Towards a complete solution of two-dimensional quantum chromodynamics

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Quantum chromodynamics in two space-time dimensions with massless fermions is studied using the path-integral approach. Performing a non-Abelian chiral change of variables, the fermion determinant is solved exactly. Apart from a gluon mass term, it contains a Wess-Zumino anomaly term, also present in the solution of other two-dimensional models. Physical properties are discussed in terms of the resulting effective Lagrangian.

Interest in two-dimensional field theories has been renewed after the advances originated in recent works by Polyakov and Weigman,[1] Alvarez,[2] and Witten.[3] A remarkable feature revealed by these works is the connection between two-dimensional models (such as the nonlinear σ model) and the Wess-Zumino functional[4] originally constructed as an effective action for chiral anomalies in $d=4$ space-time dimensions.

Here we study quantum chromodynamics in $d=2$ dimensions (QCD_2) with massless fermions. Using a recently developed[5] path-integral version of the bosonization technique (particularly adequate for non-Abelian theories) we solve exactly the fermion determinant. We show that this determinant consists of two terms: one corresponding to a gluon mass term (the non-Abelian extension of the Schwinger mechanism); remarkably, the other term is precisely the analog (for $d=2$) of the Wess-Zumino functional. This term is endowed with deep topological meaning since it corresponds to the Chern-Simons secondary invariant in differential geometry.[6,7] Since our results were obtained by exploiting the non-Abelian chiral anomaly, it is no surprise to see the Chern-Simons term arise in this context, in light of recent works.[8]

The evaluation of the fermion determinant is an important step towards a complete solution of QCD_2 with massless fermions, a model where only partial results are known (see Ref. 5 for an incomplete list of works on this model). As we shall see, our approach shows that fermions are completely decoupled from the gluon sector. This last is described by an effective Lagrangian consisting of N^2-1 [for the SU(N) case] massive scalars and N^2-1 massless gauge excitations. The presence of the Wess-Zumino term makes the scalars self-interacting. These results show interesting connections between QCD_2 and other two-dimensional models such as the nonlinear σ model, the chiral Gross-Neveu model, etc. Also, resemblances with current-algebra effective Lagrangians[4,8] become apparent.

We start by sketching how the fermion determinant can be computed using the technique developed in Ref. 5. As we shall see, this approach overcomes the usual difficulties characteristic of bosonization in non-Abelian theories, providing a very economical way of analyzing two-dimensional fermion models.

The generating functional for QCD_2 with massless fermions reads (in Euclidean space-time, which will be taken as a large two-sphere S^2)

$$Z=\int \mathscr{D}\bar{\psi}\,\mathscr{D}\psi\,\mathscr{D}A_\mu \exp\left[-\int(\bar{\psi}\not{D}\psi+\tfrac{1}{4}F_{\mu\nu}{}^2+\text{gauge-fixing term})d^2x\right], \tag{1}$$

where $\gamma_\mu D_\mu=\not{D}=i\not{\partial}+g\not{A}$ and A_μ takes values in the Lie Algebra of SU(2) [the extension to SU(N) is trivial]. The massless fermions are taken in the fundamental representation of SU(2).

Exactly as it happens in the Abelian case (the Schwinger model) there is a change in the fermion variables which completely decouples fermions from gauge fields, at the classical level. Although this is possible in an arbitrary gauge, it is simpler and more instructive to work in what we shall call the *decoupling gauge.* In this gauge one can always find a field $\phi=\phi^a t^a$ [taking values in the Lie algebra of SU(2), generated by the t^a's] such that[5]

$$\not{A}=\frac{-i}{g}(\not{\partial}e^{\gamma_5\phi})e^{-\gamma_5\phi}=\frac{-i}{g}[\not{\partial}U_5(x)]U_5{}^{-1}(x) \tag{2}$$

[Eq. (2) becomes $A_\mu=(1/g)\epsilon_{\nu\mu}\partial_\nu\phi$ in the Abelian case, and hence the decoupling gauge coincides with the Lorentz gauge for the Schwinger model].

It is straightforward to check that in terms of the new fermion variables

$$\chi=U_5{}^{-1}(x)\psi,\quad \bar{\chi}=\bar{\psi}U_5{}^{-1}(x)\ , \tag{3}$$

the fermion Lagrangian becomes completely decoupled from gauge fields,

$$L_F=\bar{\psi}\not{D}\psi=\bar{\chi}i\not{\partial}\chi\ . \tag{4}$$

That the choice of the decoupling gauge is possible can be proved, following the work of Roskies,[9] by considering the $j=i\gamma_5$ complexification of SU(2), SL(2,C). Indeed, $U_5(x)$ can be taken as an element of the form $U_5(x)=\exp[-ij\phi(x)]$, that is, a positive-definite Hermitian matrix of determinant one. [We shall call G_5 the set of all such elements, $G_5\subset$SL(2,C).]

Defining $x_\pm=x_1\pm jx_0$, Eq. (2) can be written as

$$A_- = -2i(\partial_- U_5)U_5^{-1} . \quad (5)$$

Note that in our approach the role of fermion currents, which in the operator method are written after bosonization in terms of scalar fields (for example, $J_\mu = \epsilon_{\nu\mu}\partial_\nu \phi$ in the Abelian case), is played by the gauge field. We then see that in our way to bosonization we have arrived at the analog of currents $J_\pm$ introduced in Refs. 1 and 3, except for the fact that $x_\pm$ are not truly light-cone coordinates. Precisely this difference simplifies considerably our treatment.

The quantum aspect of the decoupling is taken into account by the change in the fermionic measure under transformation (3):

$$\mathscr{D}\bar\psi\,\mathscr{D}\psi = J\,\mathscr{D}\bar\chi\,\mathscr{D}\chi . \quad (6)$$

It is important to note that the Jacobian J coincides with the fermionic determinant

$$\det \not{D} = \int \mathscr{D}\bar\psi\,\mathscr{D}\psi \exp\left[-\int \bar\psi \not{D}\psi d^2x\right]$$
$$= J\int \mathscr{D}\bar\chi\,\mathscr{D}\chi \exp\left[-\int \bar\chi i\,\partial\!\!\!/ \chi\, d^2x\right] = J \det i\partial\!\!\!/ . \quad (7)$$

In order to evaluate J, one considers an extended U_5 transformation depending on a parameter t ($t \in [0,1]$):

$$U_5(x,t) = \exp[(1-t)\gamma_5\phi] . \quad (8)$$

The whole transformation (2) is then built up by iteration from the infinitesimal one, varying t from 0 to 1. Following the method described in Refs. 5 and 10, we get

$$\ln(\det\not{D}/\det i\partial\!\!\!/) = \ln J = \frac{-g^2}{2\pi}\int d^2x\,\mathrm{tr}\left[\tfrac{1}{2}\not{A}\not{A} + \int_0^1 dt\,\gamma_5 \not{A}_t \phi \not{A}_t\right] \quad (9)$$

with

$$\not{A}_t = \frac{-i}{g}[\partial\!\!\!/ U_5(x,t)]U_5^{-1}(x,t) . \quad (10)$$

The first term in (9) shows that the gauge fields have gotten a mass m ($m^2 = g^2/2\pi$). That is, the Schwinger mechanism takes place in QCD_2 as in the Abelian case. We shall now show that the second term corresponds to the analog of the Wess-Zumino functional[4,8] in two dimensions, also appearing in the nonlinear σ model solution of Polyakov and Wiegman[1] and the simplified model solved by Alvarez.[2] Indeed, using Eq. (10) the second term in Eq. (9) can be written as

$$W_2 = \frac{-g^2}{2\pi}\mathrm{tr}\int \gamma_5 \not{A}_t \phi \not{A}_t dt\, d^2x = \frac{i}{2\pi}\int_0^1 dt\int d^2x\,\mathrm{tr}[(\partial_t U)U^{-1}(\partial_\mu U)U^{-1}(\partial_\nu U)^{-1}\epsilon_{\mu\nu}] \quad (11)$$

with

$$U = U(x,t) = \exp[t\phi(x)] . \quad (12)$$

Consider for a moment the analytic continuation of U to an element U_c of SU(2):

$$U_c(x,t) = \exp[it\phi(x)] . \quad (13)$$

Since we are taking space-time as a large sphere and $\pi_2(\mathrm{SU}(2)) = 0$, we can think of U_c as a mapping from a solid ball B (whose boundary is S^2) into the SU(2) manifold. We shall take as coordinates in B (whose boundary is precisely S^2) the parameter t and the two-space-time coordinates (writing $t = \cos\alpha$ we can think of B as the upper hemisphere on S^3). The analytically continued W_2 reads

$$W_2^c = 4\pi i\Gamma , \quad (14)$$

where Γ is the Wess-Zumino functional

$$\Gamma = \frac{1}{24\pi^2}\int_B d^3x\,\epsilon^{ijk}\mathrm{tr}[(\partial_i U_c)U_c^{-1}(\partial_j U_c)U_c^{-1} \times (\partial_k U_c)U_c^{-1}] , \quad (15)$$

which has the very important property of being defined modulo 2π, the ambiguity being related to the topological inequivalent ways of extending a given mapping $U_c : S^2 \to \mathrm{SU}(2)$ into a mapping from B into SU(2); the topologically distinct possibilities are classified by $\pi_3(\mathrm{SU}(2)) = Z$.[3,6,7]

Coming back to our actual problem, the extension from S^2 to B arose naturally when we constructed the finite chiral transformation $U_5(x)$ from $U_5(x,t) \in G_5$. Any other extension than the one defined by Eqs. (8) or (12) would have yielded the same W_2 since there are no ambiguities [of the kind described above for the case of $U_c \in \mathrm{SU}(2)$] for the elements of the set $G_5 \subset \mathrm{SL}(2,C)$. This can be seen by noting that G_5 is homotopically equivalent to R^3 and $\pi_3(R^3) = 0$. We then conclude that any other extension of U_5 different from that of Eq. (8) [for example, by changing $(1-t)$ in the exponential by an arbitrary function $f(t)$ satisfying $f(0)=1$, $f(1)=0$] would lead to the same result. Then, we shall write the Jacobian as

$$\ln J = \ln\left[\frac{\det\not{D}}{\det i\partial\!\!\!/}\right] = \frac{1}{\pi}\mathrm{tr}\left[\int d^2x(\partial_\mu U)(\partial_\mu U^{-1}) + \frac{i}{2}\epsilon_{\mu\nu}\int_0^1 dt\int d^2x(\partial_t U)U^{-1}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}\right] . \quad (16)$$

This result coincides with that obtained in Ref. 1 when solving the nonlinear σ model; it is also related to the determinant evaluated in Ref. 2 for a simplified version of QCD_2. Note that the second term has been obtained by carefully evaluating the change in the fermion measure under non-Abelian chiral transformations. This shows that the non-Abelian anomaly, which precisely coincides with the secondary Chern-Simons invariant, can be thought of as arising from the noninvariance of the functional measure under these transformations (compare this aspect with Ref. 11).

It is instructive to make a perturbative expansion

$$U=1+\phi^a t^a+O(\phi^2)\ .$$

The Jacobian reads

$$\ln J=\frac{-1}{\pi}\mathrm{tr}\int d^2x[(\partial_\mu\phi)^2+\tfrac{1}{6}\epsilon_{\mu\nu}\phi\,\partial_\mu\phi\,\partial_\nu\phi] + \text{higher-order terms}\ . \tag{17}$$

This Jacobian is a resemblant to the effective Lagrangian discussed by Witten[8] in order to describe low-energy hadron phenomenology. However, in the present case, the effective Lagrangian contains the ${F_{\mu\nu}}^2$ term and reads

$$L_{\mathrm{eff}}=\frac{1}{g^2}\mathrm{tr}\left[\phi\left[\nabla^2\nabla^2+\frac{g^2}{2\pi}\nabla^2\right]\phi+\frac{g^2}{6\pi}\phi\,\partial_\mu\phi\,\partial_\nu\phi\epsilon_{\mu\nu} +2\phi\,\partial_\mu\nabla^2\phi\,\partial_\nu\phi\epsilon_{\mu\nu}\right]\ . \tag{18}$$

As usual in the path-integral approach to bosonization[5] one gets an effective Lagrangian with high-order derivative terms. It corresponds to $N^2-1=3$ massive scalars (with mass $m=-g/\sqrt{2\pi}$) and three massless gauge excitations,[13] since the propagator Δ associated to Lagrangian (18) reads

$$\Delta=\Delta_F(m,x)-\Delta_F(0,x),\quad \Delta_F(m,x)=-K_0(mx)\ . \tag{19}$$

In contrast with QED_2,[12,13] where the massive scalar is free, here a self-interaction (given by the Wess-Zumino term and the commutator part of the ${F_{\mu\nu}}^2$ term) is present. Due to the Wess-Zumino term, the Lagrangian violates both naive parity operation ($P_0\equiv x_0\to x_0$, $x_1\to -x_1$, $U\to U$) and (modulo 2) boson number N_B conservation [$(-1)^{N_B}\equiv U\to U^{-1}$ or $\phi^a\to-\phi^a$], but is invariant under the product $P_0(-1)^{N_B}$ which corresponds to $x_0\to x_0$, $x_1\to -x_1$, $\phi\to-\phi$.

As we pointed out above, the fermion determinant (17) coincides with the one computed by Polyakov and Wiegman[1] in their solution of the nonlinear σ model. However, their effective Lagrangian has no ${F_{\mu\nu}}^2$ term, and hence bosons remain massless. (It is the presence of the kinetic term which makes the bosons massive in QCD_2.) Similar to the σ-model case, the solution of the chiral Gross-Neveu model leads, in the path-integral approach, to a theory of fermions interacting with an effective vector field (see Ref. 14), and after decoupling, the bosons remain massless.

In summary, we have computed the fermion determinant for QCD_2 and stressed the appearance of a Wess-Zumino term, analogous to those discovered in other two-dimensional models. We have shown that the fermions completely decouple from bosons, these last being massive self-interacting scalars. Concerning fermion Green's functions, the decoupling implies that at short distances fermions are free. In order to complete the physical picture of the theory (testing screening at long distances and analyzing boson Green's functions), one has to explicitly write the decoupling gauge condition in order to treat the gauge-field sector. Work on this aspect is in progress.

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