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Mass-derivative formula and the singularity structure in thermo field dynamics

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We introduce a mass-derivative formula and show in the examples of the daisy diagrams and the effective potential in the manner of Coleman and Weinberg that there is no $\delta^N(k^2-m^2)$-type singularity in the perturbative calculations in thermo field dynamics. Other derivative formulas are also mentioned.

I. INTRODUCTION

The present note is concerned with an important feature regarding the nature of the perturbation expansion in the real-time formulation of finite-temperature field theory, known as thermo field dynamics (TFD).[1-12] Unlike the imaginary-time formalism,[13-18] in TFD $T=0$ and $T\neq 0$ parts are separated at the outset. The latter is accompanied by a δ function and thus one must deal with products of δ functions at higher orders.

The propagator in TFD has a 2×2 matrix structure involving two sets of fields, the ordinary field ϕ and the so-called tilde field $\tilde{\phi}$. For example, the scalar propagator $\Delta(k)$ is

$$-i\Delta(k)=U_B(k_0)\Delta_0(k)U_B(k_0)$$
$$=\Delta_\alpha(k)+\Delta_\beta(k)=\begin{pmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix}, \tag{1.1}$$

where U_B denotes a matrix for the Bogoliubov transformation, i.e.,

$$U_B(\omega)=\begin{pmatrix}(1-e^{-\beta\omega})^{-1/2} & (e^{\beta\omega}-1)^{-1/2}\\ (e^{\beta\omega}-1)^{-1/2} & (1-e^{-\beta\omega})^{-1/2}\end{pmatrix}, \tag{1.2}$$

and

$$\Delta_0(k)=\frac{\tau}{k^2-m^2-i\delta\tau}\quad(\delta\to+0), \tag{1.3a}$$

$$\Delta_\beta(k)=-2\pi i\delta(k^2-m^2)\frac{1}{e^{\beta|k_0|}-1}\begin{pmatrix}1 & e^{\beta|k_0|/2}\\ e^{\beta|k_0|/2} & 1\end{pmatrix}, \tag{1.3b}$$

$$\tau=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.3c}$$

The perturbative expansion then proceeds in the usual way with this propagator and the vertices generated by the Lagrangian $\mathscr{L}(\phi,\tilde{\phi})=\mathscr{L}(\phi)-\mathscr{L}(\tilde{\phi})$. Then in the higher-order calculation one finds δ^N $(N\geq 2)$ singularities. Our main concern here is to show that in TFD such singularities cancel with each other in contrast to the old real-time formalism.[17] (See also Refs. 7, 10, and 12 for explicit calculations.)

For such analysis it is of vital importance to note that we have, in (1.1), $U_B(k_0)$ rather than $U_B(\omega_k)$ with $\omega_k=(\vec{k}^2+m^2)^{1/2}$. Because of $\delta(k^2-m^2)$ no difference arises between them unless products of singular functions appear. But if they do, the choice of $U_B(k_0)$ becomes essential. For instance, the two-point function obtained through perturbative calculation is consistent with the one based on general considerations,[3] only when $U_B(k_0)$ is chosen.

II. THE MASS-DERIVATIVE FORMULA

The formula we will make use of to show the absence of a δ^N singularity is

$$\frac{1}{N!}\left(i\frac{\partial}{\partial m^2}\right)^N\Delta\tau=(\Delta\tau)^{N+1}, \tag{2.1}$$

which one can easily prove by noting, for instance,

$$2\pi\delta(k^2-m^2)=\frac{i}{k^2-m^2+i\delta}-\frac{i}{k^2-m^2-i\delta}. \tag{2.2}$$

In a zero-temperature perturbation calculation the formula similar to (2.1) has been used frequently, for instance, when one tries to extract the finite part of the Feynman diagrams. It is interesting, however, that it also holds at finite temperature, provided the propagator is replaced by the propagator given by the two by two matrix of Eq. (1.5). The formula will be used extensively to examine the singularity structure of many Feynman diagrams in TFD. We shall apply it in two typical cases where the δ^N singularity appears when the propagator in (1.1) is made use of. (For the use of the mass-derivative formula in the nonrelativistic case see Ref. 19.)

III. THE DAISY DIAGRAMS (REF. 17) WITH TWO AND N PETALS

Since we are concerned only with the physical sector the external legs in Fig. 1 are constructed from the ϕ fields and do not contain any $\tilde{\phi}$ contribution. The right-hand side (RHS) of Fig. 1 expresses explicitly the diagram in terms of

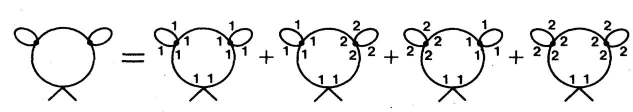

FIG. 1. Detailed structure of the daisy diagram with two petals in TFD.

the individual matrix elements of the propagator Δ given by Eq. (1.5). The petals are easy to calculate:

$$\int \frac{d^n k}{(2\pi)^n}\Delta_{11}=\int \frac{d^n k}{(2\pi)^n}\Delta_{22}=-\frac{m^2}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+1-\ln m^2\right]+\frac{m^2}{2\pi^2}\int_1^\infty dx\frac{(x^2-1)^{1/2}}{e^{\beta m x}-1}\ , \tag{3.1}$$

where $\epsilon=4-n$. The rest of the diagrams are identical to the (11) element of the RHS of (2.1) with $N=2$, which is given by

$$[(\Delta\tau)^3]_{11}=(\Delta_{11})^3-2(\Delta_{12})^2\Delta_{11}+(\Delta_{12})^2\Delta_{22}\ , \tag{3.2}$$

and reproduces the diagrams in Fig. 1. Here we used the relation $\Delta_{12}=\Delta_{21}$. One may simplify the calculation of the diagrams in Fig. 1 by using the mass-derivative formula as follows:

$$\frac{(-i)\lambda^3}{2}\left[\frac{\partial}{\partial m^2}\right]^2\int\frac{d^n k}{(2\pi)^n}\Delta_{11}=\frac{(-i)\lambda^3}{2}\left[\frac{1}{(4\pi)^2}\frac{1}{m^2}+\frac{1}{2\pi^2}\int_1^\infty dx(x^2-1)^{1/2}\left[\frac{\partial}{\partial m^2}\right]^2\frac{m^2}{e^{\beta m x}-1}\right]\ . \tag{3.3}$$

This shows that, although each diagram appearing on the RHS of Fig. 1 carries a δ^3 singularity, the result of the sum of these diagrams is well defined.

It is straightforward to generalize the calculation to the N-petal case (see Fig. 2) where one faces a δ^{N+1} singularity. The result is

$$\left[\frac{1}{(4\pi)^2}\left[\frac{\partial}{\partial m^2}\right]^N m^2\ln m^2+\frac{1}{2\pi^2}\int_1^\infty dx(x^2-1)^{1/2}\left[\frac{\partial}{\partial m^2}\right]^N\frac{m^2}{e^{\beta m x}-1}\right]\left[\frac{m^2}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+1-\ln m^2\right]+\frac{m^2}{2\pi^2}\int_1^\infty dx\frac{(x^2-1)^{1/2}}{e^{\beta m x}-1}\right]^N\ . \tag{3.4}$$

Another typical case where one faces the problem of δ^N is when one tries to evaluate the finite-temperature effective potential in the manner of Coleman and Weinberg.[20] The diagrams are given in Fig. 3. All of the external lines carry zero four-momenta.

It should be noted that Fig. 3 contains a diagram not included in Ref. 20. While the omission of this diagram at zero temperature is not of any importance in the calculation of the effective potential (at $T=0$) it does contribute a finite-temperature part which obviously cannot be absorbed in the zero-temperature renormalization.[21] The prescription for extracting the physical sector is to fix one of the legs to be ϕ and draw all diagrams compatible with the rules in TFD. (It does not matter which leg is chosen to be ϕ.) As for the first diagram in Fig. 3, one chooses the Δ_{11} propagator integrated over m^2. That this is correct can be seen by taking the derivative of the diagram and comparing it with the diagram in the tadpole method. The diagrams obtained in this manner are in one to one correspondence with daisy diagrams, except the first diagram, and thus coincide with the (11) element of (2.1) ($N=0,1,2,\ldots,\infty$). Their finite-temperature contribution is given by

$$\begin{aligned}\sum_{N=1}^\infty\frac{1}{2N}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{i}{2}\lambda\phi^2\right]^N[(\Delta\tau)^N-(\Delta(\beta=\infty)\tau)^N]_{11}\\ =\sum_{N=1}^\infty\frac{-i}{2N}\frac{1}{(N-1)!}\left[\frac{\lambda}{2}\phi^2\right]^N\left[\frac{\partial}{\partial m^2}\right]^{N-1}[\Delta_\beta\tau]_{11} \qquad (3.5)\\ =\sum_{N=1}^\infty\frac{-i}{2N}\frac{1}{(N-1)!}\left[\frac{\lambda}{2}\phi^2\right]^N\left[\frac{\partial}{\partial m^2}\right]^N\frac{2}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln\{1-\exp[-\beta(k^2+m^2)^{1/2}]\}\ . \qquad (3.6)\end{aligned}$$

The first diagram gives

$$-\frac{i}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln\{1-\exp[-\beta(k^2+m^2)^{1/2}]\}\ . \tag{3.7}$$

Adding (3.6) and (3.7) we obtain the well-known result

$$\sum_{N=1}^\infty\frac{-1}{\beta}\frac{[(\lambda/2)\phi^2]^N}{N!}\left[\frac{\partial}{\partial m^2}\right]^N\int\frac{d^3k}{(2\pi)^3}\ln\{1-\exp[-\beta(k^2+m^2)^{1/2}]\}=\frac{-i}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln\{1-\exp[-\beta(k^2+m^2+\tfrac{1}{2}\lambda\phi^2)^{1/2}]\}\ . \tag{3.8}$$

In this way the proof regarding the absence of δ^N terms is considerably simplified with the use of the mass-derivative formula.

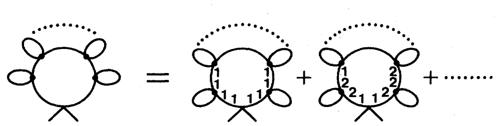

FIG. 2. Detailed structure of the daisy diagram with N petals in TFD.

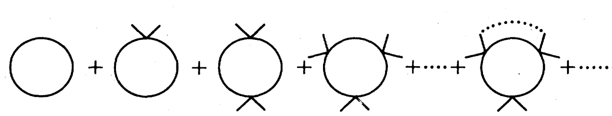

FIG. 3. Diagrams for finite-temperature effective potential.

In a similar manner the derivative formula is useful in calculating those tadpole diagrams[12] in which the external ϕ is attached to an internal line.

IV. OTHER DERIVATIVE FORMULAS

In this section we wish to demonstrate how certain other derivative formulas in zero-temperature field theory may be extended to finite temperature in a straightforward manner by means of TFD.

A. Fermion Mass-Derivative Formula

We have the relation

$$\frac{1}{N!}\left(i\frac{\partial}{\partial m}\right)^N S=(S)^{N+1} \ , \tag{4.1}$$

with

$$S(k)=i[S_0(k)+S_\beta(k)]=U_F S_0 U_F{}^{-1} \ , \tag{4.2}$$

where

$$U_F(\omega)\equiv\begin{pmatrix}[1-f_F(|\omega|)]^{1/2} & \sigma f_F{}^{1/2}(|\omega|)\\ -\sigma f_F{}^{1/2}(|\omega|) & [1-f_F(|\omega|)]^{1/2}\end{pmatrix} \ , \tag{4.3}$$

$$S_0(k)=(k+m)\frac{1}{k^2-m^2+i\delta\tau} \ , \tag{4.4}$$

$$S_\beta(k)=-(k+m)2\pi i\delta(k^2-m^2)\begin{pmatrix}f_F(|k_0|) & \sigma g_F(|k_0|)\\ \sigma g_F(|k_0|) & -f_F(|k_0|)\end{pmatrix} \ , \tag{4.5}$$

$$g_F(\omega)=\frac{e^{\beta\omega/2}}{e^{\beta\omega}+1},\quad f_F(\omega)=\frac{1}{e^{\beta\omega}+1} \ . \tag{4.6}$$

There are two choices for σ: $\sigma=1$ or $\epsilon(k_0)$ (1 for $k_0>0$ or -1 for $k_0<0$). The proof of (4.1) proceeds in a similar manner as (2.1). One may make use of this formula to repeat calculations similar to those in the preceding section.

B. Momentum-Derivative Formulas

We can prove the following relations:

$$i\frac{\partial}{\partial k_i}S=S\gamma_i S \ , \tag{4.7}$$

$$i\frac{\partial}{\partial k_i}\Delta\tau=2(\Delta\tau)k_i(\Delta\tau) \ , \tag{4.8}$$

where $i=1,2,3$. The proof for these formulas is the same as the one for the mass-derivative formulas.

The fact that the various zero-temperature-derivative formulas may be extended to finite temperature using TFD is rather nice since one would expect that, as in the zero-temperature case, they provide a useful computational device.

Incidentally, the derivative formulas (4.7) and (4.8) do not hold when $i=0$, because U_B and U_F depend on k_0. This reflects the fact that the temperature effect violates the Lorentz covariance through the appearance of f and g which contain k_0 only.

There is a different version of the k_0-derivative formulas. They are

$$[S(k)\gamma_0]^N S(k)=\frac{1}{N!}U_F(k_0)\left[\left(i\frac{\partial}{\partial k_0}\right)^N S_0(k)\right]U_F^\dagger(k_0) \ , \tag{4.9}$$

and

$$2^N[\Delta(k)\tau k_0]^N\Delta(k)=\frac{1}{N!}U_B(k_0)\left[\left(i\frac{\partial}{\partial k_0}\right)^N \Delta_0(k)\tau\right]U_B(k_0) \ . \tag{4.10}$$

These relations follow from $-iS=U_F S_0 U_F^\dagger$ and $-i\Delta=U_B\Delta_0 U_B$ when $U_F^\dagger U_F=1$ and $U_B\tau U_B=\tau$ are used. Note that the left-hand side (LHS) of these formulas describes a fermion line with many γ_0 vertices or a boson line with many k_0 vertices, each vertex having zero energy-momentum transfer. Therefore, these derivative formulas are sufficient for proving that such boson and fermion lines do not have any δ^N singularities.

V. CONCLUDING REMARKS

The main conclusion of this paper is that in perturbation expansion in TFD the δ^N singularities tend to cancel among themselves. In showing this, use was made of the mass-derivative formula in order to simplify the computations.

In closing we like to add a further observation. The formula (4.1) is closely related to the Ward-Takahashi (WT) relation of the form

$$-ik^\mu S'\left(p-\frac{k}{2}\right)\Gamma_\mu\left(p-\frac{k}{2},\ p+\frac{k}{2};\ k\right)S'\left(p+\frac{k}{2}\right)=S'\left(p+\frac{k}{2}\right)-S'\left(p-\frac{k}{2}\right) \ . \tag{5.1}$$

Here S' and Γ_μ contain quantum corrections. In particular, when S' and Γ_μ do not contain any quantum correction, the above equation reads

$$-ik^\mu S\left(p-\frac{k}{2}\right)\gamma_\mu S\left(p+\frac{k}{2}\right)=S\left(p+\frac{k}{2}\right)-S\left(p-\frac{k}{2}\right) \ . \tag{5.2}$$

At first glance, it would seem that this, with the limit $k\to 0$, would immediately lead to the derivative formula of the form $(i\partial/\partial k_\mu)S=S\gamma_\mu S$ for every μ. However, the real situation is not as simple as that. To show this, we first prove (5.2) for a nonzero k. The proof is as follows:

$$-ik^\mu S\left(p-\frac{k}{2}\right)\gamma_\mu S\left(p+\frac{k}{2}\right)=iS\left(p-\frac{k}{2}\right)\left\{\left[\left(p-\frac{k}{2}\right)\cdot\gamma+m\right]-\left[\left(p+\frac{k}{2}\right)\cdot\gamma+m\right]\right\}S\left(p+\frac{k}{2}\right)=S\left(p+\frac{k}{2}\right)-S\left(p-\frac{k}{2}\right) \ . \tag{5.3}$$

However, the limit $k\to 0$ in (5.2) is very tricky, because we cannot always equate

$$\lim_{k\to 0}k^\mu S\left(p-\frac{k}{2}\right)\gamma_\mu S\left(p+\frac{k}{2}\right) \tag{5.4}$$

to

$$\lim_{k \to 0} k^\mu S(p)\gamma_\mu S(p) \ , \tag{5.5}$$

when $S(p)$ contains a $\delta(p^2-m^2)$ term. The $k \to 0$ limit of the product of $\delta((p-k/2)^2-m^2)$ and $\delta((p+k/2)^2-m^2)$ or of the product of $\delta((p-k/2)^2+m^2)$ and $[(p+k/2)^2-m^2+i\delta]^{-1}$ is quite tricky. Therefore, we need to take particular care in order to obtain the derivative formulas from the WT relations. On the other hand, in TFD, $S(p)\gamma_\mu S(p)$ in (5.5) does not have any difficulty caused by $\delta(p^2-m^2)$ terms because the relation $U_F(p_0)U_F^\dagger(p_0)=1$ makes

$$S(p)\gamma_\mu S(p) = U_F(p_0)S_0(p)\gamma_\mu S_0(p)U_F(p_0) \ .$$

The latter is proportional to $U_F(p_0)[(\partial/\partial k_\mu)S_0(p)]U_F(p_0)$ because $S_0(p)$ satisfies all the derivative formulas. This explains why many of the derivative formulas hold in TFD. In TFD, $\delta(k^2+m^2)$ terms are created only through the Bogoliubov matrices, U_B or U_F. The products of $\delta(k^2-m^2)$ terms disappear in TFD, because the relations $U_B(k_0)\tau U_B(k_0)=\tau$ and $U_F(k_0)U_F^\dagger(k_0)=1$ eliminate all the Bogoliubov matrices associated with the inner vertices.[7] Although no δ^N singularities appear in TFD, the Bogoliubov matrices associated with both ends of a line do not disappear. Since these Bogoliubov matrices depend on k_0, they modify the form of k_0-derivative formulas as was seen in Sec. IV.

The above argument indicates also that we should be careful in reading the above WT relation; we presume that both sides of it are to be smeared out with certain square-integrable functions.

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[1]Y. Takahashi and H. Umezawa, Collect. Phenom. **2**, 55 (1975); L. Leplae, F. Mancini, and H. Umezawa, Phys. Rep. **10C**, 151 (1974).
[2]H. Matsumoto, Fort. Schr. Phys. **25**, 1 (1977).
[3]H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).
[4]I. Ojima, Ann. Phys. (N.Y.) **137**, 1 (1981).
[5]H. Matsumoto, Y. Nakano, H. Umezawa, F. Mancini, and M. Marinaro, Prog. Theor. Phys. **70**, 599 (1983).
[6]H. Matsumoto, Y. Nakano, and H. Umezawa, Phys. Rev. D **28**, 1931 (1983); **29**, 1116 (1984); H. Matsumoto, M. Nakahara, Y. Nakano, and H. Umezawa, in *Supersymmetry in Physics,* proceedings of the Los Alamos workshop, 1983 [Physica D (to be published)].
[7]H. Matsumoto, I. Ojima, and H. Umezawa, Ann. Phys. (N.Y.) **152**, 348 (1984).
[8]J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **37**, 452 (1951); **37**, 455 (1951); B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser *et al.* (MIT, Cambridge, Mass., 1970); J. Iliopoulos, C. Itzykson, and A. Martin, Rev. Mod. Phys. **12**, 875 (1972); G. Jona-Lasinio, Nuovo Cimento **34**, 1790 (1964); S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973); R. Jackiw, *ibid.* **9**, 1686 (1974); S. Weinberg, *ibid.* **7**, 2887 (1973); T. D. Lee and G. C. Wick, *ibid.* **9**, 2291 (1974); R. Fukuda, Prog. Theor. Phys. **56**, 258 (1976); Y. Fujimoto, L. O'Raifeartaigh, and G. Paravicini, Nucl. Phys. **B212**, 268 (1983); D. J. E. Callaway and D. J. Maloof, Phys. Rev. D **27**, 406 (1983).
[9]G. Semenoff and H. Umezawa, Nucl. Phys. **B220**[FS8], 196 (1983).
[10]Y. Fujimoto and J. H. Yee, Phys. Lett. **114B**, 359 (1982).
[11]A. Niemi and G. Semenoff, Ann. Phys. (N.Y.) **152**, 130 (1984).
[12]Y. Fujimoto, R. Grigjanis, and R. Kobes, University of Alberta Report No. 23, 1983 (unpublished); Y. Fujimoto, R. Grigjanis, and H. Nishino, Phys. Lett. **141B**, 83 (1984); Y. Fujimoto and R. Grigjanis, University of Alberta Report 1984 (unpublished).
[13]T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).
[14]A. A. Abrikosov, L. Gor'kov, and I. Dyzaloshinski, Zh. Eksp. Teor. Fiz. **36**, 900 (1959) [Sov. Phys. JETP **9**, 636 (1959)]; *Method of Quantum Field Theory in Statistical Physics* (Pergamon, Oxford, 1965); H. Ezawa, Y. Tomozawa, and H. Umezawa, Nuovo Cimento **5**, 810 (1957).
[15]A. D. Linde, Phys. Lett. **42B**, 471 (1972).
[16]C. Bernard, Phys. Rev. D **9**, 3312 (1974).
[17]L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[18]S. Weinberg, Phys. Rev. D **9**, 3357 (1974).
[19]J. Whitehead, H. Matsumoto, and H. Umezawa, Phys. Rev. B **25**, 4737 (1982).
[20]S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
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