Comment on the ultraviolet behavior of gauge-dependent Green's functions in QCD

Geoffrey B. West
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 30 January 1984)

The ultraviolet behavior of gauge-dependent covariant Green's functions in QCD is shown to have a *discontinuous* behavior in both the number of flavors (n_f) and the gauge parameter (α). In particular, we show that for $n_f<10$ and $\alpha\neq 0$ the behavior is, up to a normalization constant, that of pure free field theory *without* the usual logarithmic power modulation normally associated with asymptotic freedom.

In this paper we show that the generally accepted view concerning the ultraviolet (UV) behavior of gauge-dependent Green's functions in quantum chromodynamics (QCD) is incorrect. It is typically assumed and often stated, for instance, that in Lorentz gauges this behavior is given by free field theory modulated by powers of logarithms.[1] We show, however, that in actual fact there is *no* logarithmic modulation when the number of flavors (n_f) is less than 9 provided one avoids the Landau gauge. In other words, under these conditions, the asymptotic behavior of a typical gauge-dependent Green's function is, up to a normalization constant, precisely that of unadulterated free field theory. Only when the number of flavors exceeds 9 does the "usual" logarithmic behavior enter. This rather bizarre dependence on n_f is, however, entirely absent in the Landau gauge where the usual logarithmic result is valid regardless of the number of flavors. As a specific example consider the transverse part of the gluon propagator $D(q^2,\alpha)$, α being the conventional Lorentz gauge parameter; $\alpha=0$ corresponds to the Landau gauge. We show that its UV (i.e., $q^2\to\infty$) behavior has the following structure:

$$n_f<10:\ D(q^2,\alpha)\to\frac{1}{q^2}\times\begin{cases}\text{constant}, & \alpha\neq 0\\ (\ln q^2)^p, & \alpha=0\ ,\end{cases} \tag{1a}$$

$$n_f\geq 10:\ D(q^2,\alpha)\to\frac{(\ln q^2)^p}{q^2}\ , \tag{1b}$$

where $p=(n_f-\frac{39}{4})/(33-2n_f)$ is derived from the anomalous dimension in the Landau gauge *only*. Notice that p does *not* depend upon α. A more detailed version of this is given in Eq. (11) below.

To see how this comes about, consider a dimensionless Green's function $f(t,g,\alpha)$ that depends on a logarithmic momentum scale $t\equiv\ln q^2/\mu^2$, the gauge coupling g, and the gauge parameter α, both normalized at the scale μ. For the gluon propagator mentioned above, $f\equiv q^2D$; the argument can straightforwardly be applied to any similar gauge-dependent Green's function. The standard renormalization-group equation reads[1]

$$-\frac{\partial f}{\partial t}+\beta(g,\alpha)\frac{\partial f}{\partial g}+\delta(g,\alpha)\frac{\partial f}{\partial\alpha}-\gamma(g,\alpha)f=0\ , \tag{2}$$

where $\beta(g,\alpha)=\mu\partial g/\partial\mu$, $\delta(g,\alpha)=\mu\partial\alpha/\partial\mu$, and $\gamma(g,\alpha)$ is the anomalous dimension. In this equation, f is to be considered as a function of the independent variables t, g, and α so that the partial derivatives have their usual meaning; however, it should be noted that all derivatives are to be taken keeping the implicit unrenormalized quantities (including any cutoff) fixed. It is well known that gauge invariance requires that

$$\delta(g,\alpha)=-2\alpha\gamma(g,\alpha)\ .$$

Furthermore it is always possible to define $\beta(g,\alpha)$ to be gauge invariant, i.e., independent of α, as in the minimal subtraction scheme.[1,2] We shall not dwell on the problem of scheme dependence here though we shall return to the more general case of a gauge-dependent β function at the end of the paper.[2]

We shall demonstrate below that the solution to Eq. (2) can be expressed in the following way:

$$f(t,g,\alpha)=\left[\frac{\bar{\alpha}(t,g,\alpha)}{\alpha}\right]^{-1/2}f[0,G(t,g),\bar{\alpha}(t,g,\alpha)]\ . \tag{3}$$

As usual the running coupling constant $G(t,g)$ is given by

$$\int_g^{G(t,g)}\frac{dg'}{\beta(g')}=t\ , \tag{4}$$

whereas the running gauge parameter $\bar{\alpha}(t,g,\alpha)$ satisfies

$$\frac{\partial\bar{\alpha}}{\partial t}=-2\bar{\alpha}\gamma[G(t,g),\bar{\alpha}] \tag{5}$$

subject to the boundary condition $\bar{\alpha}(0,g,\alpha)=\alpha$. Perhaps the most salient feature of the solution (3) is the apparent absence of the usual explicit anomalous-dimension exponential factor. Indeed, this will prove to be the source of the curious nature of the UV behavior exemplified in Eqs. (1). This factor is, in fact, buried in the running gauge parameter $\bar{\alpha}$ as can be seen by examining the Landau-gauge ($\alpha=0$) limit. It is not difficult to verify that[3]

$$\lim_{\alpha\to 0}\left[\frac{\bar{\alpha}(t,g,\alpha)}{\alpha}\right]^{1/2}=\exp\left[-\int_g^{G(t,g)}[\gamma_0(g')/\beta(g')]dg'\right]\ , \tag{6}$$

where $\gamma_0(g)\equiv\gamma(g,0)$ is the anomalous dimension in the

Landau gauge. Inserting this in (3) then leads to the standard solution[1]

$$f(t,g,0)=\exp\left[\int_g^{G(t,g)}[\gamma_0(g')/\beta(g')]dg'\right]\times f[0,G(t,g),0]\ . \tag{7}$$

As usual the small-g behavior of the β function, namely,

$$\beta(g)\underset{g\to 0}{\simeq}-bg^3$$

with

$$b\equiv\frac{1}{16\pi^2}(11-\tfrac{2}{3}n_f)\ ,$$

leads to asymptotic freedom provided $n_f\leqq 16$; thus

$$G^2(t,g)\underset{t\to\infty}{\simeq}(2bt)^{-1}.$$

Using this in Eq. (7) then gives the well-known result that

$$f(t,g,0)\underset{t\to\infty}{\simeq}(2bt)^{a/2b} \tag{8}$$

Here $a[=(1/96\pi^2)(4n_f-39)]$ is the coefficient of the leading term in the perturbative expansion of $\gamma_0(g)$: i.e.,

$$\gamma_0(g)\underset{g\to 0}{\simeq}ag^2\ .$$

Thus, in the Landau gauge Eq. (3) does indeed reduce to the standard result which exhibits the well-known logarithmic power modulation of free field theory. The naive extension of this result to arbitrary α, typically implied in standard works,[1] is simply to replace the anomalous-dimension coefficient a as it occurs in Eq. (8) by its generalization $a(1-\alpha/\alpha_0)$, where

$$\alpha_0=\tfrac{32}{3}\pi^2a=\tfrac{1}{9}(39-4n_f)\ .$$

This is not, in general, correct.

To see what actually happens in other gauges ($\alpha\neq 0$) we need to examine the UV behavior of $\bar{\alpha}$. As we shall show explicitly below, this is given by[3]

$$\bar{\alpha}(t,g,\alpha)\underset{t\to\infty}{\simeq}\frac{\alpha_0}{[1+e^{-2\phi(g,\alpha)}(2bt)^{a/b}]}\ , \tag{9}$$

where $\phi(g,\alpha)$ enters via a boundary condition. This equation is the crux of the problem since the result of the limit taking clearly depends critically on the sign of a and this changes once n_f exceeds 9. Explicitly, Eq. (9) gives

$$\bar{\alpha}(t,g,\alpha)\underset{t\to\infty}{\simeq}\begin{cases}\alpha_0 & \text{when } n_f\leq 10\\ \alpha_0e^{2\phi(g,\alpha)}(2bt)^{-a/b}\to 0 & \text{when } n_f>10\ .\end{cases} \tag{10}$$

This formula is valid provided $\alpha\neq 0$ since in that case one can show that

$$e^{2\phi(g,\alpha)}\underset{\alpha\to 0}{\sim}\alpha$$

and Eq. (9) is simply replaced by (6). Using this in (3) straightforwardly yields

$$\lim_{t\to\infty}f(t,g,\alpha)\sim\begin{cases}(\alpha/\alpha_0)^{1/2} & \text{when } n_f<10\ ,\\ (\alpha/\alpha_0)^{1/2}e^{-\phi(g,\alpha)}(2bt)^{a/2b} & \text{when } n_f>10\ ,\end{cases} \tag{11}$$

where the free-field normalization $f(t,0,\alpha)=1$ has been used [as it was in Eq. (8)]. This, together with Eq. (8), is a more detailed version of the result quoted in Eq. (1) and exhibits the curious discontinuities in both α and n_f.

We will now show that Eq. (3) is indeed a solution to Eq. (1). To do so we shall first use the method of characteristics to derive the general solution.[4,5] The characteristics are obtained by simultaneously solving the ordinary differential equations

$$dt=-\frac{dg}{\beta(g)}=\frac{d\alpha}{2\alpha\gamma(g,\alpha)}=\frac{df}{f\gamma(g,\alpha)}\ . \tag{12}$$

Three independent characteristic surfaces are

$$\text{(i)}\quad f\alpha^{-1/2}=c_1\ , \tag{13}$$

$$\text{(ii)}\quad t+K(g)=c_2,\quad\text{where } K(g)=\int^g\frac{dg}{\beta(g)}\ , \tag{14}$$

and

$$\text{(iii)}\quad \phi(g,\alpha)=c_3\ , \tag{15a}$$

this being the parametric solution to

$$\frac{d\alpha}{dg}=-2\alpha\frac{\gamma(g,\alpha)}{\beta(g)}\ . \tag{15b}$$

The c_i in these equations are parameters whose variation sweeps out the characteristic surfaces. The general solution is simply the most general "intersection" of these surfaces and is given by $\Phi(c_1,c_2,c_3)=0$, where Φ is an arbitrary function. Using Eqs. (13)–(15) this can be recast into

$$f(t,g,\alpha)=\alpha^{1/2}\mathcal{F}[t+K(g),e^{-\phi(g,\alpha)}]\ , \tag{16}$$

where $\mathcal{F}$ is another arbitrary function to be determined by imposing suitable boundary conditions. The exponential form for the second variable in (16) is for convenience only as will become clear below. Actually all consequences of Eq. (1) can be derived from (16) without the need for introducing running coupling or gauge parameters. However, since these concepts have become conventional, we shall first demonstrate how they arise and thereby derive Eq. (3). Clearly we need to eliminate $\mathcal{F}$ from Eq. (16); this can be accomplished by introducing an arbitrary but convenient boundary condition. Suppose that we specify f at some new value of (t,g,α)—call it $(T,G,\bar{\alpha})$; then just as in (16),

$$f(T,G,\bar{\alpha})=\bar{\alpha}^{1/2}\mathcal{F}[T+K(G),e^{-\phi(G,\bar{\alpha})}]\ . \tag{17}$$

$\mathcal{F}$ can now be eliminated between (16) and (17) by simply choosing

$$T+K(G)=t+K(g) \tag{18}$$

and

$$\phi(G,\bar{\alpha})=\phi(g,\alpha) \tag{19}$$

to give

$$f(t,g,\alpha)=\left[\frac{\alpha}{\bar{\alpha}}\right]^{1/2} f[t+K(g)-K(G),G,\bar{\alpha}] \ , \qquad (20)$$

where we have used Eq. (18) to eliminate T. Notice that Eq. (19) can be used to eliminate either G or $\bar{\alpha}$ from (20) thereby leaving its right-hand side (RHS) dependent on only one arbitrary parameter ($\bar{\alpha}$ or G) in addition, of course, to t, g, and α. This arbitrary parameter can be chosen at will and simply reflects the freedom of choice of boundary condition required to eliminate $\mathscr{F}$. The fact that the left-hand side (LHS) of (20) is actually independent of this choice is guaranteed by Eq. (1). It is important to emphasize that at this stage t, g, α, and G (or $\bar{\alpha}$) are completely independent variables. The "conventional" form of the result, represented by Eq. (3), follows as a special case of (20): One simply chooses $T=0$, thereby forcing the arbitrary parameter (G, say) to become a function of t and g determined by Eq. (18):

$$K(G)=t+K(g) \ . \qquad (21)$$

This equation can be recognized as identical to the one defining the usual running coupling constant, $G(t,g)$, namely, Eq. (4). Furthermore, with this definition, $\bar{\alpha}$ becomes a function of t, g, and α to be determined by solving Eqs. (19) and (21); explicitly, it is the solution to

$$\phi[G(t,g),\bar{\alpha}(t,g,\alpha)]=\phi(g,\alpha) \qquad (22)$$

The final step in showing that (16) can be recast into (3) is to confirm that $\bar{\alpha}(t,g,\alpha)$ as defined by (22) satisfies (5). This can be accomplished by noting that the RHS of (22) is independent of t; Eq. (5) follows by taking $\partial/\partial t$ of Eq. (22) and using the fact that, from its definition as a characteristic, namely, Eq. (15a), $d\phi=0$.

It is of some interest to reexamine some of the features of the solution from the point of view of the most general solution as expressed in Eq. (16). We shall show that all of the above-stated results, namely, Eqs. (8) and (11), can be derived without the need for introducing running coupling or gauge parameters. First, we note that Eq. (15b) can be straightforwardly integrated when $\alpha\to 0$ to yield [see Eq. (6)]

$$\lim_{\alpha\to 0} e^{-\phi(g,\alpha)} \sim \alpha^{-1/2} e^{A(g)} \qquad (23)$$

where the indefinite integral

$$A(g)\equiv \int^g dg \frac{\gamma_0(g)}{\beta(g)} \ . \qquad (24)$$

[As noted above, it is this singular behavior in α that invalidates the extension of Eq. (10) to the Landau gauge.] Using (23) in (16) gives

$$f(t,g,0)=\lim_{\alpha\to 0} \alpha^{1/2} F[t+K(g),\alpha^{-1/2}e^{-A(g)}] \ . \qquad (25)$$

Since the limit must exist, it follows that

$$f(t,g,0)=e^{-A(g)}F[t+K(g)] \ , \qquad (26)$$

where F is an arbitrary function. This is precisely the result which would have been obtained[5] had we set $\alpha=0$ directly in Eq. (1). Now let us impose the free field boundary condition on $f(t,g,0)$, namely, that

$$f(t,0,0)=1 \ . \qquad (27)$$

Since $\beta(g)\to -bg^3$, $K(g)\to 1/2bg^2$ so this condition (27) requires

$$\lim_{g\to 0} F(1/2bg^2)=\lim_{g\to 0} e^{A(g)} \qquad (28a)$$

or, using Eq. (24),

$$\lim_{x\to\infty} F(x)\sim(2bx)^{a/2b} \ . \qquad (28b)$$

But, from Eq. (26)

$$\lim_{t\to\infty} f(t,g,0)=e^{-A(g)} \lim_{t\to\infty} F(t) \qquad (29a)$$

$$\sim e^{-A(g)}(2bt)^{a/2b} \ , \qquad (29b)$$

in agreement with Eq. (8). Notice that this result is independent of the magnitude of g, i.e., *one does not necessarily have to be in a perturbative regime for the asymptotic result to be valid.*

Now let us carry out a similar procedure in an arbitrary gauge: the UV limit of f is given by

$$\lim_{t\to\infty} f(t,g,\alpha)=\alpha^{1/2} \lim_{t\to\infty} \mathscr{F}[t,e^{-\phi(g,\alpha)}] \ . \qquad (30)$$

As in the Landau gauge this can be approached by taking $g\to 0$; however, in this case, one must keep $e^{-\phi(g,\alpha)}$ fixed (at some value y, say). Now the small-g behavior of $\phi(g,\alpha)$ can be derived from Eq. (15b) and is given by

$$e^{-\phi(g,\alpha)} \underset{g\to 0}{\sim} g^{a/b}\left[\left|\frac{\alpha_0}{\alpha}-1\right|+O(g^2)\right]^{1/2} . \qquad (31)$$

[Notice, incidentally, that this is precisely the limit that is required on the LHS of (22) in order to determine the UV behavior of $\bar{\alpha}$; indeed, using (31) in (22) will immediately confirm Eq. (9).] Now, it is clear that if $\phi(g,\alpha)$ is to be kept fixed when $g\to 0$, then either

$$\alpha\to\alpha_0 \ \ (\text{when } a<0, \text{ i.e., } n_f<10) \qquad (32a)$$

or

$$\alpha\to 0 \ \ (\text{when } a>0, \text{ i.e., } n_f\geq 10) \ . \qquad (32b)$$

Suppose $n_f<10$; then the free field boundary condition on (16) requires

$$f(t,0,\alpha_0)=1=\alpha_0^{1/2} \lim_{g\to 0} \mathscr{F}[1/2bg^2,y] \ , \qquad (33)$$

i.e.,

$$\lim_{t\to\infty} \mathscr{F}(t,y)=\alpha_0^{-1/2} \ . \qquad (34)$$

Using this in Eq. (30) then yields precisely the result already obtained in (11), namely,

$$\lim_{t\to\infty} \mathscr{F}(t,g,\alpha)=(\alpha/\alpha_0)^{1/2} \ .$$

It is straightforward to verify that for $n_f\geq 10$, (32b) yields Eq. (11b). Note, incidentally, that in deriving these results we have to be sure that the neglected terms of $O(g^2)$ in (31) do not contribute when $\alpha\to\alpha_0$; this requires $|a/b|<1$ which is guaranteed regardless of the number of flavors. We have thus demonstrated *that all of the results of asymptotic freedom can be derived without recourse* *to the concepts of running couplings.*

A few further remarks are worth making:

(i) The results exhibited in Eqs. (1) and (11) are consistent with the following expression for f:

$$f(t,g,\alpha)=\left[\frac{\alpha}{\alpha_0}\right]\left[1+e^{-2\phi(g,\alpha)}\left[2bt+\frac{1}{g^2}\right]^{a/b}\right]^{1/2}. \qquad (35)$$

Notice that this has the correct free field normalization, $f(t,0,\alpha)=1$, and can be expanded perturbatively in g^2.

(ii) Rather than introducing the running coupling, $G(t,g)$, we could work with Eq. (20), keeping G an arbitrary parameter. In that case the infrared behavior of f can be related to its *small*-coupling limit by simply taking $G\to 0$, as was done in Ref. 5 for the case of the Landau gauge. Indeed, if $\beta(g)=-bg^3$ for *all* g, then (35) is the *exact* perturbative solution to Eq. (1); it is only the absence of the next term in β ($\propto \ln g^2$) that prevents it from being the complete perturbative solution.

(iii) Since the first two terms in β are gauge invariant, the use of a gauge-dependent $\beta(g,\alpha)$ will not affect any of the UV results derived here.

---

[1]See, e.g., H. D. Politzer, Phys. Rep. **14C**, 129 (1974); W. Marciano and H. Pagels, *ibid.* **36C**, 138 (1978); C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980). The special role played by the Landau gauge appears to have first been noted by A. Hasoya and A. Sato, Phys. Lett. **48B**, 36 (1974). A sampling of other papers that have touched on this subject include T. P. Cheng *et al.*, Phys. Rev. D **10**, 2459 (1974); B. W. Lee and W. I. Weisberger, *ibid.* **10**, 2530 (1974); and most recently, K. Hagiwara and T. Yoshino, *ibid.* **26**, 2038 (1983); R. D. C. Miller, J. Phys. G **9**, 595 (1983); and A. Bechler and P. S. Kurzepa, Warsaw University Report No. IFT/11/83 (unpublished).

[2]D. J. Gross, in *Methods in Field Theory,* edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[3]Our notation for limit taking is that when a limit is taken in one variable, the other dependent variables are kept fixed.

[4]See, e.g., H. J. H. Piaggio, *Differential Equations* (Bell, London, 1949).

[5]G. B. West, Phys. Rev. D **27**, 1402 (1983).