a}$$

Case II: e_{111} and e_{122} have opposite signs and $6|e_{122}| > |e_{111}|$. Then (C12) implies

$$6|e_{122}| < |e_{111}| + 9\bar{f}^{1/2}M \ .$$

Using (C9) yields the bound

$$e_{122}{}^2 < \tfrac{1}{6}(14+3\sqrt{3})\bar{f}M^2 \ . \qquad \text{(C.13b)}$$

Case III: e_{111} and e_{122} have opposite signs and $6|e_{122}| < |e_{111}|$. Then (C9) immediately yields

$$e_{122}{}^2 < \tfrac{1}{12}\bar{f}M^2 \ . \qquad \text{(C13c)}$$

The coefficients on the right-hand sides of (C9), (C11), (C13a), (C13b), and (C13c) are all less than $\frac{16}{5}$. Consequently,

$$e_{ijk}{}^2 < \tfrac{16}{5}\bar{f}\max(M_i{}^2, M_j{}^2, M_k{}^2) \qquad \text{(C14)}$$

for any choice of i,j,k.

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