Mean-field-with-corrections approach to the mixed U(1) lattice gauge theory

Elbio Dagotto
Centro Atómico Bariloche, Comisión Nacional de Energía Atómica and Instituto Balseiro, Universidad Nacional de Cuyo, 8400 Bariloche, Argentina
(Received 6 March 1984)

The mixed U(1) lattice gauge theory is analyzed by the mean-field technique including $1/d$ corrections, d being the dimension of the lattice. In order to apply the saddle-point approximation required in the method, the convenience of considering different unconstrained degrees of freedom for each power of the link variables in the action is shown. The phase diagram given by Monte Carlo simulations is reproduced with high accuracy for $d=4$ and 5. Generalizations to other actions and gauge groups are also discussed.

I. INTRODUCTION

Lattice gauge theories[1] provide a powerful approach to the study of nonperturbative physics in QCD such as the confinement hypothesis. By Monte Carlo simulations the hadronic spectrum,[2] deconfinement temperatures,[3] glueball masses,[4] as well as many important physical magnitudes have been estimated. Nevertheless, it would be very useful to develop analytical techniques for the calculation of these quantities. Attempts in this direction are the mean-field methods,[5] variational techniques in both Hamiltonian[6,7] and Lagrangian[8] formulations, finite lattice extrapolations,[9] renormalization group,[10] and series expansions.[11]

Recently, the gauge-invariant version of the mean-field technique has been applied[12–15] to some models with great success. This approach has many advantages. Systematic corrections can be considered and Elitzur's theorem[16] is satisfied, i.e., the magnetization is zero. The phase diagrams given by Monte Carlo simulations are well reproduced despite the fact that the predicted transitions are always of first order.

In this paper we apply this technique to the study of a mixed U(1) action[17,18] in $d=4$ and 5 dimensions including $1/d$ corrections as presented in Ref. 13. Actions of the mixed type have been studied very carefully for many gauge groups. We prove that in order to apply the saddle-point technique (Sec. III) it is necessary to define different unconstrained degrees of freedom for any power of the link variable present in the action. In general, for any Abelian or non-Abelian group a different variable must be introduced for each character considered. In this way at zeroth order the naive mean-field results are reproduced and the one-loop correction vanishes when the dimension equals infinity. Although this fact is well known in statistical physics[19] it has not been taken into account in lattice gauge theories for actions where the variables are not linear. For this reason we believe that it is useful to develop one example in detail. In this way we reproduce with high accuracy the phase diagram given by Monte Carlo simulations.

The organization of the paper is as follows. In Sec. II we briefly review the main features of the U(1) mixed model. Section III is devoted to the zeroth-order calculation. The $1/d$ corrections are evaluated in detail in Sec. IV while in Sec. V the numerical results are given. We end this paper with a short discussion in Sec. VI.

II. U(1) MIXED ACTION

Mixed actions are defined by the combination of several characters of the group elements, on each plaquette of the lattice. They have been widely studied in order to verify the universality hypothesis, i.e., the continuum physics must not depend on fine details on the lattice. They have, in general, a rich and interesting phase structure, and an important test for any proposed analytical method is its capability to reproduce it.

The U(1) mixed action considered in this paper is given by the action

$$S=\beta\sum_{\text{plaq}}\frac{U_p+U_p^\dagger}{2}+\gamma\sum_{\text{plaq}}\frac{U_p{}^2+U_p{}^{2\dagger}}{2}\ , \tag{1}$$

where U_p is a standard plaquette variable belonging to the U(1) group defined on a hypercubical d-dimensional lattice. This action has been analyzed by Monte Carlo simulations,[17,18] the renormalization group,[20] microcanonical simulation, and analytical arguments.[21] The statistical averages are obtained through the partition function

$$Z=\left\{\prod_{\text{links}}\int dU_l\right\}\exp(S)\ , \tag{2}$$

where dU_l is the normalized gauge-invariant measure.

As was stated in Ref. 17 the study of this model is very important to verify the nonconfinement characteristic of QED in the continuum limit for $d=4$. The fact that with the Wilson action a deconfining transition appears[22] is not enough evidence because in an extended parameter space an analytical "window" may be found.

For completeness we review here the main features of the theory following Ref. 17. The model has many nontrivial limits. For both axes we recover the standard U(1) model with a second-order phase transition at β_c, $\gamma_c=1.005$ $(d=4)$.[22] If γ goes to infinity the link variables are constrained to take only two values ±1 so we obtain a $Z(2)$ gauge theory with a first-order phase transition at $\beta_c = 0.4407$ ($d=4$).[23]

Equation (1) has many interesting symmetries. Denoting a link variable as $U_\mu(r)$, where $r=(x_0,x_1,x_2,x_3)$ is a lattice site and $\mu=0,1,2,3$ are lattice directions ($d=4$), we consider the transformation

$$\begin{aligned} &U_0(r)\rightarrow U_0(r)\,,\\ &U_1(r)\rightarrow \begin{cases} WU_1(r) & \text{if } x_1 \text{ is odd}\,,\\ U_1(r) & \text{if } x_1 \text{ is even}\,,\end{cases}\\ &U_2(r)\rightarrow \begin{cases} WU_2(r) & \text{if } x_1+x_2 \text{ is odd}\,,\\ U_2(r) & \text{if } x_1+x_2 \text{ is even}\,,\end{cases}\\ &U_3(r)\rightarrow \begin{cases} WU_3(r) & \text{if } x_1+x_2+x_3 \text{ is odd}\,,\\ U_3(r) & \text{if } x_1+x_2+x_3 \text{ is even}\,,\end{cases} \end{aligned} \tag{3}$$

where W is a factor which appears an odd number of times for each plaquette. Taking $W=-1$ and changing $\beta\rightarrow-\beta$ the action remains unaltered and therefore the diagram is symmetric under a reflection with respect to the γ axis. Analogously, when $W=\exp(i\pi/2)$ and $\beta=0$ we may change $\gamma\rightarrow-\gamma$ without changing S. This symmetry implies that for $\gamma=-1.005$ a phase transition is present. This last phase may be easily identified if we look for a nontrivial minimum of the plaquette action

$$S_p=\beta\cos\theta_p+\gamma\cos 2\theta_p$$

in $\gamma<0$. This minimum exists and is given by

$$\cos\theta_m=-\frac{\beta}{4\gamma} \tag{4}$$

and it can be easily proved that it becomes stable below $\gamma+\beta/4=0$. So the phase transition at $\gamma<0$ may be thought of as a change in the ground state of the theory.

The complete phase diagram obtained by Monte Carlo simulations[17] is given schematically in Fig. 1. The corresponding one[18] for $d=5$ is qualitatively equivalent. Phases I and II are the usual confining and nonconfining ones in QED. Phase III is a continuation for finite γ of the confining $Z(2)$ phase. Finally, phase IV has "antiferromagnetic" characteristics due to its nontrivial ground state.

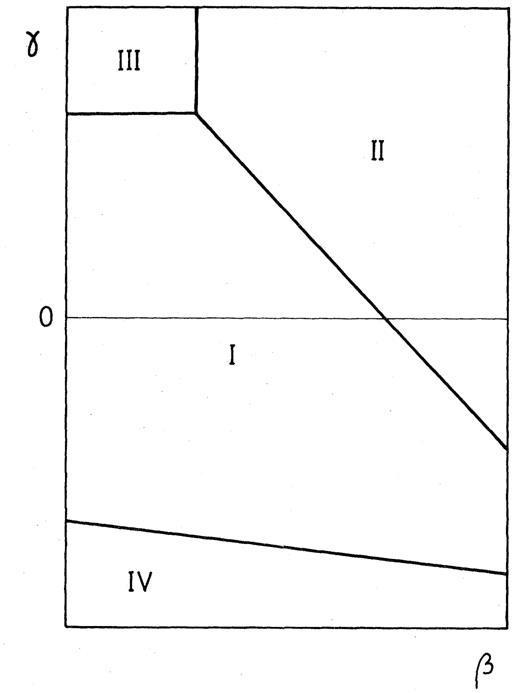

FIG. 1. Schematic representation of the U(1) extended phase diagram. An explanation of the different phases (I—IV) is given in Sec. II.

III. MEAN-FIELD APPROXIMATION (ZEROTH ORDER)

The naive mean-field approximation[24] in lattice gauge theories is obtained by replacing the link variables by their average $\langle U\rangle$ on every link of the lattice except one. By a self-consistency equation, $\langle U\rangle$ can be found. This approach forgets Elitzur's theorem which demands that $\langle U\rangle=0$ due to its non-gauge-invariant character. It has been shown[12] that in order to satisfy this point the mean-field approach must be thought of as a saddle-point approximation[25] to the partition function written in unconstrained variables. The important detail in the present model Eq. (1) is that it is necessary to define different unconstrained variables for U and U^2 if we want to get the naive mean field as zeroth order. Indeed this is a consequence of the trivial fact that $\langle U^2\rangle\neq\langle U\rangle^2$. In Ref. 26 this detail has also been mentioned in the context of a $Z(4)$ extended gauge model. The calculations are as follows.

In Eq. (2) we introduce four new variables in each link of the lattice through the identity

$$1=\prod_{\text{links}}\int_{-\infty}^{+\infty} dv_{1l}dv_{2l}d\tilde{v}_{1l}d\tilde{v}_{2l}\delta[\mathrm{Re}(U_l)-v_{1l}]\delta[\mathrm{Im}(U_l)-v_{2l}]\delta[\mathrm{Re}(U_l^2)-\tilde{v}_{1l}]\delta[\mathrm{Im}(U_l^2)-\tilde{v}_{2l}]\,. \tag{5}$$

Using the expansion

$$\delta(x)=\int_{-\infty}^{+\infty}\frac{dy}{2\pi}e^{ixy}\,,$$

the partition function can be written as

$$Z(\beta,\gamma)\prod_{\substack{\text{links}\\ a=1,2}}\int_{-\infty}^{+\infty}\frac{dv_{al}}{\sqrt{2\pi}}\frac{d\tilde{v}_{al}}{\sqrt{2\pi}}\frac{d\alpha_{al}}{\sqrt{2\pi}}\frac{d\tilde{\alpha}_{al}}{\sqrt{2\pi}}\exp(S_{\text{eff}})\,, \tag{6}$$

where

$$S_{\rm eff}=S(v,\tilde{v})-i\sum_{\rm links}(v_{al}\alpha_{al}+\tilde{v}_{al}\tilde{\alpha}_{al})+\sum_{\rm links}\omega(\alpha_{al},\tilde{\alpha}_{al})\ , \tag{7}$$

$$S(v,\tilde{v})=\beta\sum_{\rm plaq}\frac{v_p+v_p^\dagger}{2}+\gamma\sum_{\rm plaq}\frac{\tilde{v}_p+\tilde{v}_p^\dagger}{2}\ , \tag{8}$$

$$\omega(\alpha_{al},\tilde{\alpha}_{al})=\ln\left[\int_{{\rm U}(1)}dU\exp\{i[{\rm Re}(U)\alpha_{1l}+{\rm Im}(U)\alpha_{2l}+{\rm Re}(U^2)\tilde{\alpha}_{1l}+{\rm Im}(U^2)\tilde{\alpha}_{2l}]\}\right]\ , \tag{9}$$

and in v_p ($\tilde{v}_p$) each link variable is equal to $v_{1l}+iv_{2l}$ ($\tilde{v}_{1l}+i\tilde{v}_{2l}$). Note that the coupling between the v_l,α_l and $\tilde{v}_l,\tilde{\alpha}_l$ fields lies only in the ω function. $S_{\rm eff}$ is invariant under the gauge transformation

$$\begin{aligned}&v_{1l}+iv_{2l}=v_l\to v_l\exp(i\theta)\ ,\\&\tilde{v}_{1l}+i\tilde{v}_{2l}=\tilde{v}_l\to\tilde{v}_l\exp(2i\theta)\ ,\\&\alpha_{1l}+i\alpha_{2l}=\alpha_l\to\alpha_l\exp(i\theta)\ ,\\&\tilde{\alpha}_{1l}+i\tilde{\alpha}_{2l}=\tilde{\alpha}_l\to\tilde{\alpha}_l\exp(2i\theta)\ .\end{aligned} \tag{10}$$

The saddle-point approximation to Eq. (6) is obtained by minimizing $S_{\rm eff}$ with respect to all the variables. The resulting equations are highly nontrivial and in order to solve them we look for a translation-invariant solution, i.e.,

$$\begin{aligned}&v_{1l}=v,\ \ \tilde{v}_{1l}=\tilde{v}\ ,\\&v_{2l}=0,\ \ \tilde{v}_{2l}=0\ ,\\&\alpha_{1l}=-i\alpha,\ \ \tilde{\alpha}_{1l}=-i\tilde{\alpha}\ ,\\&\alpha_{2l}=0,\ \ \tilde{\alpha}_{2l}=0\ .\end{aligned} \tag{11}$$

With the help of this further approximation the resulting equations are

$$v=\frac{1}{Z}\int dU\,{\rm Re}U\exp(\hat{S}),\ \ \tilde{v}=\frac{1}{Z}\int dU\,{\rm Re}U^2\exp(\hat{S})\ , \tag{12a}$$

where

$$\alpha=2\bar{\beta}v^3,\ \ \bar{\beta}=\beta(d-1)\ , \tag{12b}$$

$$\tilde{\alpha}=2\bar{\gamma}\tilde{v}^3,\ \ \bar{\gamma}=\gamma(d-1)\ , \tag{12c}$$

$$Z=\int dU\exp(\hat{S})\ , \tag{12d}$$

and

$$\hat{S}=\alpha\,{\rm Re}U+\tilde{\alpha}\,{\rm Re}U^2\ . \tag{12e}$$

After a little thought one recognizes them as the naive mean-field equations when $\langle U\rangle$ and $\langle U^2\rangle$ are considered as "order" parameters.[27] Equation (12) has three types of solutions. The trivial one ($v=0$, $\tilde{v}=0$) in the strong-coupling regime, an "ordered" solution ($v\neq0,\tilde{v}\neq0$) associated with the nonconfining QED phase, and another one ($v=0,\tilde{v}\neq0$) which corresponds to the $Z(2)$ confining phase.

Nevertheless, the present method implies that if a degeneracy occurs between some configurations one must sum over all of them. Thus, configurations gauge equivalent to the translation-invariant one give identical contributions to the free energy and gauge-noninvariant quantities vanish when averaged over the degeneracy. Then Elitzur's theorem is satisfied showing the consistency of the approach. Repeating the above steps Eqs. (5)–(12) for a more complicated U(1) action, like

$$S=\sum_{\rm plaq}\sum_n\beta_n\frac{U^n+U^{n\dagger}}{2}\ , \tag{13}$$

the mean-field results are reobtained if we introduce different unconstrained variables for each integer n. A similar situation occurs for non-Abelian models with or without matter. In this case a different variable must be considered for each character of the gauge group and for each local power of the matter field.

The free energy per unit link to zeroth order and, in the presence of a nontrivial solution to Eq. (12), is

$$F_0=-\frac{\bar{\beta}}{2}v^4-\frac{\bar{\gamma}}{2}\tilde{v}^4+v\alpha+\tilde{v}\tilde{\alpha}-\ln\left[\int dU\exp(\hat{S})\right]\ , \tag{14}$$

while for the trivial root ($v=\tilde{v}=0$) the corresponding free energy is zero.

A similar discussion to that given in Ref. 13 shows that F_0 is the exact result in the limit $d=\infty$. We will show explicitly (Sec. IV) that the corrections vanish in this limit.

In order to obtain the phase diagram at zeroth order we must compare the free energies in different phases and look for their crossing which is interpreted as a phase transition. To zeroth order we only need to look for the solutions of the equation

$$F_0(\bar{\beta}_c,\bar{\gamma}_c)=0\ . \tag{15}$$

Numerically solving Eq. (15) we obtain the phase diagram shown in Figs. 2(a) and 2(b). Although qualitatively correct, the transitions are appreciably shifted to the weak-coupling region in comparison with Monte Carlo simulations.

Equations (12) and (14) must be slightly modified in order to consider also the "antiferromagnetic" phase. This can be done introducing a new plaquette variable θ, i.e., instead of proposing a complete translation-invariant solution Eq. (11), we make the ansatz

$$\begin{aligned}&v_l=v_{1l}+iv_{2l}=v\exp(i\theta)\ ,\\&\alpha_l=\alpha_{1l}+i\alpha_{2l}=\alpha\exp(i\theta)\ ,\\&\tilde{v}_l=\tilde{v}_{1l}+i\tilde{v}_{2l}=\tilde{v}\exp(2i\theta)\ ,\\&\tilde{\alpha}_l=\tilde{\alpha}_{1l}+i\tilde{\alpha}_{2l}=\tilde{\alpha}\exp(2i\theta)\ ,\end{aligned} \tag{16}$$

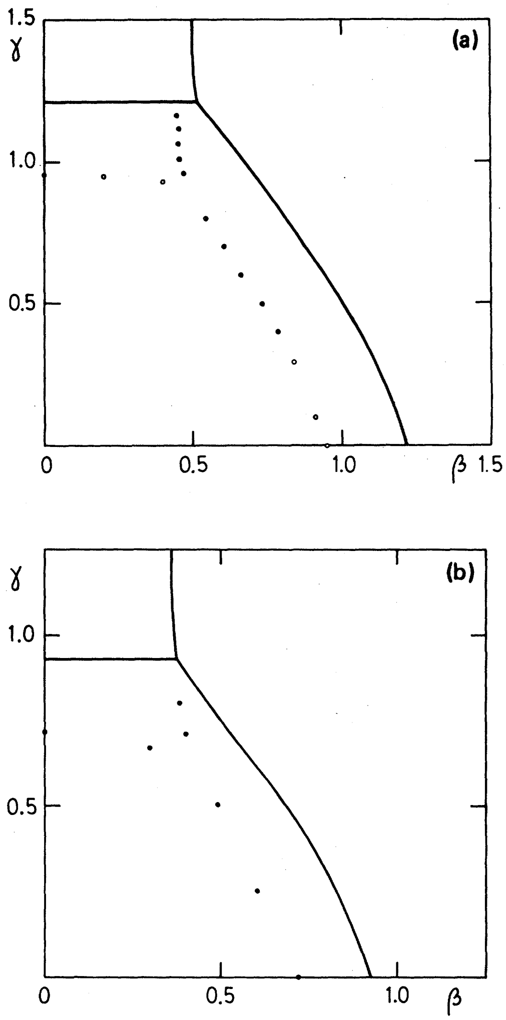

FIG. 2. The phase diagram predicted by mean-field calculations (continuous line) at zeroth order in (a) $d=4$ and (b) $d=5$ in comparison with Monte Carlo results taken from Refs. 17 and 18, respectively (a solid circle indicates a first-order transition while an open circle indicates a second-order one).

for, e.g., the same links where in Eq. (3) we apply the W factor. Stated in other words, we want the factor $\exp(i\theta)$ or $\exp(2i\theta)$ to appear only an odd number of times for each plaquette. The corresponding saddle-point equations are Eq. (12) with the change

$$\alpha \to \alpha\cos\theta ,$$

$$\tilde{\alpha} \to \tilde{\alpha}\cos 2\theta ,$$

and a new equation

$$\cos\theta=\frac{-\beta v^4}{4\gamma\tilde{v}^4} . \tag{17}$$

The free energy is

$$F_0=-\frac{\beta}{2}v^4\cos\theta-\frac{\gamma}{2}\tilde{v}^4\cos 2\theta +v\alpha+\tilde{v}\tilde{\alpha}-\ln\left[\int dU\exp(\hat{S})\right] . \tag{18}$$

Taking $\theta=\pi/2$ in the $\gamma<0$ axis we obtain the same free energy as in the $\gamma>0$ ($\theta=0$) case. Then a phase transition appears in the $\gamma<0$ region giving a qualitatively complete phase diagram. In the next section it is shown that the inclusion of $1/d$ corrections improves these results appreciably.

We remark that another approach is possible[28,29] without introducing an unconstrained variable for U^2. It consists of the factorization of a $Z(2)$ variable for each plaquette in the partition function, i.e.,

$$Z(\beta,\gamma)=\prod_{\text{links}}\left[\int dU_l\frac{1}{2}\sum_{\sigma_l=+1}\right]\times\exp\left[\beta\sum_{\text{plaq}}\sigma_p\mathrm{Re}U_p+\gamma\sum_{\text{plaq}}\mathrm{Re}U_p^2\right] . \tag{19}$$

The saddle-point equations have now the expected three phases in $\gamma>0$. Nevertheless, we evaluate the one-loop corrections in this case. They are finite when $d=\infty$ and the agreement with the Monte Carlo phase diagram is poor.

IV. $1/d$ CORRECTIONS

In this section corrections around the zeroth-order solution are evaluated by the steepest-descent method. For simplicity we deal only with the $\gamma>0$ region. Although the calculation is tedious we quote it here in detail because it is very illustrative on some technical aspects like the treatment of the zero-frequency mode. We begin with the corrections in phase II (Fig. 1). The corresponding ones for phases I and III are simpler and they are discussed at the end of the section.

The first step consists in a shift of the integration variables

$$\begin{aligned} v_{1l}&=v+z_{1l}, \quad \tilde{v}_{1l}=\tilde{v}+\tilde{z}_{1l} ,\\ v_{2l}&=0+z_{2l}, \quad \tilde{v}_{2l}=0+\tilde{z}_{2l} ,\\ \alpha_{1l}&=-i\alpha+\mu_{1l}, \quad \tilde{\alpha}_{1l}=-i\tilde{\alpha}+\tilde{\mu}_{1l} ,\\ \alpha_{2l}&=0+\mu_{2l}, \quad \tilde{\alpha}_{2l}=0+\tilde{\mu}_{2l} , \end{aligned} \tag{20}$$

and an expansion of the resulting effective action up to quadratic terms in the new variables (there are no linear terms because we are expanding around a minimum of the action). The integration in the $\mu,\tilde{\mu}$ variables is straightforward due to its local character. The resulting expression for the partition function is

$$Z(\beta,\gamma)=\frac{\exp(-F_0)}{(D_1D_2)^{Nd/2}}\left[\prod_{\text{links}}\int_{-\infty}^{+\infty}\frac{dz_{al}}{\sqrt{2\pi}}\frac{d\tilde{z}_{al}}{\sqrt{2\pi}}\right]\times\exp(S_{\text{eff(quad)}}) , \tag{21}$$

where we have defined

$$S_{\text{eff(quad)}}=S_{\text{quad}}-\sum_{\substack{\text{links}\\ a=1,2}}(E_a z_{al}{}^2+F_a\tilde{z}_{al}{}^2-G_a z_{al}\tilde{z}_{al}) , \tag{22}$$

$$D_a=\hat{Q}_a{}^2-4Q_a\tilde{Q}_a\ , \quad (23)$$

$$E_a=\tilde{Q}_a/D_a\ ,$$

$$F_a=Q_a/D_a\ , \quad (24)$$

$$G_a=\hat{Q}_a/D_a\ ,$$

and the integrals $Q_a,\tilde{Q}_a,\hat{Q}_a$ are listed in the Appendix. $S_{\rm quad}$ is the action Eq. (13) expanded up to quadratic terms. Note that the coupling between the fields z and $\tilde{z}$ is present only through the G_a factor. The information that U and U^2 are, in fact, related variables lies somehow in this coupling.

Next we turn to momentum space. This calculation has been done in Ref. 13 for the U(1) Wilson action so we quote here only the result:

$$Z(\beta,\gamma)=\frac{\exp(-F_0)}{(D_1D_2)^{Nd/2}}\left[\prod_{p,\mu,a}\int_{-\infty}^{+\infty}\frac{dz_{a\mu}(p)}{\sqrt{2\pi}}\frac{d\tilde{z}_{a\mu}(p)}{\sqrt{2\pi}}\right]$$
$$\times\exp\left\{-\sum_{\mu,\nu=1}^{d}\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}[z_{1\mu}\Lambda_{\mu\nu}z_{1\nu}+\tilde{z}_{1\mu}\tilde{\Lambda}_{\mu\nu}\tilde{z}_{1\nu}+z_{2\mu}\Omega_{\mu\nu}z_{2\nu}+\tilde{z}_{2\mu}\tilde{\Omega}_{\mu\nu}\tilde{z}_{2\nu}-\delta_{\mu\nu}(G_1z_{1\mu}\tilde{z}_{1\nu}+G_2z_{2\mu}\tilde{z}_{2\nu})]\right\}, \quad (25)$$

where

$$\Lambda_{\mu\nu}(p)=\delta_{\mu\nu}\left[E_1+\frac{\alpha}{2(d-1)v}\left[1-\sum_\lambda\cos p_\lambda\right]\right]-\frac{\alpha}{2(d-1)v}(1+e^{ip_\mu})(1+e^{-ip_\nu})\ , \quad (26a)$$

$$\Omega_{\mu\nu}(p)=\delta_{\mu\nu}\left[E_2+\frac{\alpha}{2(d-1)v}\left[1-\sum_\lambda\cos p_\lambda\right]\right]-\frac{\alpha}{2(d-1)v}(1-e^{ip_\mu})(1-e^{-ip_\nu})\ , \quad (26b)$$

$$\tilde{\Lambda}_{\mu\nu}(p)=\Lambda_{\mu\nu}(p)\ \ (E_1\to F_1,\alpha/v\to\tilde{\alpha}/\tilde{v})\ , \quad (26c)$$

$$\tilde{\Omega}_{\mu\nu}(p)=\Omega_{\mu\nu}(p)\ \ (E_2\to F_2,\alpha/v\to\tilde{\alpha}/\tilde{v})\ , \quad (26d)$$

and $z_\mu(p)$ and $\tilde{z}_\mu(p)$ are the Fourier transformations of z_l and $\tilde{z}_l$. Note that in the spirit of the forthcoming $1/d$ expansion we neglect in Λ and $\tilde{\Lambda}$ a $\cos p_\mu$. The resulting eigenvalues p,p_0 of Λ and q,q_0 of Ω are easily found because they can be written as a scalar plus a rank-1 matrix. We give them here because they will be used in subsequent calculations:

$$p=E_1-\frac{\alpha}{2v}+\frac{\alpha}{2(d-1)v}\sum_\lambda(1-\cos p_\lambda)\ ,$$
$$p_0=E_1-\frac{\alpha}{2v}-\frac{\alpha}{(d-1)v}\sum_\lambda(\cos p_\lambda)\ ,$$
$$q=E_2-\frac{\alpha}{2v}+\frac{\alpha}{2(d-1)v}\sum_\lambda(1-\cos p_\lambda)\ , \quad (27)$$
$$q_0=E_2-\frac{\alpha}{2v}\ ,$$

where p and q are $(d-1)$ degenerate while p_0 and q_0 are not degenerate. The corresponding eigenvalues for $\tilde{\Lambda}$ and $\tilde{\Omega}$ (denoted by $\tilde{p},\tilde{p}_0$ and $\tilde{q},\tilde{q}_0$, respectively) are found from Eq. (27) changing $E_a\to F_a$ and $\alpha/v\to\tilde{\alpha}/\tilde{v}$.

We define the $2d$-dimensional vectors Y and W and the $(2d\times2d)$-dimensional matrices A and B as follows:

$$Y=\begin{bmatrix} z_{11}\\ \vdots\\ z_{1d}\\ \tilde{z}_{11}\\ \vdots\\ \tilde{z}_{1d}\end{bmatrix},\quad W=\begin{bmatrix} z_{21}\\ \vdots\\ z_{2d}\\ \tilde{z}_{21}\\ \vdots\\ \tilde{z}_{2d}\end{bmatrix}, \quad (28)$$

$$A=\begin{bmatrix}\Lambda & -\frac{G_1}{2}1\\ -\frac{G_1}{2}1 & \tilde{\Lambda}\end{bmatrix},\quad B=\begin{bmatrix}\Omega & -\frac{G_2}{2}1\\ -\frac{G_2}{2}1 & \tilde{\Omega}\end{bmatrix} \quad (29)$$

[the momentum dependence is understood and the first index in Y, W corresponds to real (1) or imaginary (2) fluctuations]. Then the exponent in Eq. (25) can be written as

$$-\sum_{\rho,\epsilon=1}^{2d}\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}(Y_\rho A_{\rho\epsilon}Y_\epsilon+W_\rho B_{\rho\epsilon}W_\epsilon)\ . \quad (30)$$

As usual for continuous groups due to the local gauge invariance of the model a zero-frequency mode appears. We shall show explicitly that it occurs among the imaginary fluctuations, i.e., in the B matrix. In order to avoid it we must introduce a gauge condition which selects only fluctuations orthogonal to the zero mode. This can be performed by the standard Faddeev-Popov trick, i.e.,

$$1=\Delta_{\rm FP}\prod_x\int_0^{2\pi}d\phi_x\delta(\text{gauge condition }(\phi_x))\ , \quad (31)$$

where the "natural" gauge is deduced once we know the eigenfunction of the zero mode.

The corresponding eigenvalues λ for the matrices A and B can be easily deduced using Eq. (27). They are given by

$$\lambda_{01}^{A}=\tfrac{1}{2}\{p_0+\tilde{p}_0\pm[(p_0-\tilde{p}_0)^2+G_1{}^2]^{1/2}\}\ ,$$
$$\lambda_{23}^{A}=\tfrac{1}{2}\{p+\tilde{p}\pm[(p-\tilde{p})^2+G_1{}^2]^{1/2}\}\ ,$$
$$\lambda_{0}^{B}=0\ , \tag{32}$$
$$\lambda_{1}^{B}=q_0+\tilde{q}_0\ ,$$
$$\lambda_{23}^{B}=\tfrac{1}{2}\{q+\tilde{q}\pm[(q-\tilde{q})^2+G_2{}^2]^{1/2}\}\ .$$

Each eigenvalue involving $p,\tilde{p},q,\tilde{q}$ is $(d-1)$ degenerate while the rest are nondegenerate. All are positive definite. The existence of a zero eigenvalue in matrix B was demonstrated numerically in the whole β-γ plane. An analytic proof of this fact is not easy because the integrals $Q,\tilde{Q}$ and $\hat{Q}$ do not have a simple closed expression. The zero-mode eigenfunction is the $2d$-dimensional normalized vector

$$\psi^{(0)}(p)=\frac{1}{\left[2(1+k^2)\sum_{\lambda=1}^{d}(1-\cos p_\lambda)\right]^{1/2}}\begin{bmatrix}1-e^{ip_1}\\ \vdots\\ 1-e^{ip_d}\\ k(1-e^{ip_1})\\ \vdots\\ k(1-e^{ip_d})\end{bmatrix}, \tag{33}$$

where

$$k=\frac{G_2/2}{F_2-\tilde{\alpha}/2\tilde{v}}=\frac{E_2-\alpha/2v}{G_2/2}\ . \tag{34}$$

Closely following Ref. 13 we expand the fluctuations Eq. (28) in eigenfunctions of A and B ($\psi_A^{(n)}$ and $\psi_B^{(n)}$, respectively)

$$Y_\rho(p)=\sum_{n=1}^{2d}C_n^A(p)\psi_{A,\rho}^{(n)}(p)\ ,$$
$$W_\rho(p)=\sum_{n=1}^{2d}C_n^B(p)\psi_{B,\rho}^{(n)}(p)\ , \tag{35}$$

where the Jacobian of the transformation $Y_\rho,W_\rho\to C_n^A,C_n^B$ is one.

Introducing Eq. (31) in the partition function Eq. (25) we obtain

$$Z(\beta,\gamma)=\frac{\exp(-F_0)}{(D_1D_2)^{Nd/2}}\prod_{p,n}\int\frac{dC_n^A(p)}{\sqrt{2\pi}}\frac{dc_n^B(p)}{\sqrt{2\pi}}\Delta_{\mathrm{FP}}\prod_p\delta(\text{g.f.})\exp\left[-\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}\sum_{n=1}^{2d}(C_n^{A\,2}\lambda_n^A+C_n^{B\,2}\lambda_n^B)\right]\ . \tag{36}$$

Of course, the gauge still remains to be selected. We would need the δ function to be proportional to $\delta(C_0^B(p))$ in order to eliminate the divergence caused by the zero mode. This is the natural gauge condition. It can be easily proved that the condition

$$f(z,\tilde{z})=\sum_{\mu=1}^{d}(z_{2x,\mu}-z_{2x+\mu,\mu})+k\sum_{\mu=1}^{d}(\tilde{z}_{2x,\mu}-\tilde{z}_{2x+\mu,\mu})=0\ , \tag{37}$$

for all sites of the lattice, satisfies this requirement because in momentum space it can be written as

$$\prod_x\delta[f(z,\tilde{z})]=\prod_p\frac{\delta[C_0^B(p)]}{\left[2(1+k^2)\sum_\lambda(1-\cos p_\lambda)\right]^{1/2}}\ . \tag{38}$$

Once the gauge condition is known Δ_{FP} can be evaluated. In our $1/d$ expansion it is approximately given by

$$\Delta_{\mathrm{FP}}=\det C,\quad C_{x,y}=\sum_{\mu=1}^{d}(v+2k\tilde{v})(2\delta_{x,y}-\delta_{x,y+\mu}-\delta_{x,y-\mu})\ . \tag{39}$$

Then the integrations over $C_n^A(p)$ and $C_n^B(p)$ can be performed. Collecting all the contributions one gets for the free energy per unit link in phase II

$$F_{\mathrm{II}}=F_0+\frac{1}{2d}\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}[\ln(4\lambda_2^A\lambda_3^A)(d-1)+\ln(4\lambda_0^A\lambda_1^A)]+\frac{1}{2d}\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}[\ln(4\lambda_2^B\lambda_3^B)(d-1)+\ln(2\lambda_1^B)]$$
$$+\frac{1}{2d}\ln\left[\frac{2\pi(1+k^2)}{(v+2k\tilde{v})^2}\right]-\frac{1}{2d}\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}\ln\left[\sum_\mu 2(1-\cos p_\mu)\right]+\tfrac{1}{2}\ln(D_1D_2)\ , \tag{40}$$

where the first term comes from the real fluctuations, the second one from the $(2d-1)$ nonzero modes in the imaginary fluctuations, the third and fourth terms from the integration on the zero mode including the Faddeev-Popov factor and the group volume, and the last one from the $\alpha,\tilde{\alpha}$ integration.

The integrals must be evaluated in a $1/d$ expansion as in Ref. 13. The final result is

$$F_{\rm II}=F_0+\frac{1}{2d}\left\{\ln\left[1+2\left[\frac{\alpha}{v}Q_1+\frac{\tilde\alpha}{\tilde v}\tilde Q_1\right]\right]+\left[R-\frac{R^2}{4}\right]+\ln\left[\frac{2(q_0+\tilde q_0)(1+k^2)(-D_2)}{(v+2\tilde v k)^2}\right]+\left[T-\frac{T^2}{4}\right]-\ln(4\pi d)\right\}, \qquad (41)$$

where

$$R=-2\frac{\alpha}{v}Q_1,\quad T=-2\left[\frac{\alpha}{v}Q_1+\frac{\tilde\alpha}{\tilde v}\tilde Q_2\right].$$

In the β axis it can be proved that F is equal to the free energy given in Ref. 13.

Equation (41) will be used in the following section for the evaluation of the phase diagram of the model. The corrections to the other phases are simple to calculate. In phase III, for example, the free energy per unit link including β^2 corrections to lowest order can be written approximately[28] as

$$F_{\rm III}\approx-\frac{\bar\beta^2}{8d}+F_{\rm II}\left[0,\bar\gamma+\frac{\bar\beta^2}{4d}\right], \qquad (42)$$

where on the right-hand side $F_{\rm II}$ represents the $1/d$ approximation to the free energy in the γ axis with $\bar\gamma$ replaced by $\bar\gamma+\bar\beta^2/4$. The inclusion of the β^2 corrections are necessary in order that the transition between phases I and III has the correct curvature.

For phase I the correction to the $v,\alpha,\tilde v,\tilde\alpha=0$ solution is

$$F_{\rm I}=-\frac{\bar\beta^2+\bar\gamma^2}{8d}. \qquad (43)$$

Note that higher-order loop correlations also include $1/d$ terms. However, in Ref. 13 it has been conjectured that they vanish in the β axis [γ axis] as $(1-v)^2[(1-\tilde v)^2]$. This fact can be easily extended to the whole plane.

V. RESULTS

The phase diagram at zeroth order was obtained in Sec. III solving the equation

$$F[\bar\beta_c,\bar\gamma_c,v(\bar\beta_c,\bar\gamma_c),\tilde v(\bar\beta_c,\bar\gamma_c)]=0. \qquad (44)$$

Following Ref. 13 the shifts in critical parameters $(\Delta\bar\beta_c,\Delta\bar\gamma_c)$ due to the inclusion of the $1/d$ corrections are evaluated for large d from the expressions

$$\Delta\bar\beta_c\,|_{\rm I\text{-}II}=-\frac{2}{v^4}[\Delta F_{\rm I}(\bar\beta_c,\bar\gamma_c)-\Delta F_{\rm II}(\bar\beta_c,\bar\gamma_c)], \qquad (45a)$$

$$\Delta\bar\beta_c\,|_{\rm II\text{-}III}=-\frac{2}{v^4}[\Delta F_{\rm III}(\bar\beta_c,\bar\gamma_c)-\Delta F_{\rm II}(\bar\beta_c,\bar\gamma_c)], \qquad (45b)$$

$$\Delta\bar\gamma_c\,|_{\rm I\text{-}III}=-\frac{2}{\tilde v^4}[\Delta F_{\rm I}(\bar\beta_c,\bar\gamma_c)-\Delta F_{\rm III}(\bar\beta_c,\bar\gamma_c)], \qquad (45c)$$

where ΔF are the $1/d$ corrections in each phase.

Equation (45a) corresponds to the correction to $\bar\beta_c$ for the transition between phases I and II (Fig. 1). We also evaluate the corresponding one for $\bar\gamma_c$ proving that both ways give similar results for the predicted phase diagram.

Equations (45b) and (45c) are the corrections between phases II-III and I-III, respectively.

The results are shown in Fig. 3(a) for $d=4$ and in Fig. 3(b) for $d=5$. The agreement with Monte Carlo simulations[30] in a 4^4 lattice is excellent in both cases. Note that a more careful analysis[22] using a 6^4 lattice gives $\beta_c(\gamma_c)\approx1.005$ $(d=4)$ in the $\beta(\gamma)$ axis improving the agreement with our mean-field result. Nevertheless, all the transitions are incorrectly of first order as usual in mean-field techniques applied to gauge models. For example, the v parameter jumps from 0 to a value greater than 0.9 between phases I and II.

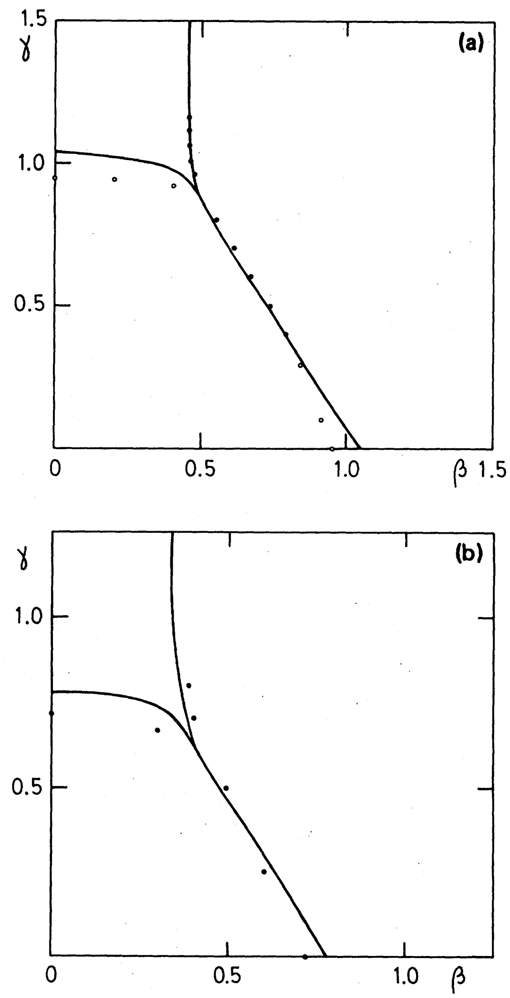

FIG. 3. The phase diagram predicted by mean-field calculations (solid curve) including $1/d$ corrections in (a) $d=4$ and (b) $d=5$. Monte Carlo results are from Refs. 17 and 18, respectively (a solid circle indicates a first-order transition while an open circle indicates a second-order one).

VI. CONCLUSIONS

In this paper we applied the gauge-invariant version of the mean-field technique (including $1/d$ corrections) to the U(1) extended model in $d=4$ and 5. The predicted phase diagram is in excellent agreement with the results from Monte Carlo simulations. For the application of the saddle-point technique we proved that it is necessary to introduce different unconstrained variables for U_l and U_l^2. The generalization to other gauge groups or in the presence of matter is immediate.

We remark that in spite of the satisfactory results obtained (Figs. 3) we do not prove that higher-order corrections are small. Indeed $d=4$ and 5 are not large enough.

We finish this section with a comment on future applications of the method. It has been proved that mean-field techniques are very accurate in the prediction of phase diagrams, but due to its local nature the continuum limit, where long-range fluctuations are important, is beyond its capabilities. For instance, in Ref. 31 it was explicitly proved that the predictions for Wilson loops are not accurate even for a small one (2×1 links). Perhaps the only way to obtain physical results (string tension, glueball masses, etc.) is to reach the scaling region for β not too large. A similar observation is valid for other local techniques like the variational one in Lagrangian and Hamiltonian formulation with local trial states. One way to implement this program could be to obtain the consistency condition for a large number of exactly treated variables (Bethe-Peierls approach).[7] However, it is not known how to adapt the saddle-point trick to this case. This fact deserves further study.

After completion of this work we received a report[32] where results identical to ours are obtained using the covariant gauge.

ACKNOWLEDGMENTS

We would like to thank A. García and L. Masperi for a careful reading of this manuscript. We are also grateful to G. Bhanot for stimulating correspondence. The author is supported by the Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

APPENDIX

The combination of integrals Q_a, $\tilde{Q}_a$, $\hat{Q}_a$ which appear in Sec. IV are listed here:

$$Q_1=\tfrac{1}{2}(\langle\cos\phi\rangle^2-\langle\cos^2\phi\rangle)\ , \tag{A1}$$

$$Q_2=-\tfrac{1}{2}\langle\sin^2\phi\rangle\ , \tag{A2}$$

$$\tilde{Q}_1=\tfrac{1}{2}(\langle\cos2\phi\rangle^2-\langle\cos^2 2\phi\rangle)\ , \tag{A3}$$

$$\tilde{Q}_2=-\tfrac{1}{2}\langle\sin^2 2\phi\rangle\ , \tag{A4}$$

$$\hat{Q}_1=(\langle\cos\phi\rangle\langle\cos2\phi\rangle-\langle\cos\phi\cos2\phi\rangle)\ , \tag{A5}$$

$$\hat{Q}_2=-\langle\sin\phi\sin2\phi\rangle\ , \tag{A6}$$

where

$$\langle f(\phi)\rangle=\frac{\int_0^{2\pi}\frac{d\phi}{2\pi}e^S f(\phi)}{\int_0^{2\pi}\frac{d\phi}{2\pi}e^S}$$

and

$$S=\alpha\cos\phi+\tilde{\alpha}\cos2\phi\ .$$

All these integrals Eqs. (A1)—(A6) admit closed expressions as a function of modified Bessel functions of order ν, $I_\nu(x)$.

Using the Fourier expansion

$$e^{a\cos x}=\sum_{k=-\infty}^{+\infty} I_k(a)e^{ikx}\ ,$$

it can be easily proved that

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^S=\sum_{k=-\infty}^{+\infty} I_{2k}(\alpha)I_k(\tilde{\alpha})\ .$$

Similarly it may be shown, for example, that

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\cos\phi\, e^S=\tfrac{1}{2}\sum_{k=-\infty}^{+\infty}[I_{2k+1}(\alpha)I_k(\tilde{\alpha})+I_{2k-1}(\alpha)I_k(\tilde{\alpha})]$$

and

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\sin\phi\, e^S=\tfrac{1}{2}\sum_{k=-\infty}^{+\infty}[I_{2k+1}(\alpha)I_k(\tilde{\alpha})-I_{2k-1}(\alpha)I_k(\tilde{\alpha})]\ .$$

[1]J. Kogut, Rev. Mod. Phys. **55**, 775 (1983).
[2]H. Hamber and G. Parisi, Phys. Rev. D **27**, 208 (1983).
[3]L. McLerran and B. Svetitsky, Phys. Lett. **98B**, 195 (1981).
[4]B. Berg, A. Billoire, and C. Rebbi, Ann. Phys. (N.Y.) **142**, 185 (1982).
[5]H. Flyvbjerg, Acta Phys. Pol. B **14**, 179 (1983); J. M. Drouffe and J. B. Zuber, Phys. Rep. **102C**, 1 (1983); G. Batrouni, Nucl. Phys. **B208**, 12 (1982); E. Dagotto, Phys. Lett. **136B**, 60 (1984).
[6]S. Drell, H. Quinn, B. Svetitsky, and M. Weinstein, Phys. Rev. D **19**, 619 (1979); D. Boyanovsky, R. Deza, and L. Masperi, *ibid.* **22**, 3034 (1980); E. Dagotto and A. Moreo, *ibid.* **29**, 2350 (1984).
[7]E. Dagotto and A. Moreo, Phys. Rev. D **29**, 300 (1984).
[8]A. Patkós and F. Deak, Z. Phys. C **9**, 359 (1981).
[9]A. Irving and C. Hamer, J. Phys. A **16**, 829 (1983).
[10]K. Bitar, S. Gottlieb, and C. Zachos, Phys. Rev. D **26**, 2853 (1982).
[11]J. Kogut, R. Pearson, and J. Shigemitsu, Phys. Rev. Lett. **43**, 484 (1979).
[12]J. M. Drouffe, Nucl. Phys. **B205** [FS5], 27 (1982).
[13]V. Alessandrini, V. Hakim, and A. Krzywicki, Nucl. Phys. **B215** [FS7], 109 (1983).
[14]E. Brezin and J. M. Drouffe, Nucl. Phys. **B200** [FS4], 93

(1982).
[15]H. Flyvbjerg, B. Lautrup, and J. B. Zuber, Phys. Lett. **110B**, 279 (1982); K. Ghoroku, Prog. Theor. Phys. **70**, 1091 (1983).
[16]S. Elitzur, Phys. Rev. D **12**, 3978 (1975).
[17]G. Bhanot, Nucl. Phys. **B205** [FS5], 168 (1982).
[18]G. Bhanot, Phys. Lett. **117B**, 431 (1982).
[19]E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6, p. 125.
[20]K. Bitar, S. Gottlieb, and C. Zachos, Phys. Lett. **121B**, 163 (1983).
[21]U. Heller and N. Seiberg, Phys. Rev. D **27**, 2980 (1983); D. Horn, M. Karliner, E. Katznelson, and S. Yankielowicz Phys. Lett. **113B**, 258 (1982); D. Horn and E. Katznelson, *ibid.* **121B**, 349 (1983).
[22]B. Lautrup and M. Nauenberg, Phys. Lett. **95B**, 63 (1980).
[23]M. Creutz, Phys. Rev. Lett. **42**, 1390 (1979).
[24]R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D **10**, 3376 (1974).
[25]A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956), pp. 39 and 51; C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 448.
[26]V. Alessandrini and Ph. Boucaud, Nucl. Phys. **B225**, 303 (1983).
[27]D. Pritchard, Phys. Lett. **106B**, 193 (1981).
[28]J. M. Alberty, H. Flyvbjerg, and B. Lautrup, Nucl. Phys. **B220**, 61 (1983).
[29]J. M. Drouffe, Phys. Lett. **105B**, 46 (1981); H. Flyvbjerg and E. Marinari, *ibid.* **132B**, 385 (1983).
[30]In Ref. 18, Fig. 1 the plaquette action considered was

$$S_p=\beta\frac{U+U}{2}+\delta\left[\frac{U+U^{+}}{2}\right]^2,$$

where $\delta=2\gamma$ in our nomenclature.
[31]N. Kawamoto and K. Shigemoto, NBI Report No. 83/02 (unpublished).
[32]H. Ceccato (unpublished).