Cosmological solutions of $N=1$ supergravity in 11 dimensions

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Freund and Rubin's ansatz for the ground state of $N=1$ supergravity in 11 dimensions can be easily generalized by allowing a cosmological time dependence of the vacuum expectation value of the antisymmetric tensor field. Some numerical solutions of the corresponding field equations (in the classical approximation) are presented, and the resulting physical picture of the primeval universe is discussed.

In a Kaluza-Klein picture of unification, the gauge degrees of freedom are the low-energy manifestation of spacetime symmetries in the extra dimensions. Local supersymmetry is a particularly attractive guiding principle in the search for a fundamental theory in an extended spacetime and, in particular, $N=1$ supergravity in $d=11$ dimensions[1] is in some respects unique,[2] and thus worthy of being studied in some detail.

A notorious candidate for the vacuum in this theory is given by the Freund-Rubin[3] ansatz, with the original 11 dimensions naturally split into $4+7$. If this point of view is seriously taken though, all this needs to be generalized to the primeval universe,[1] by allowing a dependence of the vacuum expectation values on the cosmic time. The "cosmological ground state" of the theory would then be a solution of the equations of motion allowing for "dynamical compactification" of spacetime.

The aim of this note is to begin the study of this problem.[4] The crudest approach will be used: quantum effects will be completely neglected, and the sections $\{t=\text{const}\}$ of spacetime assumed of the form $V_3\times V_7$, where V_n is the maximally symmetric space of dimension n.

The action of $N=1, d=11$ supergravity is[5]

$$S=\int d^{11}x\left[-\frac{e}{4\kappa^2}R(\omega)-\frac{ie}{2}\bar{\psi}_m\Gamma^{mnp}D_n\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_p-\frac{e}{48}F_{mnpq}F^{mnpq}+\frac{2\kappa}{(144)^2}\epsilon^{ijklmnopqrs}F_{ijkl}F_{mnop}A_{qrs}\right.$$
$$\left.+\frac{\kappa e}{192}(\bar{\psi}_r\Gamma^{rsmnpq}\psi_s+12\bar{\psi}^m\Gamma^{pq}\psi^n)(F_{mnpq}+\hat{F}_{mnpq})\right]\ . \tag{1}$$

The gravitational constant of the theory κ has mass dimension $-\frac{9}{2}$. In spite of the fact that the action contains the potential A_{mnp}, the classical equations of motion can be written in terms of gauge-invariant quantities only:

$$R_{rm}-\tfrac{1}{2}Rg_{rm}=\frac{\kappa^2}{3}\left(\tfrac{1}{8}F_{snpq}F^{snpq}g_{rm}-F_{rnpq}F_m{}^{npq}\right)\ , \tag{2a}$$

$$e\,\partial_r F^{rstu}=-\frac{\kappa}{576}\epsilon^{ijklmnopstu}F_{ijkl}F_{mnop}\ . \tag{2b}$$

In order to study its ground state, the "gravitinos" ψ_m will be frozen out to zero, and, moreover, the antisymmetric tensor field F_{mnpq} assumed to be different from zero on V_4 only, i.e., when all its indices run from 0 to 3 ($\alpha,\beta,\ldots=0,\ldots,3$). If the Levi-Cività and metric tensor densities on V_4 are represented by $\epsilon_4{}^{\alpha\beta\gamma\delta}$ and $g_4\equiv\det g_{4\alpha\beta}$, the Freund-Rubin ansatz can be written somewhat symbolically as

$$F^{mnpq}=\frac{\epsilon_4{}^{\alpha\beta\gamma\delta}}{\sqrt{g_4}}F(t) \tag{3}$$

and the metric in $R\times V_3\times V_7\equiv V_{11}$ (Ref. 6) is

$$ds^2=dt^2-R(t)^2\tilde{g}_{ij}dx^idx^j-A^2(t)\tilde{g}_{ab}dy^ady^b\ . \tag{4}$$

By direct substitution of (3) in (2b) a single equation for the function F is obtained:

$$\dot{F}+7\frac{\dot{A}}{A}F=0\ , \tag{5a}$$

fixing the time dependence of F as[7]

$$F(t)=F_0(A_0/A(t))^7\ . \tag{5b}$$

The other field equation then reduces to

$$3\frac{\ddot{R}}{R}+7\frac{\ddot{A}}{A}=-\tfrac{4}{3}\kappa^2F^2\ , \tag{6a}$$

$$2\dot{R}^2+R\ddot{R}+7R\dot{R}\frac{\dot{A}}{A}+2\kappa_1=-\tfrac{4}{3}\kappa^2F^2R^2\ , \tag{6b}$$

$$6\dot{A}^2+A\ddot{A}+3A\dot{A}\frac{\dot{R}}{R}+2\kappa_2=\tfrac{2}{3}\kappa^2F^2A^2\ . \tag{6c}$$

The components of the Ricci tensor in V_3 and V_7 are

$$R_{ij}=-\tfrac{4}{3}\kappa^2F^2R^2\tilde{g}_{ij}\ , \tag{7a}$$

$$R_{ab}=\tfrac{2}{3}\kappa^2F^2A^2\tilde{g}_{ab}\ . \tag{7b}$$

The manifold V_3 is then open ($\kappa_1=-1$) and V_7 is closed ($\kappa_2=+1$). Bianchi identities imply a functional relationship among (6), in such a way that a first integral can be written, which, by introducing the convenient variables $\Theta_1\equiv\dot{R}/R$ and $\Theta_2=\dot{A}/A$, reads

$$\Theta_1{}^2+7\Theta_1\Theta_2+7\Theta_2{}^2=\lambda_{\text{eff}}(t)\ , \tag{8a}$$

$$\lambda_{\text{eff}}(t)\equiv\frac{1}{3}\left[\kappa^2F_0{}^2\left(\frac{A_0}{A}\right)^{14}-\frac{7}{A^2}+\frac{3}{R^2}\right]\ . \tag{8b}$$

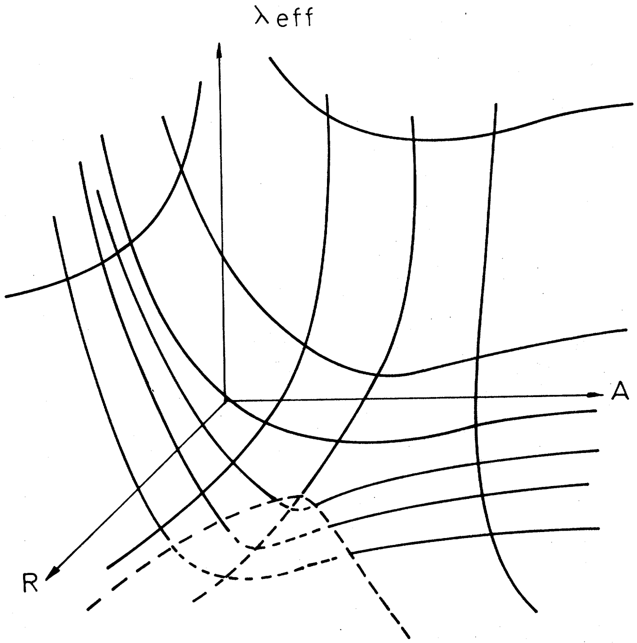

FIG. 1. The effective cosmological constant, in terms of the scale factors R and A.

The effective cosmological constant $\lambda_{\rm eff}(t)$ is depicted in Fig. 1 as a function of A and R. As long as $A > 6^{-1/12}(\kappa F_0 A_0{}^7)^{1/6}$ there is an $\bar{R}$ such that $\lambda_{\rm eff} < 0$ for $R > \bar{R}$. As has been stressed in Ref. 1, this implies a certain stability, in the sense that if $\lambda_{\rm eff} < 0$ and moreover $\Theta_1 > 0$, $\Theta_2 < 0$, the universe will continue to increase R and decrease A as long as $\lambda_{\rm eff} < 0$; this is a region of dynamical compactification.

A linear combination of (6) gives

$$\dot{\Theta}_2 = 2\Theta_1{}^2 + 11\Theta_1\Theta_2 + 7\Theta_2{}^2 - \frac{2}{R^2} + \frac{8}{3A^2}\,, \tag{9a}$$

$$\dot{\Theta}_1 = -7\Theta_1{}^2 - 28\Theta_2{}^2 - 35\Theta_1\Theta_2 + \frac{6}{R^2} - \frac{28}{3A^2}\,. \tag{9b}$$

When $A \to \infty$ and $R \to 0$ while keeping Θ_1 and Θ_2 bounded, those equations imply that $\dot{\Theta}_2 < 0$, $\dot{\Theta}_1 > 0$, recovering the result of the preceding paragraph that $\lambda_{\rm eff} < 0$. When $A \to 0$ and $R \to \infty$ the situation is just the opposite: $\dot{\Theta}_2 > 0$, $\dot{\Theta}_1 < 0$. When both A and $R \to \infty$ the system (9) can be completely analyzed in the plane (Θ_1, Θ_2). But when the four quantities $(\Theta_1, \Theta_2, A, R)$ are comparable the flow of (9) is quite complicated. A numerical integration of (6) has then been performed.

TABLE I. Set of initial conditions for the numerical integration, expressed as a point $Y = (A, R, \dot{A}, \dot{R})$.

Curve		Unit of time
	Forward integration (Fig. 2) $Y(0)$	
1	(1,1,0,0)	10^{-44} sec
2	$(10^{10},10^4,0,10^4)$	10^{-114} sec
3	$(10^8,10^2,10,10^3)$	10^{-100} sec
	Backwards (Fig. 3) $Y(1)$	
4	$(1,10^{10},0,10^{10})$	10^{-53} sec
5	$(10^2,10^8,-1,1)$	10^{-65} sec
6	$(10^2,10^8,0,10^8)$	10^{-67} sec

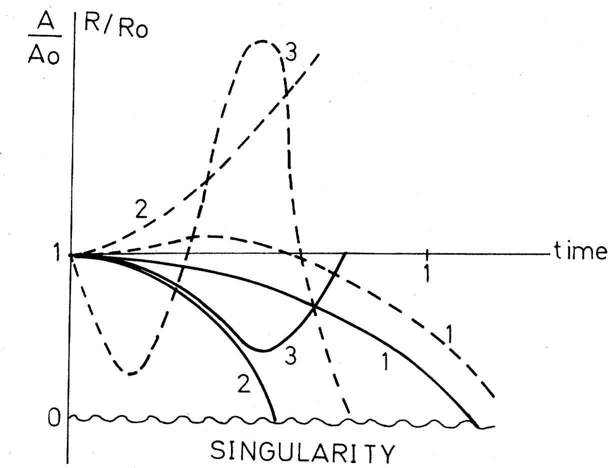

FIG. 2. Forward integration of the field equations. The values of A are represented by the continuous line; R is the discontinuous curve. What is represented is always the relative values A/A_0, R/R_0. The numbers on the curves are those of Table I.

There is a mass scale in the problem given by the product of the gravitational constant in eleven dimensions, and the initial vacuum expectation value of the antisymmetric tensor field $M \equiv \kappa F_0$. It can be analytically proved that there are no solutions to Eqs. (6) with $\dot{R} = \dot{A} = 0$; there are some with $\dot{A} = 0$, but with oscillatory behavior of R.[8] It can be also proved that there are no power solutions of the type $R \sim t^r$, $A \sim t^a$, except the already mentioned ones. Nor, in fact, are there any solutions of the "de Sitter" type, i.e.,

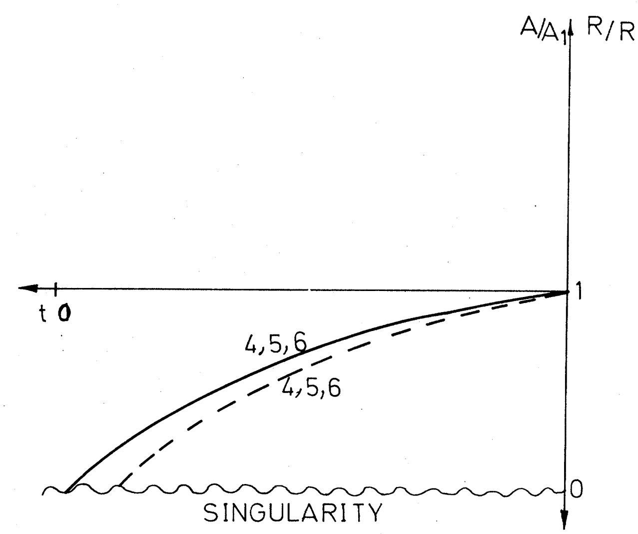

FIG. 3. Backwards integration of the field equations. The conventions are the same as those of Fig. 1, except that the relative values are now with respect to $A(1)$ and $R(1)$. The three curves 4, 5, and 6 are almost identical, the only difference being the scale, as indicated in Table I.

with constant expansion or contraction, i.e., $\Theta_1=\text{const}$, $\Theta_2=\text{const}$.

The numerical integration was made both forward and backwards, starting with different initial conditions [the values of $Y\equiv(A,R,\dot{A},\dot{R})$ displayed in Table I[9]]. In all the cases studied a point is reached with A (or $R)=0$, which is a true spacetime singularity, since the scalar curvature diverges quadratically.[10] If this theory makes sense, these singularities are presumably avoided either by quantum effects not considered here, or by the presence in the ground state of a nonzero value for some fermion condensate made up from the eleven-dimensional gravitinos.

The behavior of the scale factors is quite complicated, and in particular there is not a nonsingular solution with dynamical compactification. The first piece of the curve number 2 (Fig. 2), whose initial values are $R_0/A_0\sim10^{-6}$, can perhaps be taken as an indication that in the full quantum theory this kind of behavior will be possible. The backwards integration has been done in order to get a feeling of the figures which are necessary to achieve a compactified situation $[A(1)/R(1)\ll 1]$. This is only possible just after a spacetime singularity—which could, of course, be taken with hindsight as the initial condition.

More complicated classical configurations should be examined before definite conclusions can be drawn, because the ansatz[4] for the metric is unduly restrictive (there is no evidence for this kind of symmetry in the extra dimensions). It will not be possible, in the general case, to maintain (3) so that the problem of the full set of equations (2) has to be faced. The most important drawback of the present computation from the physical point of view is the neglect of any quantum effect, presumably very important, even for the determination of the ground state; after all, if $N=1, d=11$ supergravity has any sense as a standard quantum field theory, this would only be possible thanks to essentially nonperturbative effects.[11]

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[1]P. G. O. Freund, Nucl. Phys. **B209**, 146 (1982); E. Alvarez and M. B. Gavela, Phys. Rev. Lett. **51**, 931 (1983) (where further references can be found).

[2]E. Witten, Nucl. Phys. **B186**, 412 (1981).

[3]P. G. O. Freund and M. Rubin, Phys. Lett. **97B**, 233 (1980).

[4]Some work on this line has already been done by Freund in Ref. 1.

[5]E. Cremmer, B. Julia, and J. Scherk, Phys. Lett. **76B**, 409 (1978). The notation is $m,n,\ldots=0,1,\ldots,10$ are curved indices on the manifold (signature -9). The tensor F is the antisymmetric curl of the potential $A_{mnp}, F_{mnpq}\equiv4\partial_{[m}A_{npq]}$. The *vielbein* $e_m{}^a$ $(a,b,\ldots=0,1,\ldots,10$ are flat indices on the Lorentzian tangent space) has determinant e. The antisymmetric product of Γ matrices is $\Gamma^{a_1\cdots a_p}\equiv\Gamma^{[a_1}\ldots\Gamma^{a_p]}$. The supercovariant quantities $\hat{F}$ and $\hat{\omega}$ are defined by

$$\hat{F}_{mnpq}\equiv F_{mnpq}-3\kappa\bar{\psi}_{[m}\Gamma_{np}\psi_{q]}\ ,$$

$$\hat{\omega}_{mab}\equiv\omega_{mab}+\frac{i\kappa^2}{4}\bar{\psi}_p\Gamma_{mab}{}^{pq}\psi_q\ .$$

The curvature conventions are such that

$$R(\omega,e)\equiv e^{ma}e^{nb}(\partial_m\omega_{nab}-\partial_n\omega_{mab}+\omega_{mac}\omega_n{}^c{}_b-\omega_{nac}\omega_{mcb})\ .$$

[6]Local coordinates in V_3 will be denoted by x^i, $i=1,2,3$ and corresponding ones in V_7 by y^a, $a=4,\ldots,10$. The quantities κ_1 and κ_2 are ±1 or 0, depending on the curvatures of V_3 and V_7.

[7]In (5b) both F_0 and A_0 are constants. The point $t=0$ is not necessarily taken as the one in which R or A are zero.

[8]Some of these solutions have been discussed by Freund in Ref. 1.

[9]The time scale of the figures was obtained by assuming (somewhat arbitrarily) that $M\sim M_p\equiv G^{1/2}$, i.e., that the vacuum expectation value of the antisymmetric tensor field is of the order of $F_0\sim\kappa^{-1}M_p$.

[10]As a matter of fact, a straightforward generalization of the Hawking-Ellis theorem [S. Hawking and G. F. R. Ellis, Phys. Lett. **17**, 246 (1975)] shows that every cosmological solution of Einstein's equations for the metric (4) with "reasonable" matter is a singular one.

[11]E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B227**, 252 (1983).