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Infrared divergences in three-dimensional gauge theories

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The loop expansion in massless three-dimensional gauge theories develops infrared divergences starting at the two-loop level. However, we show that the divergences in QCD_3 which arise through the three-loop level (g^6) can be removed by using a residual gauge freedom in covariant α gauges. Divergences in gauge-invariant amplitudes arise at g^8 order, as shown explicitly for the Wilson loop. Massless scalar QED develops a gauge-invariant singularity at order g^4, which can be canceled by changing the perturbative vacuum.

I. INTRODUCTION

Three-dimensional field theories have been studied because they provide interesting toy models in particular for the study of confinement,[1] and of nontrivial topological structure.[2] Also, in finite-temperature field theories the infinite-temperature behavior is governed by the three-dimensional theory.[3]

A theory that is renormalizable in four dimensions becomes superrenormalizable in three dimensions; only a mass renormalization may be necessary. In some cases, as QED_3 and QCD_3, no renormalization is needed. In three dimensions the gauge coupling constant acquires a dimension $\dim(g)=(\text{mass})^{1/2}$ which is the cause of infrared (IR) divergences arising in the off-mass-shell amplitudes in QED_3, QCD_3, and massless scalar electrodynamics ($SQED_3$). That is, for dimensional reasons, the higher-order diagrams must involve a high power of a momentum variable in the denominators of loop integrals which then become IR divergent.

Some previous studies of the IR problem in three dimensions have found, by resummation techniques, that nonanalytic terms (logarithms of the coupling constant) appear in the two-point functions.[4–6] In this paper we use the method of dimensional regularization to show a correspondence between these nonanalyticities and the divergences in the loop expansion: nonanalytic terms appear together with poles in the dimensional plane. To eliminate the poles in QED_3 and QCD_3 we perform a special gauge transformation by exploiting a residual gauge freedom in covariant α gauges. The relationship between gauge symmetry and IR divergences in the loop expansion is the main result of this paper, Secs. II B and III B.

In Sec. II we review QED_3 (Ref. 7) and in Sec. III we study QCD_3. In both cases the first IR divergences appear at the two-loop (g^4) level when the lowest-order self-energy is inserted into a free gauge field propagator. At the three-loop level there are no new divergences and to this order we find all divergences to be gauge dependent. Singularities in gauge-invariant amplitudes can appear at four-loop (g^8) and higher-loop orders according to arguments based on the operator-product expansion (OPE).[4,6] We confirm those arguments by studying the Wilson loop, Sec. III C. In Sec. IV we investigate gauge-invariant divergences in $SQED_3$ which are produced by soft scalar lines in the two-loop photon self-energy. These can be canceled by fixing ambiguities in the lowest-order effective action. That is, we choose an appropriate vacuum state among the degenerate vacua of the free massless theory.

II. MASSLESS QED_3

A. IR divergences in dimensional regularization

We consider a theory of a massless fermion ψ coupled to an Abelian gauge field A_μ, QED, in D dimensions. The theory in covariant gauges α is governed by

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}{}^2+\bar{\psi}\gamma^\mu(i\partial_\mu-eA_\mu)\psi-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 ,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{2.1}$$

with massive coupling constant

$$e=e'\mu^{2-D/2} , \tag{2.2}$$

where μ is a unit of mass and e' is a number. In dimensions $D<4$ the theory is superrenormalizable. When $D\to 3$, IR divergences arise in the off-shell amplitudes because of the presence of massless fermions. Symmetries of the Lagrangian (2.1) prevent perturbative mass generation: P and T invariance in $D=3$ and γ_5 invariance in $D=4$. Gauge-boson mass generation and spontaneous breaking of P and T invariance in three-dimensional gauge theories containing fermions have been discussed in Ref. 2. Similar effects have not been found in pure Yang-Mills theory in three dimensions, which is relevant for finite-temperature calculations.

We first show results of the loop expansion for self-energies in a covariant α gauge, using dimensional regularization.[8] When dealing with fermions one has to define Tr1 in D dimensions. One requires $\mathrm{Tr}1=2$ for $D=3$ (the γ matrices are 2×2 Pauli matrices) and $\mathrm{Tr}1=4$ for $D=4$. Any smooth interpolation is acceptable, since the ambiguity can be absorbed into a redefinition of the parameters. The propagators are given by

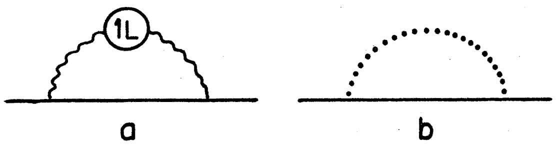

FIG. 1. The two-loop fermion self-energy: (a) IR-divergent contribution; (b) the counterterm.

$$D_{\mu\nu}(p)=\frac{-iP_{\mu\nu}}{p^2+i\epsilon-\Pi(p^2)}-\frac{i\alpha p_\mu p_\nu}{(p^2+i\epsilon)^2}\,,$$
$$P_{\mu\nu}=g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\,, \qquad (2.3)$$

$$S(p)=\frac{i}{\not p-\Sigma(p)}\,. \qquad (2.4)$$

The one-loop self-energies $\Pi_{\mu\nu}=P_{\mu\nu}\Pi(p^2)$ and $\Sigma(p)$ are [7]

$$\Pi^{(1)}_{\mu\nu}(p)=-P_{\mu\nu}e'^2p^2\left[\frac{\mu^2}{-p^2}\right]^{2-D/2}\times\frac{2\,\mathrm{Tr}1\,\Gamma(2-D/2)B(D/2,D/2)}{(4\pi)^{D/2}}\,, \qquad (2.5)$$

$$\Sigma^{(1)}(p)=-e'^2\alpha\not p\left[\frac{\mu^2}{-p^2}\right]^{2-D/2}\times\frac{(D-2)\Gamma(2-D/2)B(D/2,D/2-1)}{(4\pi)^{D/2}}\,, \qquad (2.6)$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$. At $D=3$ both expressions are finite. Notice that the expansion parameter $e'^2(\mu^2/-p^2)^{1/2}$, being proportional to $1/p$, will produce high powers in denominators of the loop integrals at higher orders.

The two-loop photon self-energy at $D=3$ is finite, but the fermion self-energy exhibits an IR divergence which appears in the diagram of Fig. 1(a). We evaluate this diagram, making use of (2.5), and then set $D\to 3$:

$$\Sigma^{(2)}(p)=\frac{e'^4\not p}{96\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{4-D}[\Gamma(D-3)+O(1)] \qquad (2.7a)$$

$$=\frac{e'^4\not p}{96\pi^2}\left[\frac{\mu^2}{-p^2}\right]\left[\frac{1}{D-3}-\ln\frac{\mu^2}{-p^2}+O(1)\right]\,, \qquad (2.7b)$$

where $O(1)$ is a number. The divergence appears in a form familiar from the UV regularization: a pole in the dimensional plane and a logarithm of momentum. Although α independent, (2.7) is not gauge invariant. We will show how a special gauge transformation generates the counterterm, depicted in Fig. 1(b), which cancels the pole in (2.7b) and leaves the logarithm with an arbitrary scale.

B. Residual gauge freedom and IR counterterms

In general, the IR problem is related to a large- (infinite) distance behavior of the theory, or equivalently, to the existence of soft- (zero) momentum quanta of the massless fields. In the example above the singularity is produced by a soft contribution to the photon propagator

$$D_{\mu\nu}(k)=\int d^Dx\,\exp(ikx)\langle 0\,|\,TA_\mu(x)A_\nu(0)\,|\,0\rangle\,,$$

i.e., it is related to the behavior of $A_\mu(x\to\infty)$. Since $A_\mu(x\to\infty)$ is a pure gauge, we expect that the IR contribution can be eliminated without changing the physical content of the theory. This is strictly true up to three loops in QED_3, as shown below.

We start from the Lagrangian (2.1) that defines the generating functional Z with usual boundary conditions on the set of fields $A_\mu,\psi,\bar\psi$,

$$Z=\int\mathscr{D}A_\mu\mathscr{D}\psi\,\mathscr{D}\bar\psi\exp\left[i\int d^Dx\,\mathscr{L}(A_\mu,\psi,\bar\psi)\right]\,,$$
$$A_\mu,\psi,\bar\psi\to 0\,,\ \text{when } x\to\infty\,. \qquad (2.8)$$

To change boundary conditions from $A_\mu(x\to\infty)\to 0$ to $A'_\mu(x\to\infty)\to B_\mu$, where $B_\mu=\text{const}$ we use a gauge transformation

$$A_\mu(x)\to A'_\mu(x)=A_\mu(x)+\partial_\mu\omega(x) \qquad (2.9a)$$

and require that (2.9a) belong to the residual gauge group. Therefore, $\omega(x)$ is specified by the following two conditions: transformation (2.9a) must respect the gauge-fixing term in (2.1), $\partial^\mu A_\mu=\partial^\mu A'_\mu$, and $A'_\mu(x\to\infty)\to B_\mu$, yielding

$$\partial^2\omega(x)=0\,, \qquad (2.9b)$$

$$\partial_\mu\omega(x\to\infty)=B_\mu\,. \qquad (2.9c)$$

Obviously, $\omega(x)=B_\mu x^\mu$ satisfies (2.9b) and (2.9c), and the transformation, allowed by a residual gauge freedom in (2.1), is

$$A_\mu\to A'_\mu=A_\mu+B_\mu\,,$$
$$\psi\to\psi'=\exp(-i\,eB_\mu x^\mu)\psi\,. \qquad (2.10)$$

In the new gauge, the functional form of Z in terms of the new fields is unchanged, except for the boundary conditions: $A'_\mu\to B_\mu$ and $\psi,\bar\psi\to 0$, when $x\to\infty$. To derive Feynman rules in the new gauge it is more convenient to change variables $A'_\mu=A_\mu+B_\mu$ and treat B_μ as a new field, then (2.8) becomes

$$Z=\int\mathscr{D}A_\mu\mathscr{D}\psi'\mathscr{D}\bar\psi'\exp\left[i\int d^Dx\,\mathscr{L}(A_\mu+B_\mu,\psi',\bar\psi')\right]$$
$$=\int\mathscr{D}A_\mu\mathscr{D}\psi\,\mathscr{D}\bar\psi\times\exp\left[i\int d^Dx[\mathscr{L}(A_\mu,\psi,\bar\psi)-eB_\mu\bar\psi\gamma^\mu\psi]\right]\,, \qquad (2.11)$$

where we have dropped primes on the dummy variables $\psi',\bar\psi'$.

Equation (2.11) defines a class of gauges parametrized by the value of B_μ. We now define an average over these gauges with a weight $\exp(iB^2/2\Delta)$, where Δ is a parameter to be determined later. That is, we use the following generating functional:

$$Z=\int d^D B\,\mathscr{D}A_\mu \mathscr{D}\psi\,\mathscr{D}\bar\psi \times\exp\left[\frac{iB^2}{2\Delta}+i\int d^D x[\mathscr{L}(A_\mu,\psi,\bar\psi) -eB_\mu\bar\psi\gamma^\mu\psi]\right]. \qquad (2.12)$$

The Feynman rules that follow from (2.12) are ordinary QED rules with additional vertex and propagator for the B_μ field, which carries zero momentum, Fig. 2. For a more rigorous treatment see Ref. 7, Appendix A.

This formal construction of the zero-momentum photon can be interpreted in the following way. QED, in the noninteracting limit, has a degenerate vacuum because the addition of a zero-momentum photon does not change the energy. When the interaction is treated perturbatively, a typical problem of degenerate perturbation theory arises. To avoid divergences one must take a suitable linear combination of the degenerate states to form a new ground state. In this sense we can interpret above transformation as a change of the vacuum state from a conventional one where $B_\mu=0$ to an average over all values of B_μ. Clearly, the same argument holds for any massless field, but only for gauge fields the two vacua are related by a gauge transformation. An example with a massless scalar field is considered in Sec. IV.

The rules of Fig. 2 directly give the diagram in Fig. 1(b). If we set

$$\Delta=\frac{ie'^2\mu}{96\pi^2}\left[\frac{1}{D-3}+O(1)\right] \qquad (2.13)$$

then the sum of the two diagrams in Fig. 1 is finite when $D=3$ and we get

$$\Sigma^{(2)}(p)=\frac{e'^4\mu^2\not p}{96\pi^2 p^2}\left[\ln\frac{\mu^2}{-p^2-i\epsilon}+\text{const}\right] \qquad (2.14)$$

in agreement with the result obtained by resummation technique.[5] In addition, from (2.13) and (2.14) it follows that the logarithm is gauge dependent, since the finite constant $O(1)$, in (2.13) can be chosen arbitrarily, giving an arbitrary scale to the logarithm.

The three-loop photon self-energy is finite since divergences arising in individual diagrams cancel in the sum. At this order Σ receives a divergent contribution only from the subdiagram of Fig. 1(a) and no additional counterterms are necessary. Consequently, in gauge-invariant amplitudes to order e^6 both poles and logarithms cancel as explicitly checked for the three-loop vacuum polarization.[5] At higher orders the leading divergences come from diagrams with a maximum number of subdiagrams of the type shown in Fig. 1(a). The counterterm of Fig. 1(b) is sufficient to cancel poles, leaving gauge-dependent leading logarithms.[7] This is a surprising result since the leading logarithms, being α independent, are usually believed to have a gauge-invariant origin. At order e^8 and higher, gauge-invariant divergences can arise. This is because the degree of IR divergence, for superrenormalizable interactions, increases with the loop order leading to the breakdown of perturbation theory at sufficiently high orders. On the basis of OPE analysis, we expect this to happen at four loops in QED_3. For further discussion of gauge-invariant singularities see Sec. III C.

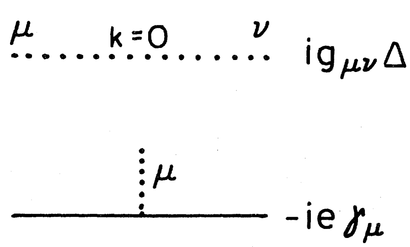

FIG. 2. Feynman rules for the B photons.

III. QCD_3

A. One- and two-loop self-energies

We consider an SU(N) gauge theory coupled to massless fermions, governed by the following Lagrangian, with gauge fixing and ghost term added:

$$\mathscr{L}=-\tfrac{1}{4}(F^a_{\mu\nu})^2+\bar\psi\gamma^\mu(i\partial_\mu-igA^a_\mu T^a)\psi-\frac{1}{2\alpha}(\partial_\mu A^\mu_a)^2 +\partial_\mu\bar u_a(\partial^\mu u_a-gf_{abc}A^\mu_b u_c),$$
$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-gf^{abc}A^b_\mu A^c_\nu. \qquad (3.1)$$

The color algebra

$$[T^a,T^b]=f^{abc}T^c \qquad (3.2a)$$

is normalized such that the T^a in the fundamental representation satisfy

$$\mathrm{Tr}(T^a_f T^b_f)=-\tfrac{1}{2}\delta^{ab}. \qquad (3.2b)$$

A representation R involves a Casimir eigenvalue Q_R,

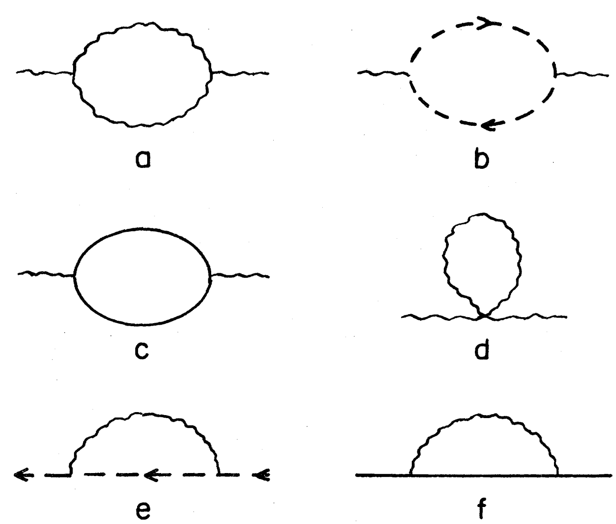

FIG. 3. The one-loop self-energies of (a)—(d) gluon, (e) ghost, and (f) fermion.

$$T^aT^a=-Q_R\ ,$$

$$\mathrm{Tr}(T^aT^b)=-C_R\delta^{ab}\ , \tag{3.2c}$$

$$C_R=Q_R\frac{\text{dim of representation}}{\text{dim of group}}\ .$$

Gluons are in the adjoint representation, which for SU(N) gives

$$f^{abc}f^{a'bc}=N\delta^{aa'}\ . \tag{3.3}$$

We study the two-point functions in D dimensions, and then set $D=3$. Using the same notation as in Sec. II with $g=g'\mu^{2-D/2}$ and the Feynman rules in Fig. 7 (without B gluons), for the lowest-order self-energies, Fig. 3, we get

$$\Pi^{(1)}_{\mu\nu}(p)=\tfrac{1}{2}ig'^2\mu^{2-D/2}N\int\frac{d^Dq}{(2\pi)^D}D^{\rho\tau}(q)D^{\lambda\sigma}(p+q)V_{\mu\rho\sigma}(-p,p+q,-q)V_{\nu\lambda\tau}(p,q,-p-q)$$
$$+ig'^2\mu^{2-D/2}N\int\frac{d^Dq}{(2\pi)^D}\frac{q_\mu(p+q)_\nu}{q^2(p+q)^2}+ig'^2\mu^{2-D/2}C_F\int\frac{d^Dq}{(2\pi)^D}\gamma_\mu\frac{1}{\not p+\not q}\gamma_\nu\frac{1}{\not q}\ , \tag{3.4}$$

$$\Pi^{(1)}_G(p)=-g'^2\mu^{2-D/2}N\int\frac{d^Dq}{(2\pi)^D}\frac{p^\mu(p+q)^\nu}{(p+q)^2}D_{\mu\nu}(q)\ , \tag{3.5}$$

$$\Sigma^{(1)}_F(p)=-ig'^2\mu^{2-D/2}Q_F\int\frac{d^Dq}{(2\pi)^D}\gamma^\mu\frac{1}{\not p+\not q}\gamma^\nu D_{\mu\nu}(q)\ , \tag{3.6}$$

where

$$D_{\mu\nu}(q)=\frac{-i}{q^2}\left[g_{\mu\nu}-(1-\alpha)\frac{q_\mu q_\nu}{q^2}\right],\quad V_{\alpha\beta\gamma}(p,q,r)=g_{\alpha\beta}(p-q)_\gamma+g_{\beta\gamma}(q-r)_\alpha+g_{\gamma\alpha}(r-p)_\beta\ . \tag{3.7}$$

The group factors are defined in (3.2) and the diagram d in Fig. 3 is zero in dimensional regularization. A straightforward calculation yields

$$\Pi^{(1)}_{\mu\nu}(p)=P_{\mu\nu}g'^2a_1p^2\left[\frac{\mu^2}{-p^2}\right]^{2-D/2},$$
$$a_1=\left[N\left[\frac{3D-2}{D-1}+(2D-7)(1-\alpha)+(1-D/4)(1-\alpha)^2\right]-C_F\frac{D-2}{D-1}\mathrm{Tr}1\right]\frac{\Gamma(2-D/2)B(D/2,D/2-1)}{(4\pi)^{D/2}}\ , \tag{3.8}$$

$$\Pi^{(1)}_G(p)=g'^2b_1p^2\left[\frac{\mu^2}{-p^2}\right]^{2-D/2},\quad b_1=\frac{N}{2}\left[\frac{4}{D-2}+D-5-\alpha(D-3)\right]\frac{\Gamma(2-D/2)B(D/2,D/2-1)}{(4\pi)^{D/2}}\ , \tag{3.9}$$

$$\Sigma^{(1)}_F(p)=g'^2c_1\not p\left[\frac{\mu^2}{-p^2}\right]^{2-D/2},\quad c_1=-\alpha Q_R(D-2)\frac{\Gamma(2-D/2)B(D/2,D/2-1)}{(4\pi)^{D/2}}\ . \tag{3.10}$$

In the limit $D=3$ everything is finite. $\Pi^{(1)}_{\mu\nu}$ has the characteristic opposite sign of its QED counterpart (2.5) if C_F is not too large. This sign would produce a tachyonic pole in the gluon propagator for $p\sim O(g'^2\mu)$, i.e., for the expansion parameter $g'^2(\mu^2/-p^2)^{1/2}\sim O(1)$, but in this momentum region the one-loop result cannot be trusted. Similarly, the ghost propagator would exhibit a tachyonic pole in the same momentum region according to (3.9).

It is interesting to notice how $\Pi^{(1)}_{\mu\nu}$ and $\Pi^{(1)}_G$ depend on the gauge parameter α. $\Pi^{(1)}_G$ becomes α independent in $D=3$, while $\Pi^{(1)}_{\mu\nu}$ acquires a term quadratic in α for $D\neq4$. In particular

$$a_1(D=3)=\frac{N}{64}[(1+\alpha)^2+10]-\frac{C_F}{16}>0$$

for real α in the pure Yang-Mills theory ($C_F=0$). The impossibility of setting a_1 to zero has been discussed, both in the covariant and the axial gauge.[9] By allowing α to be complex one can set $a_1(\alpha_c)=0$, therefore eliminating all IR divergences from two- and three-loop diagrams.[4] This has the disadvantage that for this complex α the gauge-fixing term, that in general acts as a convergence factor in the functional integral, will now make this integral diverge. In addition, $\mathscr{L}$ will not be Hermitian. We will take another approach: keep α real and use a residual gauge freedom to eliminate singularities coming from insertions of $\Pi^{(1)}_{\mu\nu}$ in the internal lines. It will be clear later that conclusions obtained by working with complex α can be obtained by this method also.

The first singularities are encountered at the two-loop level. The potentially divergent diagrams of the gluon self-energy are shown in Fig. 4. By power counting, only diagrams (a) and (b) of Fig. 4 can have IR divergences, while the remaining graphs can develop only ultraviolet (UV) divergences. Since QCD_3 is a superrenormalizable theory, the UV terms either sum to zero or combine with

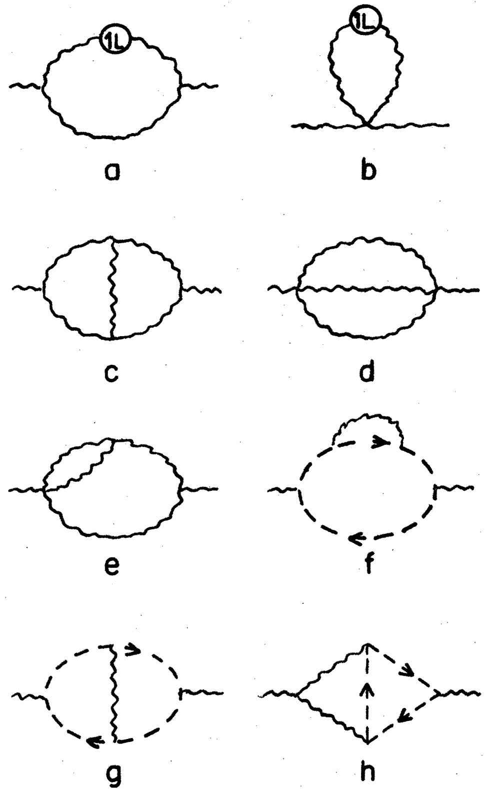

FIG. 4. Potentially divergent contributions to $\Pi^{(2)}_{\mu\nu}$.

IR terms. Figure 4(b) is zero in dimensional regularization, so we compute only Fig. 4(a). Using (3.8) as the $1L$ subdiagram, we evaluate the loop integral and then set $D=3$ wherever possible, retaining only the singular terms:

$$\Pi^{(a)}_{\mu\nu}(p)=g'^2\mu^2\frac{a_1N(2+\alpha)}{6\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{3-D}\times[P_{\mu\nu}\Gamma_{\rm IR}(D-3)+g_{\mu\nu}\Gamma_{\rm UV}(3-D)+O(1)]\ , \tag{3.11}$$

where a_1 is defined in (3.8) and subscripts IR and UV in the Γ functions stand for the momentum region responsible for the singularity. Since the IR term is transverse, the second term in (3.11) proportional to $g_{\mu\nu}$ must cancel against similar terms from remaining diagrams of Fig. 4; those cannot have divergent terms proportional to $p_\mu p_\nu$. Therefore, the total two-loop contribution to the gluon self-energy is

$$\Pi^{(2)}_{\mu\nu}(p)=P_{\mu\nu}g'^4\mu^2\frac{a_1N(2+\alpha)}{6\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{3-D}\times[\Gamma(D-3)+O(1)]$$
$$=P_{\mu\nu}g'^4\mu^2\frac{a_1N(2+\alpha)}{6\pi^2}\left[\frac{1}{D-3}-\ln\frac{\mu^2}{-p^2}+O(1)\right]\ . \tag{3.12a}$$

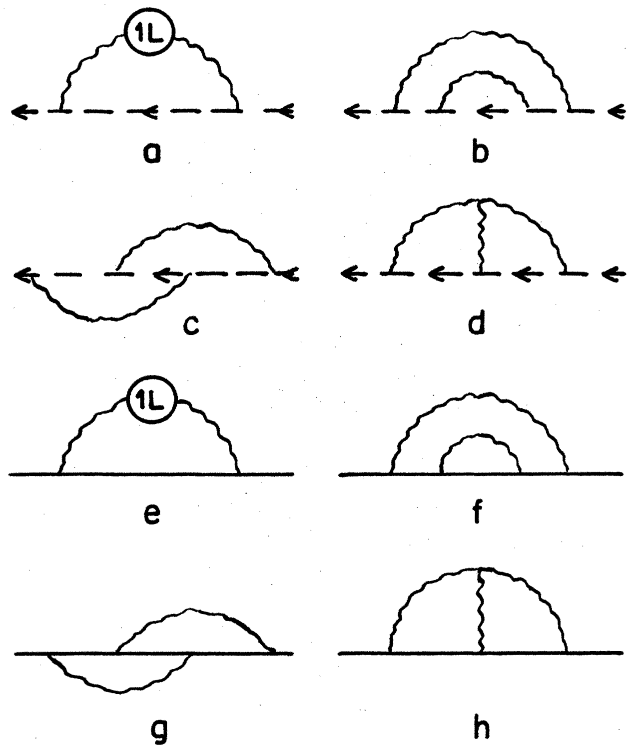

FIG. 5. The two-loop contribution to (a)–(d) ghost self-energy $\Pi^{(2)}_G$ and (e)–(h) fermion self-energy $\Sigma^{(2)}_F$.

The ghost and fermion self-energies are depicted in Fig. 5. Singularities appear only in Figs. 5(a) and 5(e):

$$\Pi^{(2)}_G(p)=g'^4\mu^2\frac{a_1N}{6\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{3-D}[\Gamma(D-3)+O(1)]$$
$$=g'^4\mu^2\frac{a_1N}{6\pi^2}\left[\frac{1}{D-3}-\ln\frac{\mu^2}{-p^2}+O(1)\right]\ , \tag{3.12b}$$

$$\Sigma^{(2)}_F(p)=g'^4\not p\frac{a_1Q_F}{6\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{4-D}[\Gamma(D-3)+O(1)]$$
$$=-g'^4\mu^2\frac{a_1Q_F\not p}{6\pi^2p^2}\left[\frac{1}{D-3}-\ln\frac{\mu^2}{-p^2}+O(1)\right]\ . \tag{3.12c}$$

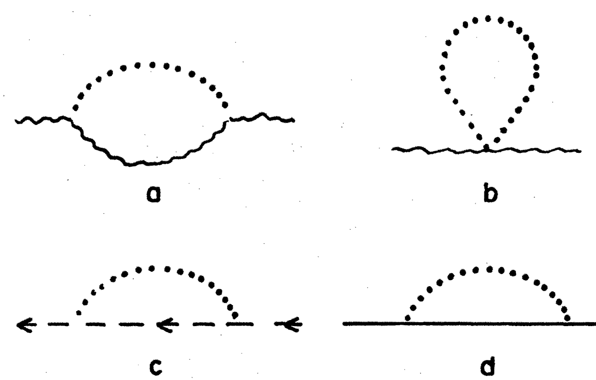

FIG. 6. The counterterms for (a) and (b) $\Pi^{(2)}_{\mu\nu}$, (c) $\Pi^{(2)}_G$, and (d) $\Sigma^{(2)}_F$.

To cancel poles at $D=3$ we construct counterterms (3.13) shown in Fig. 6 where dotted lines represent zero-momentum gluons (B gluons) obtained from a residual gauge symmetry in α gauges. Using Feynman rules, Fig. 7, we find

$$\Pi^{\rm ct}_{\mu\nu}(p)=P_{\mu\nu}g'^2\mu N(2+\alpha)i\Delta \ , \qquad (3.13a)$$

$$\Pi^{\rm ct}_{G}(p)=g'^2\mu N i\Delta \ , \qquad (3.13b)$$

$$\Sigma^{\rm ct}_{F}(p)=-g'^2\mu Q_F\frac{\not p}{p^2}i\Delta \ . \qquad (3.13c)$$

By comparing with (3.12), one sees that the pole at $D=3$ can be canceled by choosing

$$\Delta=\frac{ig'^2\mu a_1}{6\pi^2}\left[\frac{1}{D-3}+O(1)\right] \qquad (3.13d)$$

and we finally obtain

$$\Pi^{(2)}_{\mu\nu}+\Pi^{\rm ct}_{\mu\nu}=-P_{\mu\nu}g'^4 \times\mu^2\frac{a_1N(2+\alpha)}{6\pi^2}\left[\ln\frac{\mu^2}{-p^2-i\epsilon}+{\rm const}\right] , \qquad (3.14a)$$

$$\Pi^{(2)}_{G}+\Pi^{\rm ct}_{G}=-g'^4\mu^2\frac{a_1N}{6\pi^2}\left[\ln\frac{\mu^2}{-p^2-i\epsilon}+{\rm const}\right] , \qquad (3.14b)$$

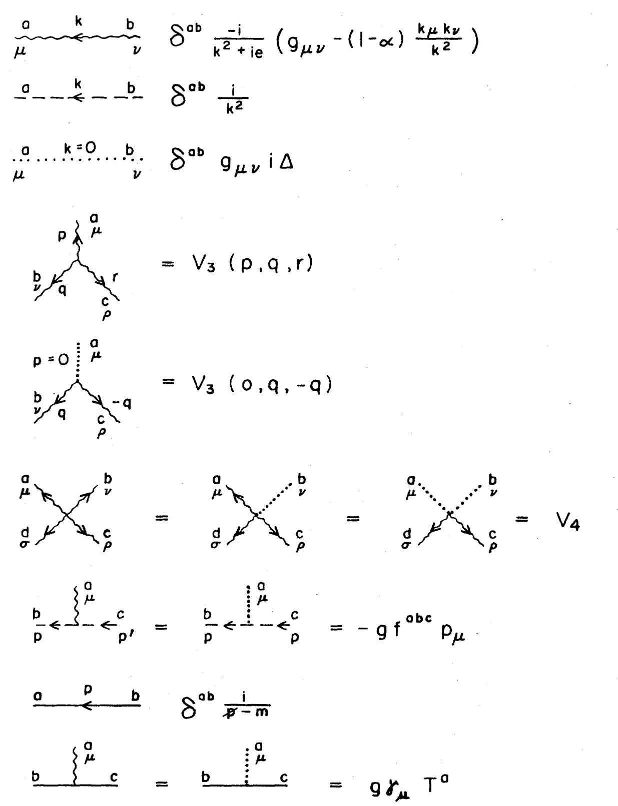

FIG. 7. Feynman rules for QCD with the B gluons: curly lines—gluons; dashed lines—ghosts; solid lines—fermions; dotted lines—B gluons. Vertices V_3 and V_4 are given by (3.18).

$$\Sigma^{(2)}_{F}+\Sigma^{\rm ct}_{F}=g'^4\mu^2\frac{a_1Q_F\not p}{6\pi^2p^2}\left[\ln\frac{\mu^2}{-p^2-i\epsilon}+{\rm const}\right] , \qquad (3.14c)$$

where the const stands for the finite diagrams and an arbitrary finite part of the counterterm. Results (3.14) agree with those obtained by resummation techniques.[5] This method, however, offers computational simplicity and shows explicit gauge dependence of the nonanalytic terms (i.e., the logarithms of the coupling constant).

B. Residual gauge freedom and IR counterterms

We shall now give a heuristic derivation of the Feynman rules for the zero-momentum gluons exploiting a residual gauge freedom in (3.1). The argument, although formal, closely follows the Abelian case, Sec. II B, and the final results give correct counterterms (3.13).

We start from the effective Lagrangian (3.1) with the usual boundary conditions on the set of fields ($A_\mu\equiv A^a_\mu T^a$):

$$A_\mu(x\to\infty)\to(\text{pure gauge})\to 0 \ ;$$

$\psi,\bar\psi,u,\bar u\to 0$, when $x\to\infty$. A gauge transformation (deformable to unity),

$$A_\mu\to A'_\mu=UA_\mu U^{-1}+\frac{i}{g}(\partial_\mu U)U^{-1} \qquad (3.15a)$$

can be used to modify $A_\mu(x\to\infty)$. We impose two conditions on $U(x)$: invariance of the gauge-fixing term in (3.1), and

$$A'_\mu(x\to\infty)\to B^a_\mu T^a \ ,$$

that yield

$$\partial^\mu\left[UA_\mu U^{-1}+\frac{i}{g}(\partial_\mu U)U^{-1}\right]=\partial^\mu A_\mu \ , \qquad (3.15b)$$

$$U(x\to\infty)=\exp(-igT^aB^a_\mu x^\mu) \ , \qquad (3.15c)$$

where $B^a_\mu=n^aB_\mu$, B_μ=const, and n^a is a unit vector. Equations (3.15) are non-Abelian counterparts of (2.9). Notice that presence of the non-Abelian term in (3.15b) does not allow a simple solution, as in QED. Fortunately, the exact form of $U(x)$ that satisfies (3.15b) and (3.15c) is not necessary for the construction of the counterterms. If we recall that IR behavior is governed by $A'_\mu(x\to\infty)$, then condition (3.15c) is sufficient for our purposes.

Since the Faddeev-Popov determinant (and consequently ghosts) is invariant under (3.15) and the fermion behaves similarly as in QED, the generating functional in terms of new fields becomes

$$Z=\int\mathscr{D}(A^{a'}_\mu,\psi',\bar\psi',u,\bar u)\times\exp\left[i\int d^Dx\,\mathscr{L}(A^{a'}_\mu,\psi',\bar\psi',u,\bar u)\right] ,$$

$$A^{a'}_\mu(x\to\infty)=B^a_\mu \ . \qquad (3.16)$$

To derive Feynman rules in this new gauge, we express $A^{a'}_\mu$ as a superposition of a new field $\tilde A^a_\mu$ vanishing at infinity and the pure gauge $B_\mu n^a$, and take an average over different gauges defined by different values of $B_\mu n^a$, with a weight $\exp(iB^2/2\Delta)$,

$$Z=\int d^DB\,dn^a\mathscr{D}(\tilde{A}^a_\mu,\psi,\bar{\psi},u,\bar{u})\times\exp\left[\frac{iB^2}{2\Delta}+i\int d^Dx\,\mathscr{L}(\tilde{A}^a_\mu+B^a_\mu,\psi,\bar{\psi},u,\bar{u})\right]. \quad (3.17)$$

The Feynman rules that follow from (3.17) are ordinary QCD rules with additional vertices and the propagator for the zero-momentum gluon (dotted lines), Fig. 7, where Δ is given by (3.14d) and

$$V_3(p,q,r)=gf^{abc}[g_{\mu\nu}(p-q)_\rho+g_{\nu\rho}(q-r)_\mu+g_{\rho\mu}(r-p)_\nu], \quad (3.18)$$

$$V_4=-ig^2[f^{xab}f^{xcd}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})+\text{perm}].$$

Notice that vertices with three or four B gluons vanish since $B^a_\mu T^a$ commutes with itself.

It is now obvious that to each divergent diagram, in Figs. 4 and 5, corresponds a counterterm obtained by replacing the internal line (with $1L$ insertion) with the B gluon. This is valid to all orders—a soft line is automatically canceled by the counterterm. Further justification of this method comes from the OPE results, discussed below.

C. Higher orders, OPE, and gauge-invariant singularities

So far we have considered self-energies to order g^4 and found only gauge-dependent divergences. Now we analyze $3L$ diagrams starting with $\Pi^{(3)}_{\mu\nu}$. There are two classes of potentially divergent graphs: The first class contains lines with $1L$ insertion, already encountered at the $2L$ level; those are taken care of to all orders. The second class contains lines with $2L$ insertion, Fig. 8, where $2L$ means connected, not necessarily one-particle irreducible (1PI). By power counting, it is linearly divergent $\sim g'^6\int d^3q\,q^{-4}f(p,q)$ as $q\to 0$, $f(p,q)$ finite, but it turns out to be finite for $D=3$. In general, diagrams of this form, with $2L$ replaced by $2nL$, $n=1,2,\ldots$, are finite. Therefore, all divergences at the $(2n+1)$-loop level are those from $2nL$ subdiagrams which can be canceled by counterterms, order by order. Then it follows that the leading and the first subleading logarithms are gauge dependent, to all orders. This is also true for the ghost and fermion self-energies. We can summarize the $3L$ result in the expression for $\Pi_{\mu\nu}$,

$$\Pi^{(3)}_{\mu\nu}(p)=P_{\mu\nu}g'^6a_3(\alpha)p^2\left[\frac{\mu^2}{-p^2}\right]^{3/2}\left[\ln\frac{\mu^2}{-p^2-i\epsilon}+\text{const}\right], \quad (3.19)$$

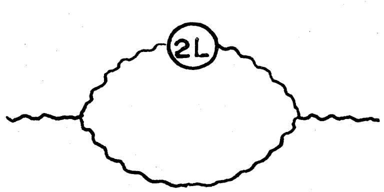

FIG. 8. A potentially divergent contribution to $\Pi^{(3)}_{\mu\nu}$.

where $a_3(\alpha)$ is a calculable constant. In order to simplify the analysis at the $4L$ level, it is convenient to completely cancel the contribution of the divergent $3L$ diagrams by setting $\alpha=\alpha_c$ such that $a_1(\alpha_c)=0$ in (3.8). This can be justified as a finite part fixing, in our counterterm procedure. For $\alpha=\alpha_c$ we get

$$\Pi^{(3)}_{\mu\nu}(p)=P_{\mu\nu}g'^6\tilde{a}_3(\alpha_c)p^2\left[\frac{\mu^2}{-p^2}\right]^{3/2}, \quad (3.20a)$$

$$\Pi^{(3)}_G(p)=g'^6\tilde{b}_3(\alpha_c)p^2\left[\frac{\mu^2}{-p^2}\right]^{3/2}, \quad (3.20b)$$

$$\Sigma^{(3)}_F(p)=g'^6\not{p}\,\tilde{c}_3(\alpha_c)\left[\frac{\mu^2}{-p^2}\right]^{3/2}, \quad (3.20c)$$

where $\tilde{a}_3$, $\tilde{b}_3$, and $\tilde{c}_3$ are calculable constants.

Now we consider $4L$ diagrams. In addition to finite and trivially divergent (if $\alpha\neq\alpha_c$) graphs, we also have dangerous ones, shown in Fig. 9. The soft-momentum flow through the internal line with a $3L$ insertion produces a divergence $\sim g'^8\int d^3q\,q^{-5}f(p,q)$ as $q\to 0$. These diagrams are analyzed most simply by choosing $\alpha=\alpha_c$, so that the 1PI $3L$ subdiagram is given by the finite expression (3.20a). The existing counterterms cannot be adjusted so to cancel new divergences, without spoiling the cancellation at $2L$ and $3L$ levels. We conclude that diagrams in Fig. 9 are the first signal of breakdown of the loop expansion in QCD_3 and we shall argue below that singularities at order g^8 are gauge invariant. One can write $\Pi^{(4)}_{\mu\nu}$ up to a calculable constant $a_4(\alpha_c)$

$$\Pi^{(4)}_{\mu\nu}(p)=P_{\mu\nu}g'^8a_4(\alpha_c)p^2\left[\frac{\mu^2}{-p^2}\right]^2\times\left[\frac{1}{D-3}-\ln\frac{\mu^2}{-p^2-i\epsilon}+O(1)\right], \quad (3.21)$$

where the logarithm now has a gauge-invariant meaning.

The gauge dependence of the IR singularities can be understood in the language of the OPE.[4,6] Since the coupling constant is dimensional, the loop expansion will

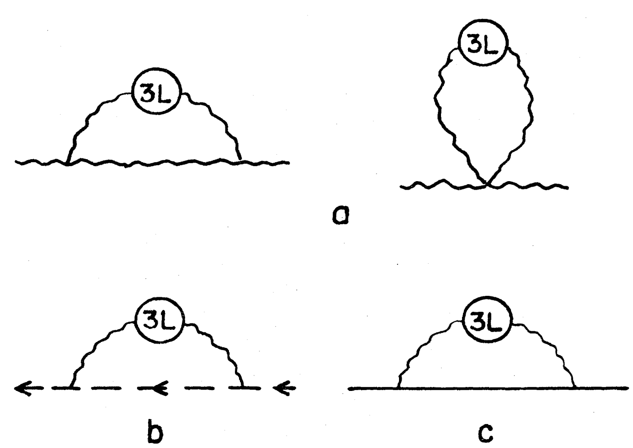

FIG. 9. The four-loop self-energies that contain leading gauge-invariant divergences.

eventually lead to operators of higher dimension than the unit operator. The occurrence of divergences signals the necessity of new operators in the OPE, which can be identified by dimensional analysis. At the $2L$ level $\Pi_{\mu\nu}$ develops a singularity canceled by the B gluon, where dimensions are as follows: (B-gluon propagator)$\sim\Delta\sim\mu$, $\Pi^{(2)}_{\mu\nu}\sim g^4\sim\mu^2$, so g^2 is needed to couple B gluon to the gluon line, Fig. 6. In the OPE language the new operator appearing in $\Pi^{(2)}_{\mu\nu}$ is the vacuum expectation value $\langle A^2\rangle$, because $A^2\sim\mu$ and g^2 couples $\langle A^2\rangle$ to the gluon line giving appropriate dimension μ^2. Clearly, there is a direct correspondence between our B-gluon propagator and $\langle A^2\rangle$ in the OPE. Gauge dependence of the two-loop singularities is obvious in both methods.

The gauge-invariant local operator of the lowest dimension is $(F^a_{\mu\nu})^2\sim\mu^3$ and $g^2\sim\mu$ is needed to couple it to the gluon line, so the first gauge-invariant singularities appear at g^8 order. Examples are shown in Fig. 9 where the soft lines with $3L$ insertion correspond to $\langle (F^a_{\mu\nu})^2\rangle$ in the OPE. Since the self-energies are gauge dependent it is more convenient to study the gauge-invariant Wilson loop

$$W=\left\langle P\exp\left[ig\oint dx_\mu A^\mu\right]\right\rangle . \tag{3.22}$$

For the analysis of IR divergences, in the loop expansion, an appropriate contour would be a small circle of radius $R\ll(g'^2\mu)^{-1}$, Fig. 10, so that only short distances are being probed. This case was considered in Ref. 4 and the cancellation of IR divergences has been checked through order g^4 ($n=1$ in Fig. 9). We shall now extend the analysis to the g^8 order. For a planar circle of radius R in D dimensions to order g^2 ($n=0$ in Fig. 10) one finds, Eq. (24) in Ref. 10,

$$W^{(0)}=-\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}4\pi^2g^2R^2\int_0^\infty dk\,(k^2)^{D/2-3/2}J_1{}^2(kR) . \tag{3.23}$$

Since the Bessel function has the asymptotic form $J_1(x\to 0)\sim x$ and $J_1(x\to\infty)\sim x^{-1/2}$, there is only an UV singularity at $D=3$ which is irrelevant for our discussion. At higher orders, the most severe IR divergences come from the soft lines with nL insertions, Fig. 10, where nL is

$$\Pi^{(n)}_{\mu\nu}\sim P_{\mu\nu}g'^{2n}k^2(\mu^2/-k^2)^{n/2} .$$

If $\Pi^{(n)}_{\mu\nu}$ is inserted into the gluon propagator it is easily seen that (3.23) can be written as ($x\equiv kR$, $D=3$)

$$W^{(n)}\sim g'^{2(n+1)}\mu R^2\int_0^\infty dk\,(k^2)^{D/2-3/2}\left[\frac{\mu^2}{-k^2}\right]^{n/2}J_1{}^2(kR)$$
$$\sim(g'^2\mu R)^{n+1}\int_0^\infty dx\,x^{-n}J_1{}^2(x) . \tag{3.24}$$

Notice that for a small loop, $g'^2\mu R\ll 1$, the higher orders would give a small correction in the absence of IR divergences. A similar situation was encountered in the expansion of self-energies: it converges for large external momentum p, $g'^2(\mu^2/-p^2)^{1/2}\ll 1$, until IR divergences set in. Also notice that we need consider only finite insertions in (3.24) for $n<4$ since divergent ones must cancel by gauge invariance. From (3.24) it follows that $W^{(n)}$ becomes IR divergent at order g^8, which is consistent with the self-energy results and the OPE analysis.

Although we have not calculated all contributions to $W^{(3)}$, we do not expect the leading singularity (3.24) to cancel. It would require that $3L$ subdiagrams sum up to zero, that is, $\tilde{a}_3(\alpha_c)=0$. We see no reason for this to happen. In conclusion, each soft internal line with $3L$ insertion produces a gauge-invariant singularity which corresponds to the appearance of $\langle (F^a_{\mu\nu})^2\rangle$ in the OPE.

IV. MASSLESS SCALAR QED$_3$

We have seen, in the previous sections, how an unbroken gauge symmetry can be utilized to cure the IR problem at low orders in the loop expansion. Since divergences are created by soft gauge field lines, it was sufficient to modify gauge fields only, at least up to three loops. The subject of this section is the loop expansion in massless SQED$_3$ which exhibits an interesting IR behavior: it offers an example of divergences generated by the soft scalar lines; in addition, gauge-invariant divergences appear at two loops, allowing more detailed analysis. This theory, with N charged scalars, has been studied both in the loop expansion and the $1/N$ expansion.[4] Here we take a closer look at the loop expansion. There are two sources of divergences in the two-loop SQED$_3$ self-energies: soft photon lines and the soft scalar lines, both with $1L$ insertions. The soft photon lines, which contribute to $2L$ scalar self-energy, can be handled in the same way as in QED$_3$, and will not be discussed further. Instead, we shall study divergences due to soft

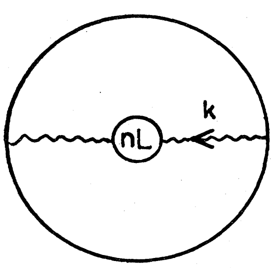

FIG. 10. A term that gives leading IR divergence in perturbation expansion of a small Wilson loop.

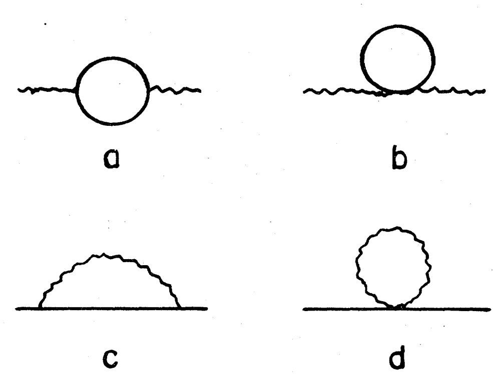

FIG. 11. The one-loop self-energies: (a) and (b) $\Pi^{(1)}_{\mu\nu}$; (c) and (d) $\Pi^{(1)}_S$.

scalar lines which first appear in the gauge-invariant photon self-energy $\Pi_{\mu\nu}^{(2)}$.

Consider a massless charged scalar ϕ coupled to a massless Abelian gauge field A_μ, in covariant gauges

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}{}^2+|(\partial_\mu+i\,eA_\mu)\phi|^2-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2\,, \qquad (4.1)$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,$$

where $e=e'\mu^{2-D/2}$. We proceed as before, by evaluating $1L$ photon and scalar self-energies, depicted in Fig. 11:

$$\begin{aligned}\Pi_{\mu\nu}^{(1)}(p)&=-e'^2\mu^{4-D}\int\frac{d^Dq}{(2\pi)^D}\frac{(p+2q)_\mu(p+2q)_\nu}{q^2(p+q)^2}\\&=-P_{\mu\nu}e'^2p^2\left[\frac{\mu^2}{-p^2}\right]^{2-D/2}\\&\quad\times\frac{2\Gamma(2-D/2)B(D/2,D/2-1)}{(D-1)(4\pi)^{D/2}}\\&=-P_{\mu\nu}\frac{e'^2p^2}{16}\left[\frac{\mu^2}{-p^2}\right]^{2-D/2}\,,\quad D=3\,,\end{aligned} \qquad (4.2)$$

$$\begin{aligned}\Pi_S^{(1)}&=-i\,e'^2\mu^{4-D}\int\frac{d^Dq}{(2\pi)^D}\frac{(2p+q)_\mu(2p+q)_\nu}{q^2(p+q)^2}\\&\quad\times\left[g_{\mu\nu}-(1-\alpha)\frac{q_\mu q_\nu}{q^2}\right]\\&=e'^2p^2\left[\frac{\mu^2}{-p^2}\right]^{2-D/2}[D-1+\alpha(3-D)]\\&\quad\times\frac{2\Gamma(2-D/2)B(D/2,D/2-1)}{(4\pi)^{D/2}}\\&=\frac{e'^2p^2}{4}\left[\frac{\mu^2}{-p^2}\right]^{2-D/2}\,,\quad D=3\,.\end{aligned} \qquad (4.3)$$

Both results are finite, but the expansion parameter $e'^2(\mu^2/-p^2)^{1/2}$ is IR sensitive, i.e., the loop expansion is valid at best for hard external momenta $p\gg e^2$. $SQED_3$, being superrenormalizable, needs only the scalar mass renormalization and the masslessness of the scalar field is implemented as a renormalization condition. However, in dimensional regularization it turns out that $\Pi_S^{(1)}$ is finite, and since we need only $\Pi_S^{(1)}$ for the two-loop diagrams, no renormalization is necessary.

The $2L$ photon self-energy $\Pi_{\mu\nu}^{(2)}$ is given by diagrams in Fig. 12. The IR-divergent contribution is to be expected from Fig. 12(a) because of the soft line with $1L$ insertion, but we find that other diagrams participate also, in order to preserve gauge invariance of the divergent term. By power counting, each diagram contributes an UV term proportional to $g_{\mu\nu}$, and the only IR term comes from Fig. 12(a), proportional to $p_\mu p_\nu$. To see how the divergent part of $\Pi_{\mu\nu}^{(2)}$ becomes transverse, we first look at a simpler, IR-finite case, with scalar mass $m\neq 0$. Explicit calculation shows that UV terms cancel in the sum, hence $\Pi_{\mu\nu}^{(2)}(m\neq 0)$ is finite as expected for a superrenormalizable theory. The same set of diagrams for $m=0$ gives the following divergent terms [Fig. 12(e) vanishes for $m=0$, and we set $D\to 3$]:

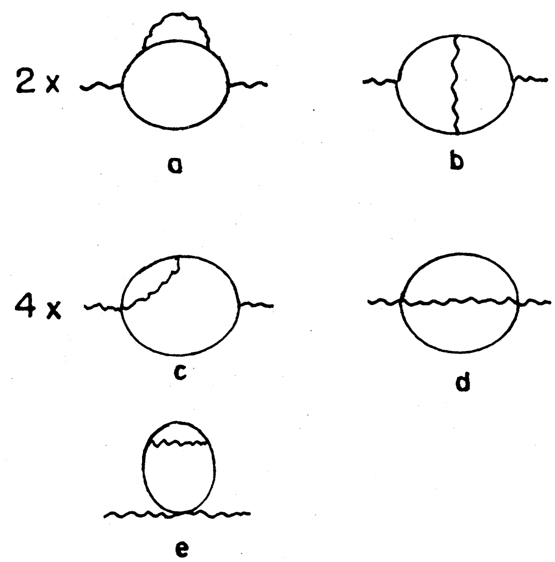

FIG. 12. The two-loop contribution to photon self-energy $\Pi_{\mu\nu}^{(2)}$.

$$\begin{aligned}\Pi_{\mu\nu}^{(a)}(p)&=-\frac{e'^4\mu^2}{8\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{3-D}\\&\quad\times\left[\tfrac{4}{3}g_{\mu\nu}\Gamma_{\rm UV}(3-D)\right.\\&\qquad\left.+\frac{p_\mu p_\nu}{p^2}\Gamma_{\rm IR}(D-3)+O(1)\right]\,,\end{aligned} \qquad (4.4)$$

$$\begin{aligned}\Pi_{\mu\nu}^{(b+c+d)}(p)&=-\frac{e'^4\mu^2}{8\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{3-D}\\&\quad\times[-\tfrac{1}{3}g_{\mu\nu}\Gamma_{\rm UV}(3-D)+O(1)]\,,\end{aligned} \qquad (4.5)$$

$$\begin{aligned}\Pi_{\mu\nu}^{(2)}(p)&=\Pi_{\mu\nu}^{(a)}+\Pi_{\mu\nu}^{(b+c+d)}\\&=P_{\mu\nu}\frac{e'^4\mu^2}{8\pi^2}\left[\frac{\mu^2}{-p^2}\right]^{3-D}[\Gamma(D-3)+O(1)]\\&=P_{\mu\nu}\frac{e'^4\mu^2}{8\pi^2}\left[\frac{1}{D-3}-\ln\frac{\mu^2}{-p^2}+\text{const}\right]\,.\end{aligned} \qquad (4.6)$$

The subscripts UV and IR to the Γ functions indicate the regions of the momentum variable causing the divergence, as determined by power counting. Notice that divergences disappear when $m\neq 0$ and in this sense they can be collectively labeled as infrared. The reason for this mixing of UV and IR terms is in the dimensional regularization method. Since massless integrals have no scale except for the overall one, the external momentum, there is no good separation between IR and UV effects. If another regularization method is used, then Figs. 12(a) and 12(e) would give a transverse IR term and the UV terms would cancel among themselves.

The final result (4.6) contains a pole at $D=3$, a logarithm, and a constant which cannot be determined in the loop expansion. Although the const looks like a mass term, it should be noticed that (4.6) is valid only for large external momentum p; in fact, the $1/N$ expansion gives vanishing $\Pi^{(2)}_{\mu\nu}(p)$ for $p \to 0$. So, perturbation theory in $SQED_3$ breaks down at order g^4. This suggests two possibilities: either the vacuum is unstable, and radiative corrections induce symmetry breaking $\langle\phi\rangle \neq 0$; or the symmetric vacuum $\langle\phi\rangle=0$ survives but perturbation theory is inadequate. The second possibility seems more likely, according to $1/N$ calculations.[4] We shall assume that this is the case and employ our counterterm technique to get a finite result from (4.6). Since the singularity is produced by a soft scalar line with $1L$ insertion, Figs. 12(a) and 12(e), the counterterms must contain a zero-momentum scalar propagator, which is equivalent to the appearance of $\langle\, |\phi|^2 \rangle$ in the OPE.

To construct additional Feynman rules, we start with the generating functional, explicitly displaying only scalar sources

$$Z[J,J^*]=\int \mathscr{D}A_\mu \mathscr{D}\phi\, \mathscr{D}\phi^* \times \exp\left[i \left[S + \int d^Dx (J\phi + \mathrm{H.c.}) \right] \right] . \quad (4.7)$$

Making a shift of fields $\phi \to \phi + \phi_c$, $\phi_c = \mathrm{const}$, we get

$$Z[J,J^*]=\int \mathscr{D}A_\mu \mathscr{D}\phi\, \mathscr{D}\phi^* \exp\left[i \left[\int d^Dx\, |\partial_\mu \phi|^2 + S_0(A_\mu) + S_I(\phi+\phi_c, A_\mu) + \int d^Dx\, J(\phi+\phi_c) + \mathrm{H.c.} \right] \right] , \quad (4.8)$$

where S_I contains interaction terms. Notice that the action in (4.8) has a symmetry $\phi \to \phi + \mathrm{const}$ in the noninteracting limit. To remove this ambiguity and ensure $\langle\phi\rangle = 0$ in perturbation theory, we introduce a small mass term and take limit $m \to 0$ at the end. By averaging (4.8) over ϕ_c with a convenient measure $\exp(i\phi_c\phi_c^*/\Delta)$, we obtain

$$Z[J,J^*]=\lim_{m\to 0} \int \mathscr{D}A_\mu \exp\left[iS_I \left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta J^*}, A_\mu \right] + iS_0(A_\mu) \right] \times \int d\phi_c d\phi_c^* \mathscr{D}\phi\, \mathscr{D}\phi^* \exp\left[\frac{i\phi_c\phi_c^*}{\Delta} + i\int d^Dx(\,|\partial_\mu\phi|^2 - m^2|\phi|^2) + i\int d^Dx [J(\phi+\phi_c) + \mathrm{H.c.}] \right] . \quad (4.9)$$

The Feynman rules that follow from (4.9), in addition to the usual ones, include a new propagator and vertices depicted in Fig. 13; dotted lines are ϕ_c fields and Δ is a parameter. Vertices involving ϕ_c are obtained from corresponding ϕ vertices by setting to zero momentum carried by the field ϕ.

This procedure is similar to a canonical transformation used by Bogolyubov to eliminate IR divergences.[11] The counterterms for the two-loop photon self-energy, Fig. 14, give

$$\Pi^{\mathrm{ct}}_{\mu\nu}(p) = P_{\mu\nu} 2e^2 i\Delta\ . \quad (4.10)$$

To cancel the pole in (4.6) we set

$$\Delta = \frac{ie'^2\mu}{16\pi^2} \left[\frac{1}{D-3} + O(1) \right] \quad (4.11)$$

and obtain the finite part of $\Pi^{(2)}_{\mu\nu}$,

$$\Pi^{(2)}_{\mu\nu} + \Pi^{\mathrm{ct}}_{\mu\nu} = -P_{\mu\nu} \frac{e'^4\mu^2}{8\pi^2} \left[\ln\frac{\mu^2}{-p^2 - i\epsilon} + \mathrm{const} \right] , \quad (4.12)$$

where const depends on the parameter Δ and cannot be fixed by perturbation theory alone. While in QED_3 and QCD_3, Δ is just another gauge parameter, here it has a nonperturbative meaning.

This result is in agreement with $1/N$ calculations.[4] The explicit construction of the counterterms confirms the OPE arguments: IR divergences arise in gauge-invariant amplitudes at low order if there is a gauge-invariant local operator of appropriate dimension in the OPE. In $SQED_3$ such an operator is $\langle\, |\phi|^2 \rangle$.

The procedure used above to construct counterterms has a simple interpretation. The original vacuum of a massless scalar theory is degenerate, leading to divergences in perturbation theory, familiar from quantum mechanics. By our transformation the generating functional (4.7) is replaced by a new one, Eq. (4.9), that gives a perturbation expansion around one particular linear combination of the degenerate vacua which differs from the original vacuum. This new perturbative vacuum is defined by a particular value of Δ.

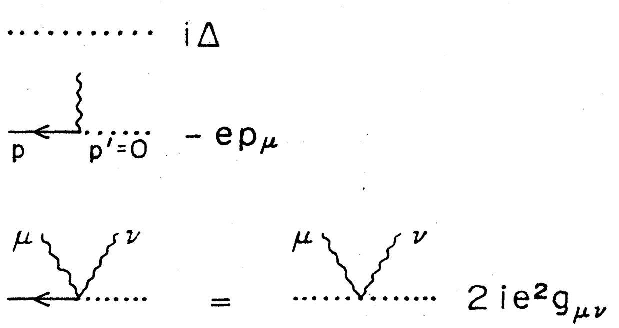

FIG. 13. Feynman rules for the ϕ_c scalars.

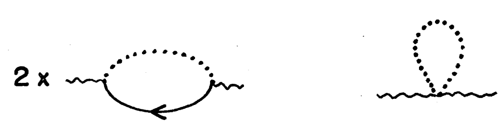

FIG. 14. The counterterms for the two-loop photon self-energy.

V. CONCLUSIONS

We have shown that IR divergences in QED_3 and QCD_3 are gauge artifacts up to three loops. The same is true for leading and first subleading divergences to all orders in perturbation theory. Using the residual gauge freedom in α gauges, we construct counterterms that cancel these singularities diagram by diagram. At fourth order the loop expansion breaks down. We find singularities of gauge-invariant origin in the Wilson loop, confirming the expectations based on the OPE analysis.

In $SQED_3$ gauge-invariant singularities already appear at two loops. In this case we generate counterterms by changing the vacuum of the free theory. However, the finite part of the counterterms cannot be determined in the loop expansion, some nonperturbative information is necessary. In both QCD_3 and $SQED_3$ we have demonstrated a one-to-one correspondence between our counterterm procedure and the OPE analysis.

Finally, we should point out that counterterms derived in Secs. II B and III B are a consequence of the residual gauge symmetry, rather than a special feature of three-dimensional theories. We expect that this method could be applicable to the IR problem in four dimensions.

ACKNOWLEDGMENTS

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