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Minimal lattice theory of fermions

K. M. Barad
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
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A new formalism for Susskind fermions is presented which provides a very efficient method for determining the quantum numbers carried by the components of the fields at various lattice sites. The addition of flavor-dependent mass terms and consequent symmetries are studied as an application. It is shown that removal of the SU(4) flavor symmetry may be accomplished without sacrificing the continuous chiral symmetry. On the basis of the insight gained about the influence of the coupling of modes on the isospin algebra, a free-theory construction with the minimal degeneracy compatible with known no-go theorems is given. That is, we achieve a theory with only two fermions in the continuum limit and a continuous chiral symmetry.

I. INTRODUCTION

It is now well accepted that a theory of lattice fermions cannot be found which is local, chirally symmetric, and circumvents the problem of spectrum multiplicity. Such no-go theorems have been discussed by Karsten and Smit and Nielsen and Ninomiya.[1,2] The latter group presents rigorous arguments concerning the topology of the Brillouin zone on the basis of the reasonable assumptions of locality, Hermiticity, and translation invariance by an integer number of lattice constants. Karsten and Smit argue that species doubling is a necessary ingredient in giving a consistent lattice regularization of the Adler-Bell-Jackiw anomaly.[3] They show that the extra excitations serve to cancel the anomaly, i.e., half the particles have chiral charge $+1$ and half have chiral charge -1. Although Karsten and Smit consider the obvious question of why the standard theories provide more than one extra species with which to cancel the anomaly, when one may suffice, their "incomplete answer," which basically amounts to an extra particle being provided for each of the four directions, leaves one curious about the possibility of finding a theory which circumvents possible conflicts with the anomaly by providing cancellation from just a single extra species. We feel that the question of such minimal theories obtainable within the constraints of the no-go theorems have not been given adequate consideration.

While the assignment of quantum numbers in the Wilson formulation is straightforward, this is not the case for Susskind fermions. The mixing of flavor symmetries with translations results in particles with different spin-parity assignments appearing as singularities at different parts of the Brillouin zone. The formalism described in Sec. II provides a method for determining the quantum numbers carried by the components of the fields at various lattice sites. Furthermore, our formalism provides an understanding of the vector vs chiral nature of the continuous symmetries which has caused some confusion in the literature.[4,5] On the basis of the insight gained about the influence of the coupling of modes on the underlying algebra, two applications are examined in Sec. III. First, a flavor-dependent mass term is added which removes the SU(4) flavor degeneracy allowing a phenomenologically realistic model of lattice gauge theories with Susskind fermions. We show that this may be achieved without sacrificing the continuous chiral symmetry of the flavor-degenerate Susskind theory, which is a problem other investigators[6–9] have encountered in devising schemes to remove the SU(4) degeneracy. That is, in these schemes, the proposed mass terms couple the modes in a way which makes the $U(1)\otimes U(1)$ invariance a purely vector symmetry, and not a chiral symmetry after all.[10] Secondly, we show that, at least for the free-theory case, a minimal theory can be constructed which has only two fermions and a continuous chiral symmetry in four dimensions. That is, the anomaly is canceled on the lattice by the introduction of a single chirally complementary species. The possibility of extending this result to the full interacting theory is discussed in Sec. IV.

II. FORMALISM

In the Susskind lattice theory of fermions[11–14] individual components of the Dirac spinors are placed on sites of the lattice. The general Euclidean form of the free action of massless particles is given by

$$S_F=\frac{1}{2}\sum_{r,\mu}\psi_r\Gamma_r^\mu(\psi_{r+\mu}-\psi_{r-\mu})\ ,$$

where the ψ_r are Grassmann variables and the lattice spacing has been set equal to unity. The usual sign factors one sees in the Susskind formulation, elements from the set Ω where

$$\Omega=\{1,(-1)^{r_i},(-1)^{r_i+r_j},(-1)^{r_i+r_j+r_u}\ ,\\ (-1)^{r_i+r_j+r_k+r_l}\,|\,i,j,k,l=1,2,3,4\}$$

(with possible factors of i to preserve Hermiticity), are suggestively labeled Γ^{μ}_{r} since they effectively play the role of the Dirac matrices in momentum space. This observation gives motivation to the following formalism which we propose to elucidate the underlying algebraic structure of the theory. For concreteness, and to avoid the cumbersome explicit expressions in four dimensions, the formalism itself is outlined for a two-dimensional theory. However, it is straightforward to generalize this method to four-dimensional Euclidean space-time.

Consider the following two-dimensional Lagrangian for the free theory:

$$S_F=\sum_{xy}\bar{\psi}_{xy}[(\psi_{x+1,y}-\psi_{x-1,y})+(-1)^x(\psi_{x,y+1}-\psi_{x,y-1})+m\psi_{xy}]\ . \tag{2.1}$$

Now let

$$\psi_{xy}=\int_{-\pi/2}^{3\pi/2}dp_x dp_y e^{ip_x x}e^{ip_y y}\phi(p_x,p_y)$$

and perform the Fourier transformation of the action (2.1). After carrying out the sum over x and y, the momentum-space action is given by the expression

$$S_F(p)=\int_{-\pi/2}^{3\pi/2}dp_x dp_y[\bar{\phi}(p_x,p_y)2i\sin p_x\phi(p_x,p_y)+\bar{\phi}(p_x+\pi,p_y)2i\sin p_y\phi(p_x,p_y)+m\bar{\phi}(p_x,p_y)\phi(p_x,p_y)]\ .$$

There are two observations to be made from the outset: (1) the replacement of the first derivative of the Dirac operator by a central difference gives rise to 2^d independent modes in momentum space, and (2) the factor of $(-1)^x$ serves to shift the momentum by π in the p_x direction. As emphasized above, modulation factors from the set Ω, such as $(-1)^x$, play a role in simulating the Dirac algebra in momentum space. In order to see this explicitly, it is useful to define the fields as an expansion in the low-frequency components defined in the first Brillouin zone $k_x,k_y\in[-\pi/2,\pi/2]$ times specific modulation factors which refer the modes to the appropriate sector of momentum space. Specifically, we define

$$\phi_{00}(k_x,k_y)=\phi(p_x,p_y)\text{ for }p_x=k_x\ ,\ \ p_y=k_y\ ,$$
$$\phi_{01}(k_x,k_y)=\phi(p_x,p_y)\text{ for }p_x=k_x\ ,\ p_y=k_y+\pi\ ,$$
$$\phi_{10}(k_x,k_y)=\phi(p_x,p_y)\text{ for }p_x=k_x+\pi\ ,\ \ p_y=k_y\ ,$$
$$\phi_{11}(k_x,k_y)=\phi(p_x,p_y)\text{ for }p_x=k_x+\pi\ ,\ \ p_y=k_y+\pi\ ,$$

and hence we express the fields as

$$\psi_{xy}=\int\int dk_x dk_y e^{ik_x x}e^{ik_y y}[\phi_{00}(k_x,k_y)+(-1)^y\phi_{01}(k_x,k_y)+(-1)^x\phi_{10}(k_x,k_y)+(-1)^{x+y}\phi_{11}(k_x,k_y)]\ . \tag{2.2}$$

This expansion lends itself very naturally to a method of reproducing the Dirac algebra by introducing Pauli matrices which act in the space of the low-frequency components. Two sets of Pauli matrices $\{\sigma_i\}$ and $\{\rho_i\}$ are needed, one for each lattice direction. We use the usual representation of the Pauli matrices and hence the action of σ_1 is to shift the momentum by π in the k_x direction, i.e.,

$$(\sigma_1\phi)_{ij}=(\sigma_1)_{ik}\phi_{kj}\ .$$

Similarly, ρ_1 produces a shift by π in the k_y direction, i.e.,

$$(\rho_1\phi)_{ij}=(\rho_1)_{jk}\phi_{ik}\ .$$

Hence σ_1 and ρ_1 simulate the factors $(-1)^x$ and $(-1)^y$, respectively. On the other hand, the action of σ_3 on the low-frequency components is nonlocal,

$$(\sigma_3\psi)_{xy}=\int dk_x dk_y e^{ik_x x}e^{ik_y y}[\phi_{00}(k_x,k_y)-(-1)^x\phi_{10}(k_x,k_y)+(-1)^y\phi_{01}(k_x,k_y)-(-1)^{x+y}\phi_{11}(k_x,k_y)]\ . \tag{2.3}$$

Notice, however, that the local operator $e^{ik_x}\sigma_3$ corresponds simply to a shift by one lattice spacing in the x direction. The analogous observation can be made that $e^{ik_y}\rho_3$ is a shift by one lattice spacing in the y direction.

Such considerations elucidate the underlying algebraic structure of Susskind fermions. In the low-frequency limit of the theory the 4×4 reducible operator

$$P+m=\sigma_3 k_x+\sigma_1\rho_3 k_y+m\ , \tag{2.4}$$

which acts on the low-frequency components, corresponds to the Lagrangian (2.1). Obviously, the $2^d\times2^d$ matrix structure derives from the $2^{d/2}$-fold degeneracy inherent in the Susskind formulation in combination with the $2^{d/2}$ Dirac components which comprise a d-dimensional spinor. By appropriate similarity transformations this operator can be reduced to block form which gives the decoupling of flavors in momentum space. For the example above, the transformation UPU^{-1}, where

$$U=\frac{1+\rho_3}{2}+\frac{1-\rho_3}{2}\sigma_3 \tag{2.5}$$

gives rise to two flavors each satisfying the Dirac equation

$$(\sigma_3 k_x+\sigma_1 k_y+m)\phi=0\ . \tag{2.6}$$

The generalization to four dimensions should be obvious. We now require four sets of Pauli matrices $\{\sigma_i\}$, $\{\rho_i\}$, $\{\pi_i\}$, and $\{\tau_i\}$. A given action will then be transcribed into a $2^4 \times 2^4$ matrix equation for the long-range components. After reduction to block form we will interpret the direct-product operators $\sigma \otimes \rho$ as γ matrices and $\pi \otimes \tau$ as isospin matrices.

It should be mentioned that a local construction of the $2^{d/2}$ quark fields has been given by Kluberg-Stern *et al.*[15] The fields are assembled directly from hypercubes in configuration space and are found to have a remnant $U(1)_A \otimes U(1)_V$ continuous symmetry whose axial part is flavor nonsinglet. On the other hand, a momentum-space construction, such as the one presented here, has a $U(2^{d/2}) \otimes U(2^{d/2})$ continuous symmetry although some of the currents are nonlocal. However, the nonlocalities are rather mild far from a given lattice site and should not be of consequence in the long-wavelength regime of the continuum limit. Furthermore, our formalism gives a clear understanding of the symmetries and as such enables us to explore the removal of the SU(4) flavor degeneracy and the construction of minimal theories with particular attention being paid to the preservation of the continuous chiral symmetry which is the advantage enjoyed by the Susskind formulation.

III. REMOVAL OF FLAVOR DEGENERACY AND CONSTRUCTION OF MINIMAL THEORIES

All actions which reproduce the Dirac algebra in momentum space are equivalent. The action in this section,

$$S_F = \sum_{xyzt} \bar{\psi}_{xyzt} [i(-1)^{x+y+z+t} (\psi_{x+1,yzt} - \psi_{x-1,yzt}) + (-1)^{z+t} (\psi_{x,y+1,zt} - \psi_{x,y-1,zt}) + i(-1)^{z+t} (\psi_{xy,z+1,t} - \psi_{zy,z-1,t}) + (\psi_{xyz,t+1} - \psi_{xyz,t-1})] , \tag{3.1}$$

which differs from the more standard choice (e.g., Ref. 14)

$$S_F = \sum_{xyzt} \bar{\psi}_{xyzt} [(\psi_{x+1,yzt} - \psi_{x-1,yzt}) + (-1)^{x} (\psi_{x,y+1,zt} - \psi_{x,y-1,zt}) + (-1)^{x+y} (\psi_{xy,z+1,t} - \psi_{xy,z-1,t}) + (-1)^{x+y+z} (\psi_{xyz,t+1} - \psi_{xyz,t-1})] , \tag{3.2}$$

simply corresponds to a different representation of the algebra, and therefore the results are in no way dependent on any particular choice of the action. For the case of massless fermions, the Susskind action has a continuous symmetry

$$\begin{aligned} \psi &\to \exp[i\alpha(-1)^{x+y+z+t}]\psi , \\ \bar{\psi} &\to \bar{\psi}\exp[i\alpha(-1)^{x+y+z+t}] , \end{aligned} \tag{3.3}$$

which is a result of the fact that odd sites are then only coupled to even sites and as such can be given separate rotations. In our formalism, the generator of this rotation is represented by $T' = \sigma_1 \rho_1 \pi_1 \tau_1$. It is an essential point that the symmetry which T' generates depends upon the coupling of the modes and can be made to be either of the vector or chiral type.

In our formalism, the fermion action (3.1) corresponds to the operator

$$P' = i\sigma_3 \sigma_1 \rho_1 \pi_1 \tau_1 k_x + \rho_3 \pi_1 \tau_1 k_y + i\pi_3 \pi_1 \tau_1 k_z + \tau_3 k_t . \tag{3.4}$$

We reduce this operator to block form by the transformation $UP'U^{-1}$, where

$$U = \left[\frac{i+\pi_2\tau_2}{2} + \frac{1-\pi_2\tau_2}{2}\rho_2 \right] \left[\frac{1+\pi_3}{2} + \frac{1-\pi_3}{2}\sigma_3\rho_3 \right] \left[\frac{1+\tau_1}{2} + \frac{1-\tau_1}{2}\sigma_1\rho_3 \right] \left[\frac{1+\sigma_3\rho_3}{2} + \frac{1-\sigma_3\rho_3}{2}\pi_3\tau_1 \right] . \tag{3.5}$$

This gives

$$P = UP'U^{-1} = i\sigma_1\rho_3\rho_1 k_x + \sigma_3 k_y + i\sigma_1\sigma_3 k_z - \sigma_1\rho_3 k_t , \tag{3.6}$$

so that $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \sigma_1\rho_1$ and $T = UT'U^{-1} = -\gamma_5\pi_3\tau_3$.

We now want to consider adding a mass term to the action. *A priori,* the only restriction on the mass matrix M is that $[P,M]=0$ so as not to jeopardize Lorentz invariance which is to be recovered in the continuum limit. Hence the most general mass term is given by

$$M = m_{00} + \sum_i m_{i0}\pi_i + \sum_i m_{0i}\tau_i + \sum_{ij} m_{ij}\pi_i\tau_j . \tag{3.7}$$

Undoing the diagonalization, the free-theory action with the most general mass term is given by

$$\begin{aligned} S_F + S_M = \sum_{xyzt} \bar{\psi}_{xyzt} [& i(-1)^{x+y+z+t}(\psi_{x+1,yzt} - \psi_{x-1,yzt}) + (-1)^{z+t}(\psi_{x,y+1,zt} - \psi_{x,y-1,zt}) \\ & + i(-1)^{z+t}(\psi_{xy,z+1,t} - \psi_{xy,z-1,t}) + (\psi_{xyz,t+1} - \psi_{xyz,t-1})] \\ & + \sum_{xyzt} \bar{\psi}_{xyzt} [m_{00}\psi_{xyzt} + im_{01}(-1)^{x+y+z}\psi_{x+1,y+1,z+1,t} + im_{02}(-1)^{x}\psi_{x+1,y,z+1,t+1} + im_{03}(-1)^{y+z}\psi_{x,y+1,z,t+1} \end{aligned}$$

$$+im_{10}(-1)^{x+y+z}\psi_{xy,z+1,t}+im_{20}(-1)^{x}\psi_{x+1,yzt}+im_{30}(-1)^{y+z}\psi_{x+1,y,z+1,t}$$

$$+m_{11}\psi_{x+1,y+1,z,t}-m_{12}(-1)^{y+z}\psi_{x+1,yz,t+1}-m_{13}(-1)^{x}\psi_{x,y+1,z+1,t+1}$$

$$+m_{21}(-1)^{y+z}\psi_{x,y+1,z+1,t}-m_{22}\psi_{xy,z+1,t+1}+m_{23}(-1)^{x+y+z}\psi_{x+1,y+1,z,t+1}$$

$$+m_{31}(-1)^{x}\psi_{x,y+1,zt}-m_{32}(-1)^{x+y+z}\psi_{xyz,t+1}+m_{33}\psi_{x+1,y+1,z+1,t+1}] \ . \tag{3.8}$$

Notice that the individual terms in M are of two types: those which anticommute with T which we label M_v, and those which commute with T which we label M_c. The designation v and c stand for vector and chiral, since $\{P,T\}=0$ and as such M_c breaks the symmetry while M_v respects the symmetry generated by T.

In particular, if it is possible to construct a mass matrix of the vector type which has two zero eigenvalues and assigns arbitrary masses to the other two flavors, then the massive quarks can be removed from the dynamics of the continuum, by taking their masses to be inversely proportional to the lattice spacing, without jeopardizing the chiral invariance of the theory. Therefore, a theory of only two massless particles is obtained while maintaining the continuous symmetry generated by T which would be broken by a mass term of the usual type, i.e., a chiral symmetry.[16] In this way, the minimal fermionic theory allowed by the no-go theorem is achieved. This may be accomplished as follows.

Since

$$M_v=m_{01}\tau_1+m_{02}\tau_2+m_{10}\pi_1+m_{20}\pi_2+m_{13}\pi_1\tau_3+m_{23}\pi_2\tau_3+m_{31}\pi_3\tau_1+m_{31}\pi_3\tau_2 \ , \tag{3.9}$$

a straightforward calculation shows that choosing $m_{ij}=m_{ji}$ for all i, j produces a mass matrix which has two zero eigenvalues and two nonzero eigenvalues

$$\lambda=\pm 2({m_{01}}^2+{m_{02}}^2+{m_{13}}^2+{m_{23}}^2)^{1/2} \tag{3.10}$$

as desired.[17] Hence, an action which has this minimal realization property is

$$S_F+S_{M_v}=\sum_{xyzt}\bar{\psi}_{xyzt}[i(-1)^{x+y+z+t}(\psi_{x+1,yzt}-\psi_{x-1,yzt})+(-1)^{z+t}(\psi_{x,y+1,zt}-\psi_{x,y-1,zt})$$

$$+i(-1)^{z+t}(\psi_{xy,z+1,t}-\psi_{xy,z-1,t})+(\psi_{xyz,t+1}-\psi_{xyz,t-1})]$$

$$+\sum_{xyzt}\bar{\psi}_{xyzt}\{im_{01}[(-1)^{x+y+z}\psi_{x+1,y+1,z+1,t}+(-1)^{x+y+z}\psi_{xy,z+1,t}]$$

$$+im_{02}[(-1)^{x}\psi_{x+1,y,z+1,t+1}+(-1)^{x}\psi_{x+1,yzt}]$$

$$+m_{13}[-(-1)^{x}\psi_{x,y+1,z+1,t+1}+(-1)^{x}\psi_{x,y+1,zt}]$$

$$+m_{23}[(-1)^{x+y+z}\psi_{x+1,y+1,z,t+1}-(-1)^{x+y+z}\psi_{xyz,t+1}]\} \ . \tag{3.11}$$

Notice, however, that this construction requires that mass terms involving shifts in one direction on the lattice have the same mass coefficient as terms involving shifts in three directions. In the full interacting theory, renormalization effects may inhibit such equalities. However, even if the mass parameters should require tuning in the full theory, the fact remains that the removal of the SU(4) degeneracy may be accomplished in this way without sacrificing the continuous chiral invariance.

IV. CONCLUSIONS

We have presented a formalism which makes the analysis of the mass spectrum and symmetries of Susskind fermions clear and compact. The role of the modulation factors Γ_μ^r in reproducing the underlying Dirac algebra is made explicit. Since in the Susskind formulation only single Dirac components are assigned to the sites of the lattice, the question arises as to how the $2^{d/2}$ quark fields are to be constructed. By considering the addition of flavor-dependent mass terms, it was made clear that the nature of the continuous symmetries is directly dependent upon this construction. In particular, one way of coupling the modes may produce a vector symmetry and another a chiral symmetry. On the basis of this understanding, an attempt has been made to remove two of the four flavors from the dynamics of the four-dimensional theory without destroying the continuous chiral symmetry. Such a construction has been given for the free theory. Whether this construction survives possible renormalization effects in the full interacting theory is an open question. Naively, it does not seem so since it is required that terms involving three-link operators have the same mass coefficient as terms with one-link operators. A further study of the discrete symmetries of this model may be helpful in this regard. On the other hand, if this prescription fails to generalize to the full interacting theory, this may be indicative of a deeper physical meaning underlying the phenomena of species doubling. Banks *et al.*[18] suggest that one identify this multiplicity with the generation structure of the observed spectrum of quarks and leptons. Their interpretation is based on the observation that the Kahler-Dirac equation no longer decouples into $2^{d/2}$ independent equations once gravity is introduced. The interesting consequences of such an interpretation are discussed in their paper. In any case, an understanding of the symmetry structure provided by this formalism shows that it is possible to remove the SU(4) flavor symmetry while retaining the continuous chiral invariance which is the essential characteristic of the Susskind formulation. Hence, Susskind fermions may be used in phenomenologically realistic studies of theories with four flavors of quarks.

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