Hadronic τ decay, pion radiative decay, and pion polarizability

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Using conserved vector current and the first Weinberg sum rule, it is shown that the "missing" 10% of the hadronic τ decay could come from the axial-vector current. This implies for $\pi \to e\nu\gamma$ decay, $F_A(0)/F_V(0)=0.50\pm0.15$, and a charged-pion polarizability $\alpha_\pi=5\times10^{-42}$ cm^3.

Using the conserved-vector-current hypothesis[1] (CVC) and the experimental data on $e^+e^- \to$ even number of pions, it is straightforward to calculate the following branching ratio (B) of the hadronic heavy-lepton τ decay:[2]

$$\begin{aligned} &B(\tau^+ \to \pi^+\pi^0\nu)=(24\pm1)\% \ , \\ &B(\tau^+ \to \pi^+\pi^+\pi^-\pi^0\nu)=(5\pm0.5)\% \ , \\ &B(\tau^+ \to \pi^+ 3\pi^0\nu)=(1.2\pm0.0)\% \ , \end{aligned} \tag{1}$$

where we have assumed $B(\tau \to e\nu\bar{\nu})=17.5\%$, compared with the experimental averaged branching ratio[3] of $(17.5\pm1.4)\%$. $B(\tau \to \pi^+\nu)$ can be calculated without any ambiguity and is equal to 11%. These results together with the experimental $B(\tau^+ \to \pi^+\rho^0\nu)=(5.4\pm1.7)\%$ enable us to compute the quantities

$$\begin{aligned} &B(\tau \to \text{one prong})=(77\pm2.4)\% \ , \\ &B(\tau \to \text{three prongs})=(10.6\pm2)\% \end{aligned} \tag{2}$$

compared with the experimental results, $(86\pm3)\%$ and 14%,[4] respectively. We have added to Eq. (2) 2% contribution of strange-particle decays (K,K^*,Q_1,Q_2) estimated by a parton-model calculation and we have neglected the π', and 5π contributions to three- and five-prong events which were shown to be very small.[5,6] From Eq. (2), it is clear that about 10% of the hadronic mode of the one-prong type is unaccounted for. The origin of this discrepancy could be due to a statistical fluctuation and/or an inaccurate measurement of e,μ branching ratio. In this article, we would like to point out that the 10% missing hadronic events could be real and in fact due to the axial-vector matrix element. This is so because of the symmetry between axial-vector and vector-current matrix elements as implied by the first Weinberg sum rule: the vector current (via CVC) contributes to a total branching ratio of $(30\pm1)\%$, while the axial-vector current contributes to a known experimental branching ratio of $(20.8\pm3.4)\%$ and hence the missing 10% is due to the axial-vector-current matrix element.[7] The missing events could come from the 3π continuum or the existence of a second axial-vector meson resonance A_1'. As a consequence of this analysis, the axial-vector form factor in $\pi \to e\nu\gamma$ is calculated. It is found that the ratio $\gamma=F_A(0)/F_V(0)=0.5$, which is independent of the nature of the missing 10% hadronic events. It agrees with one of the two solutions obtained from $\pi \to e\nu\gamma$ experiments. The charged-pion polarizability deduced from this calculation agrees in sign and magnitude with that given by a recent indirect measurement.

We begin first by writing down the formula for the hadronic decay rate:

$$\frac{\Gamma\left[\tau \to \begin{bmatrix} V \\ A \end{bmatrix} \nu\right]}{\Gamma(\tau \to e\nu\bar{\nu})} = \frac{6\pi}{m_\tau^2}\int_{m_\pi^2}^{m_\tau^2} ds \left[1-\frac{s}{m_\tau^2}\right]^2 \times \left[1+\frac{2s}{m_\tau^2}\right]\begin{bmatrix} v_1(s) \\ a_1(s)\end{bmatrix} , \tag{3}$$

where v_1 and a_1 are, respectively, the spin-1 vector and axial-vector spectral functions. The spin-0 part of the axial-vector spectral function $a_0(s)$ is given by a similar expression and is not written. From the previous estimation of the decay constant of $\pi'(1300)$, $F_{\pi'}=5$–6 MeV,[6] its branching ratio is completely negligible. We assume in the remainder of this article that this is also true for the continuum contribution to the function $a_0(s)$, which can be estimated from the quark-parton model.[8]

The first Weinberg sum rule reads

$$\int ds\, v_1(s)=\int ds\, a_1(s)+2\pi f_\pi^2 \ , \tag{4}$$

where $f_\pi=133$ MeV. The axial-vector form factor in $\pi \to e\nu\gamma$ decay is given by[9,10]

$$F_A(0)=\frac{1}{2\pi f_\pi}\int \frac{ds}{s}[a_1(s)-v_1(s)]+f_\pi\frac{\langle r^2\rangle}{3} \ . \tag{5}$$

Equations (3)–(5) represent our present knowledge of the low-energy axial-vector form-factor matrix elements. Obviously Eq. (5) is least sensitive to the high-energy behavior of the spectral function, while Eq. (4) is sensitive because it depends on how good is the high-energy cancellation between the axial-vector and vector-form factors. Fortunately, quantum chromodynamics (QCD) and experimental data on $e^+e^- \to$ hadrons provide a reasonable estimate of the region where the asymptotic freedom in QCD applies. We take a reasonable value $s=N=3$ GeV^2 above all the known isovector mesons; above N,

$v_1(s)=a_1(s)+a_0(s)$ to a good accuracy as can be shown from QCD.[8]

Using experimental data on $e^+e^-\to\pi^+\pi^-$ and $e^+e^-\to 4\pi$ and Eq. (4) we obtain

$$\int v_{2\pi}(s)ds=0.27\ \mathrm{GeV}^2\ ,$$

$$\int v_{4\pi}(s)ds=0.25\ \mathrm{GeV}^2\ ,$$

and hence

$$\int_{9m_\pi^2}^{N} ds\, a_1(s)=0.41\ \mathrm{GeV}^2\ . \tag{6}$$

Note that for our value of N, the 4π contribution is as important as the ρ contribution in the Weinberg sum rule. We now demand that the A_1 contribution to Eq. (6) in the form of $\pi\rho$ resonance yields approximately an experimental branching ratio fo 12%. A δ-function approximation for an A_1 resonance with an experimental width of 300 MeV underestimates its effect in the Weinberg sum rule by as much as 60% compared with a numerical integration of a current-algebra model[5] which correctly takes into account the appropriate phase-space factor and finite width of the A_1 resonance. If we use, however, the δ-function approximation in both the Weinberg sum rule Eq. (4) and the expression for $\tau\to A_1\nu$ decay, Eq. (3), then the error cancels out. For clarity we use the δ-function approximation.

Denoting the A_1 contribution ($m_{A_1}=1.27$ GeV) in $a_1(s)$ by

$$2\pi f_1{}^2 m_1{}^{-2}\delta(s-s_1)$$

and requiring it to yield a branching ratio of 12%, we have

$$2\pi f_1{}^2 m_1{}^{-2}=0.21\ . \tag{7}$$

(As long as m_1 is not at the edge of the τ-decay phase space, this expression is insensitive to the value of m_1 used.) Using this value in the Weinberg sum rule Eq. (4), the remaining contribution $\Delta a_1(s)$ to the axial-vector spectral function is

$$\int \Delta a_1(s)ds=0.20\ \mathrm{GeV}^2\ . \tag{8}$$

Equation (8) is the basis for the following phenomenological analysis. In the following, we consider its consequence on $\tau\to$axial-vector current$+\nu$ decay and $\pi\to e\nu\gamma$ decay.

I. τ DECAY

Because the axial-vector spectral function contribution is given in the integral form, it can be either in the form of a second axial-vector meson resonance or a continuum.

(a) Second axial-vector meson resonance A_1'. It is not out of the question to consider the possibility of the existence of a second axial-vector resonance of higher mass besides the well-known A_1 resonance. This is so because there are two strange axial-vector mesons Q_1 (1.28 GeV) and Q_2 (1.40 GeV). Using the symmetry argument, we expect to have a second A_1 resonance A_1', with essentially the same mass as Q_2. (We can ignore the difference between up-, down-, and strange-quark masses.) Denoting, the decay constant and mass of this second resonance, respectively, as f_2 and $m_2=1.4$ GeV, using Eq. (8), we have

$$2\pi f_2{}^2 m_2{}^{-2}=0.20\ \mathrm{GeV}^2\ , \tag{9}$$

which is the same value as that of A_1 resonance, Eq. (7). The τ-decay branching ratio for this second resonance is 8%, which is very near to the missing 10% of the hadronic events. If the experimental data on one prong are correct, this resonance must decay mostly to one-prong events (the neutral modes which are not detected could come from π^0, η^0, K_L). Likely candidates for the decay modes are $\eta\eta\pi^+$ for $K\bar{K}\pi$ which decays two-thirds of the time as one-prong events.

(b) Continuum contribution. The continuum contribution to $a_1(s)$ could be in the form of 3π or 5π. . . . Using current algebra, the 5π contribution was found to be very small.[5] The 3π contribution could be in the form of π "σ" where "σ" is a correlated 2π, $i=0$, S state. In this case

$$\Gamma(\tau\to\pi^+\pi^+\pi^-\nu)/\Gamma(\tau\to\pi^+\pi^0\pi^0\nu)=2\ ,$$

which is not what we need to account for the "missing" 10% one-prong events. We cannot say, however, in general, what the value of this ratio is (continuum events could also come from $\eta\eta\pi$ and $KK\pi$). Assuming the axial-vector-current continuum starts at $s_0=1.4$ GeV^2 with the value given by QCD, i.e., parton model, its branching ratio is 7.5%, which is what we need. However, within the wisdom of the QCD sum rule and phenomenology this choice of s_0 is rather low; it should be higher than the mass of the lowest axial-vector meson resonance; $s_0\geq 1.7$ GeV^2; in this case its branching ratio would be 4.5%, which would not present a convincing argument for the missing events.

II. $\pi\to e\nu\gamma$ DECAY

The remainder of this paper is devoted to study the implication of Eq. (8) on the axial-vector form factor in $\pi\to e\nu\gamma$ and hence the pion polarizability. We begin first by showing, to a good degree of accuracy, that the 2π contribution to the integral of the vector spectral function $v_{2\pi}(s)$ cancels out the $\langle r^2\rangle$ term in Eq. (5). Because each of these two terms is large [the magnitude of each is about 2.4 times larger than $F_V(0)$], we cannot use experimental data on $\langle r^2\rangle$ which has a large uncertainty. Instead, using analyticity and unitarity we can show that the integrals of $v_{2\pi}(s)$ and $\langle r^2\rangle$ are closely related and cancel out. Experimental data on the pion form factor in the timelike region and, to a lesser degree, the spacelike pion form factor can be used to control approximations and assumptions made.

There is confusion in the literature on the question of whether to use subtracted or unsubtracted dispersion relations for the pion form factor. Of course, we must use the subtracted dispersion relation. But it is important to require that the timelike pion form factor must satisfy the final-state theorem, which states that below the inelastic threshold, a condition which is practically valid for $s\lesssim 1$ GeV^2, the phase of the pion form factor is the same as the P-wave pion-pion phase shift. Once this condition is imposed together with the experimental P-wave phase shift, one cannot have the flexibility associated with the question of subtracted or unsubtracted dispersion relation as frequently discussed in the literature.

There are two steps involved in showing the cancellation. The first consists in showing the validity of the one-pole formula of Frazer-Fulco[11] or the Gounaris-Sakurai[12] formula with parameters adjusted to give the observed ρ mass and width. To do this, we can use the crossing-symmetric P-wave Roy equation[13] to study the P-wave phase shift. Using experimental data on the S and P waves in the Roy equation, it is straightforward to show that in the low-energy region, and in the vicinity of the ρ resonance, the correction due to the left-hand cut is almost canceled out by the appropriate subtraction constants and amounts to a correction of only a few degrees phase shift. Hence we can write $[s=4(\nu+m_\pi^2)]$

$$F_\pi(\nu)=\frac{\nu_\rho+m_\pi^2[1-\gamma h(-m_\pi^2)]}{\nu_\rho-\nu+\gamma\nu h(\nu)-i\gamma\nu^{3/2}/(\nu+m_\pi^2)^{1/2}}g(\nu)\ , \quad (10)$$

where $h(\nu)$ is the well-known logarithm function;[11,12] the factor multiplying $g(\nu)$ is the usual pion-form-factor formula, and $g(\nu)$ simulates the inelastic effect and possibly the polynomial ambiguity, with $g(0)=1$. Equation (10) provides an excellent description of pion-form-factor data $-4\leq s\leq 1.8$ GeV^2 (Ref. 14). From experimental data, we have $s_\rho=25.8m_\pi^2$ and $\gamma=0.184$ (s_ρ is defined such that the P-wave $\pi\pi$ phase shift is equal to 90° at 770 MeV). Using the definition of the pion radius

$$\tfrac{1}{6}\langle r^2\rangle=\frac{1}{\pi}\int_{3m_\pi^2}^{\infty}\frac{ds}{s^2}\mathrm{Im}F_\pi(s) \quad (11a)$$

and $v_{2\pi}(s)$;

$$\int_{4m_\pi^2}^{\infty}\frac{ds}{s}v_{2\pi}(s)=\frac{1}{12\pi}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\left[1-\frac{4m_\pi^2}{s}\right]^{3/2}\times|F_\pi(s)|^2\ . \quad (11b)$$

Upon comparing Eqs. (11a) and (11b), using Eq. (10) and the fact that the integrand is peaked at the ρ mass while $g(s)$ is a slowly varying function of s, we have

$$\left\langle\frac{r^2}{3}\right\rangle-\frac{1}{2\pi f_\pi^2}\int\frac{ds}{s}v_{2\pi}(s)=\left\langle\frac{r^2}{3}\right\rangle\left[1-\frac{1}{24\pi f_\pi^2}\left[\frac{s_\rho}{\gamma}\right]g(s_\rho)\right]=-0.07\left\langle\frac{r^2}{3}\right\rangle\ , \quad (12)$$

where $\langle r^2\rangle=0.21m_\pi^{-2}$, and $g(s_\rho)=1.10$.

It remains to calculate the 4π contribution to $v_1(s)$, the A_1 and A_1' or continuum contribution to the right-hand side of Eq. (5). Using Eqs. (7), (9), and (12) and experimental data on $e^+e^-\rightarrow 4\pi$, we have

$$F_A(0)=(0.021+0.017-0.020-0.005)m_\pi^{-1}=0.013m_\pi^{-1}\ , \quad (13)$$

where the first term on the right-hand side represents the A_1 contribution, the second term the A_1' contribution (this value would change slightly if we used instead the continuum contribution), and the third term the 4π state, and the last term comes from Eq. (12); Using CVC, $F_V(0)=0.0265m_\pi^{-1}$ and hence finally

$$\gamma\equiv F_A(0)/F_V(0)=0.5 \quad (14)$$

with an estimated uncertainty of ± 0.15 which comes mostly from the uncertainty of the timelike pion form factor at the ρ peak. The experimental values for γ are 0.44 ± 0.12 or -2.36 ± 0.12 (Ref. 15) and 0.26 or -1.98 (Ref. 16). The negative solution for γ is of course ruled out by our calculation. The pion polarizability α_π calculated from our value of $F_A(0)$ is[17]

$$\alpha_\pi=\frac{e^2F_A(0)}{m_\pi f_\pi}=5\times 10^{-42}\ \mathrm{cm}^3\ , \quad (15)$$

which is consistent with that obtained by an indirect measurement using the reaction $\pi^-A\rightarrow\pi^-A\gamma$.[18]

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