Gauge-invariant Ritz's method for electrodynamics

Kimichika Fukushima
Toshiba Research and Development Center, Toshiba Corporation, 4-1, Ukishima-cho, Kawasaki-ku, Kawasaki, 210, Japan
(Received 4 January 1984; revised manuscript received 2 May 1984)

Ritz's method is formulated for electrodynamics. The wave function is expanded in terms of the scalar basis function. For the classical electromagnetic field, the expansion coefficients are determined by the direct variation of the action functional. Quantization is performed by the path integral. This method is Lorentz covariant and makes it possible to use the derivative, unlike the finite-difference approach, and gives the more natural gauge-invariant formalism compared to lattice gauge theory. The present method is applied to a self-interacting scalar field and the perturbation expansion is given in real space-time. The result is compared to the Feynman diagram method, and it is shown that all order terms are reduced to the product of one-dimensional integrals. All these integrals are obtained in analytic form under some conditions.

I. INTRODUCTION

The importance of numerical calculation has become greater in physics. For field theory, Wilson's lattice gauge theory[1] is widely used. The advantage of his theory is that it is gauge invariant and can be applied to strongly coupled fields. However, the derivative cannot be used and approximated by the finite difference in his theory, so that the theory has a complicated form to satisfy gauge invariance. In fact, the action functional for the boson field is quite different from the form in the continuum limit.

Recently, Bender, Milton, and Sharp[2] applied the finite-element theory[3] to the free Dirac field. They obtained the difference equation by substituting the wave function, which is linearly interpolated in the finite element, into the differential field equation. They did not show how they constructed the gauge-invariant formalism and whether the field equations were obtained by the variational principle for the gauge field. The above two methods are not Lorentz covariant, so that it is not clear whether the obtained result is Lorentz invariant or not.

The purpose of this paper is to give a formulation which overcomes the above difficulties by the use of Ritz's method.[3] In Sec. II, the formulation is given for electrodynamics. The field is expanded in terms of the scalar basis function. This basis function interpolates the wave function linearly in the finite element of the parameter space. The field equation is obtained by the direct variation of the action functional for classical theory. Quantization is performed by the path integral.[4,5] The present method makes it possible to use the derivative and gives a more natural gauge-invariant formalism compared to lattice gauge theory. This method also has the important feature that it is Lorentz covariant.

In Sec. III, the method is applied to a self-interacting scalar field and the perturbation expansion is given. There, the comparison to the Feynman diagram method is presented. Feynman diagrams are convenient for calculation in the energy-momentum space. It will be shown that in the present method all order terms of the perturbation expansion in real space-time are reduced to the product of one-dimensional integrals. These integrals are obtained in analytic form under some conditions. This calculational procedure is simpler compared to the Monte Carlo method,[6] which needs much computational time.

II. FORMULATION FOR ELECTRODYNAMICS

A. Basis function

In the present method, the four-potential $A_\mu(x)$ of the electromagnetic field is expanded in terms of the scalar basis functions $f_a(x)$,

$$A_\mu(x)=A_{\mu a}f_a(x)\ . \tag{2.1}$$

Here, the repeated index a implies summation over discrete integers. The point x_μ in space-time is mapped to the point t_μ in four-dimensional parameter t space by

$$t_\mu=G_\mu(x)\ . \tag{2.2}$$

When the point x_μ is transformed to x'_μ under a Lorentz transformation, $G'_\mu(x')$ satisfies

$$t_\mu=G'_\mu(x')\ . \tag{2.3}$$

That is, the function $G_\mu(x)$ is a scalar. The t space is decomposed into finite elements with the form of a hypercube. The coordinate t_μ is divided into N_μ sections.

The basis function $f_a(t)$ is defined around the vertex $t_{\mu a}$ and expressed as

$$f_a(t)=h_{1i}(t_1)h_{2j}(t_2)h_{3k}(t_3)h_{0l}(t_0)\ . \tag{2.4}$$

Here, the index a refers to the vertex $(t_{1i},t_{2j},t_{3k},t_{0l})$. The function $h_m(t)$ (the indices 1, 2, 3, and 0 are omitted) is defined in the section $t_{m-1}\le t\le t_{m+1}$ and takes the following values in the section $t_{m-1}\le t\le t_m$:

$$h_m(t)=\begin{cases}0 & \text{for } m=0\\ \dfrac{t-t_{m-1}}{\xi} & \text{for } m>0\ ,\end{cases} \tag{2.5}$$

and in the section $t_m \le t \le t_{m+1}$:

$$h_m(t)=\begin{cases}\dfrac{t_{m+1}-t}{\xi} & \text{for } m<N\\ 0 \text{ for } m=N\ .\end{cases} \tag{2.6}$$

Here, ξ is the lattice spacing.

Using this basis function, the wave function $A(t)$ is, for example, represented in the one-dimensional space section $t_m \le t \le t_{m+1}$ by

$$\begin{aligned}A(t)&=A(t_m)h_m(t)+A(t_{m+1})h_{m+1}(t)\\&=\frac{1}{\xi}(t_{m+1}-t)A(t_m)+\frac{1}{\xi}(t-t_m)A(t_{m+1})\ .\end{aligned} \tag{2.7}$$

For this basis function, the expansion coefficients in Eq. (2.1) are equal to the value of the field at the vertex around where the basis function is defined and the wave function in t space is linearly interpolated in the finite element. Since the wave function is linearly interpolated, there is no rapid oscillation in the finite element. That is, the regularization can be performed by the use of this basis function.

This method has another important feature. Since the transformation function $G_\mu(x)$ in Eq. (2.2) is scalar and the parameter t space is independent of the Lorentz transformation, the basis function $f_a(x)$ is scalar. Then, the following formulation becomes Lorentz covariant.

B. Classical electromagnetic field

The action functional for the classical electromagnetic field is given by

$$\begin{aligned}&S=S_f+S_I\ ,\\&S_f=-\tfrac{1}{4}\int {F_{\mu\nu}}^2 dx\ ,\\&S_I=\int j_\mu A_\mu dx\ ,\end{aligned} \tag{2.8}$$

where $F_{\mu\nu}$ is the electromagnetic field strength, which is represented by a derivative unlike lattice gauge theory and is given by

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ . \tag{2.9}$$

Therefore, the field is local in this theory unlike the finite-difference approach. The quantity j_μ is the current four-vector.

The variation of the action functional is performed with respect to $A_{\mu p}$ which excludes the value of the field on the initial and final spacelike hypersurfaces.[7,8] Then, δS_f becomes

$$\delta S_f=-\int \partial_\mu A_\nu \partial_\mu \delta A_\nu dx+\int \partial_\nu A_\mu \partial_\mu \delta A_\nu dx\ . \tag{2.10}$$

In the second term of these integrals, we integrate by parts. Then, we have

$$\int \partial_\nu A_\mu \partial_\mu \delta A_\nu dx=\int \partial_\nu A_\mu \delta A_\nu d\sigma_\mu-\int \partial_\nu \partial_\mu A_\mu \delta A_\nu dx\ , \tag{2.11}$$

where $d\sigma_\mu$ is the hypersurface element. The variation is not performed on the initial and final spacelike hypersurfaces and the field far from the region considered becomes zero, so the first term on the right-hand side vanishes.

In order to approximate the Lorentz condition

$$\partial_\mu A_\mu(x)=0\ , \tag{2.12}$$

we use the weighted residual method[9] and impose

$$\int \partial_\mu A_\mu \delta\beta_b g_b dx=\delta\beta_b A_{\mu a}L_{\mu ab}=0\ . \tag{2.13}$$

Here,

$$\begin{aligned}&g_b(x)=\frac{\sin[N(x_0-x_{0b})]}{\pi(x_0-x_{0b})}\prod_{i=1}^{3}\frac{\sin[N(x_i-x_{ib})]}{\pi(x_i-x_{ib})}\ ,\\&L_{\mu ab}=\int \partial_\mu f_a g_b dx\ .\end{aligned} \tag{2.14}$$

The quantity N is a large number and g_b becomes

$$g_b=\delta(x_0-x_{0b})\prod_{i=1}^{3}\delta(x_i-x_{ib}) \tag{2.15}$$

when $N\to\infty$. Then, the condition becomes

$$Q_b=A_{\mu a}L_{\mu ab}=0\ . \tag{2.16}$$

This equation corresponds to the condition

$$\partial_\mu A_\mu(x_b)=0\ . \tag{2.17}$$

In Eq. (2.16), the limit $N\to\infty$ cannot be taken because there is a discontinuity of $\partial_\mu A_\mu$ at the vertex. Then, the second term of Eq. (2.11) on the right-hand side approximately vanishes.

Therefore, the variation δS_f becomes

$$\begin{aligned}\delta S_f&=-\int A_{\nu a}\partial_\mu f_a \delta A_{\nu p}\partial_\mu f_p dx\\&=-\delta A_{\nu p}M_{pa}A_{\nu a}\ ,\end{aligned} \tag{2.18}$$

where

$$M_{pa}=\int \partial_\mu f_p \partial_\mu f_a dx\ . \tag{2.19}$$

Here, index a refers to all basis functions. On the other hand, the index p excludes the basis functions defined around the vertex on the initial and final spacelike hypersurfaces.

For the action functional S_I, we obtain the variation

$$\begin{aligned}\delta S_I&=\delta A_{\nu p}\int j_\nu f_p dx\\&=\delta A_{\nu p}b_{\nu p}\ ,\end{aligned} \tag{2.20}$$

where

$$b_{\nu p}=\int j_\nu f_p dx\ . \tag{2.21}$$

With the aid of the variational principle, we get, by interchanging the indices μ and ν,

$$M_{pa}A_{\mu a}=b_{\mu p}\ . \tag{2.22}$$

The above equation corresponds to the Maxwell equation

$$-\Box A_\mu=j_\mu\ . \tag{2.23}$$

In solving the above field equation, the values of the field at the first and second time-steps are given as the initial

condition. Then, the field at the third time-step can be obtained if the first and third steps are given as the initial and the final time-steps, respectively. Iterating the above procedure, the field at all time-steps can be calculated.

C. Quantum electrodynamics

For the quantum theory, the Dirac field is expanded in the same form in Eq. (2.1),

$$\psi(x)=\psi_a f_a(x) . \tag{2.24}$$

The action functional is given by

$$S=-\tfrac{1}{4}\int F_{\mu\nu}{}^2 dx-\int \bar{\psi}(\gamma_\mu\partial_\mu+m)\psi\, dx+\int j_\mu A_\mu dx , \tag{2.25}$$

where the current four-vector j_μ is denoted as

$$j_\mu=ie\bar{\psi}\gamma_\mu\psi , \tag{2.26}$$

and γ_μ are Dirac matrices. The quantization is performed by the path integral.[4,5] Then, by the use of the approximated Lorentz condition (2.16) the transition amplitude $T_{\rm if}$ is given by

$$T_{\rm if}=\frac{1}{Z}\int D[A_{\mu p}]D[\bar{\psi}_q]D[\psi_r]\delta(Q_p)\exp(iS) . \tag{2.27}$$

Here, the indices p, q, and r exclude the values on the initial and final spacelike hypersurfaces. Quantities $\bar{\psi}_q$ and ψ_r are Grassmann numbers and Z is the normalization constant. The integration is performed, for example, by the Monte Carlo method.[6]

The above formalism is invariant under the gauge transformation

$$\begin{aligned}&\psi(x)\to\psi(x)\exp(ie\theta) ,\\ &A_\mu(x)\to A_\mu(x)+\partial_\mu\theta .\end{aligned} \tag{2.28}$$

Thus, this method gives the more natural gauge-invariant formalism compared to lattice gauge theory, in which the action functional for the boson field is quite different from the form in the continuum limit and complicated.

III. APPLICATION TO SCALAR FIELD

A. Free field

In this section, the method given in Sec. II is applied to a scalar field. For simplicity, the calculation is performed in two-dimensional space-time, and real space-time is taken so as to coincide with the parameter t space. At first, a massless scalar field is considered. The action functional is taken to be

$$S=-\tfrac{1}{2}\int(\partial_x\phi\partial_x\phi-\partial_t\phi\partial_t\phi)dx\, dt . \tag{3.1}$$

The scalar field ϕ is expanded as

$$\phi=U_a f_a=U_{m,n}f_{m,n} . \tag{3.2}$$

Here, index a refers to the vertex $(x,t)=(m\xi_x,n\xi_t)$. Quantities ξ_x and ξ_t are the lattice spacings. The integers m and n go from 0 to N_x+1 and 1 to N_t, respectively. For convenience in the discussions, the following conditions are imposed:

$$\begin{aligned}&U_{m,n}=0 \quad \text{for } m=0,\ N_x+1 ,\\ &N_t=3 .\end{aligned} \tag{3.3}$$

Then, we can regard the index m as going from 1 to N_x. The obtained results can be easily generalized in the case where these conditions are not imposed.

When the lattice spacing is taken to be $\xi_x=\xi_t=\xi$, the action functional becomes

$$\begin{aligned}&S=U_a U_b M_{ab} ,\\ &M_{ab}=-\tfrac{1}{2}\int(\partial_x f_a\partial_x f_b-\partial_t f_a\partial_t f_b)dx\, dt .\end{aligned} \tag{3.4}$$

Here, the matrix M_{ab}, necessary for the following discussions, is given by

$$M_{ab}=\begin{cases}-1 & \text{for } a=(m,2),\ b=(m\pm1,2)\\ 1 & \text{for } a=(m,2),\ b=(m,2\pm1)\\ 0 & \text{for } a=b=(m,2),\end{cases} \tag{3.5}$$

using Eqs. (2.5) and (2.6) with

$$\int h_m h_n dt=\begin{cases}\dfrac{2\xi}{3} & \text{for } m=n\\ \dfrac{\xi}{6} & \text{for } m=n\pm1\\ 0 & \text{for other case ,}\end{cases} \tag{3.6}$$

and

$$\int\frac{dh_m}{dt}\frac{dh_n}{dt}dt=\begin{cases}\dfrac{2}{\xi} & \text{for } m=n\\ -\dfrac{1}{\xi} & \text{for } m=n\pm1\\ 0 & \text{for other case .}\end{cases} \tag{3.7}$$

In the classical theory, the following equation is obtained by the variation with respect to $U_{m,2}$:

$$F_m=-U_{m-1,2}+U_{m,1}+U_{m,3}-U_{m+1,2}=0 . \tag{3.8}$$

For the initial condition

$$\begin{aligned}&U_{k,1}=U_{k+1,2}=1,\quad U_{m,1}=0\quad (m\neq k) ,\\ &U_{m,2}=0\quad (m\neq k+1) ,\end{aligned} \tag{3.9}$$

the solution becomes

$$U_{k+2,3}=1,\quad U_{m,3}=0\quad (m\neq k+2) . \tag{3.10}$$

This result shows that the pulse is propagated exactly with the velocity of light and the amplitude is conserved.

The transition amplitude in the quantum theory is given by

$$T_{if}=\frac{1}{Z}\int D[U_{m,2}]\exp(iS) . \tag{3.11}$$

If the integration is performed with respect to $U_{m,2}$ at first, the factor $\delta(F_m)$ appears, so that the transition amplitude becomes zero if Eq. (3.8) is not satisfied. Therefore, the field in quantum theory also has to satisfy, at least in this case, the field equation which has the same form in the classical theory.

B. Self-interacting field

The self-interacting scalar field is useful to study the fundamental characteristics of the interacting field and to investigate the effectiveness of the computational method. The (Euclidean) action functional is taken to be

$$\begin{aligned} &S=S_0+S_I\ , \\ &S_0=\tfrac{1}{2}\int(\partial_x\phi\partial_x\phi+\partial_t\phi\partial_t\phi+m^2\phi^2)dx\,dt\ , \\ &S_I=\frac{\lambda}{6}\int\phi^3dx\,dt\ . \end{aligned} \tag{3.12}$$

Using the expansion given by Eq. (3.2), the action functional becomes (the lattice spacings ξ_x and ξ_t are not necessarily equal in this case)

$$\begin{aligned} &S=S_0+S_I\ , \\ &S_0=U_aU_bM_{ab}\ , \\ &S_I=U_aU_bU_cN_{abc}\ , \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &M_{ab}=\tfrac{1}{2}\int(\partial_xf_a\partial_xf_b+\partial_tf_a\partial_tf_b+m^2f_af_b)dx\,dt\ , \\ &N_{abc}=\frac{\lambda}{6}\int f_af_bf_cdx\,dt\ . \end{aligned} \tag{3.14}$$

The action functional for the free field is rewritten as

$$\begin{aligned} &S_0=S_{0A}+S_{0B}\ , \\ &S_{0A}=U_vU_wM^d_{vw}\ , \\ &S_{0B}=U_pM_{pq}U_q-B_qU_q\ . \end{aligned} \tag{3.15}$$

Here,

$$\begin{aligned} &B_q=-2U_vM_{vq}=-2U_vM_{qv}\ , \\ &M^d_{vw}=\delta(v,w)M_{vw}\ , \end{aligned} \tag{3.16}$$

with

$$\delta(v,w)=\begin{cases}1 & \text{for } v=w\\ 0 & \text{for } v\neq w\ .\end{cases} \tag{3.17}$$

Hereafter, the indices v and w refer to the field values at the initial and final time-steps. On the other hand, the indices p, q, r, s, t, and u exclude these values. The indices a, b, and c refer to all field values.

Following the technique for the Gaussian integration given in Appendix A of Ref. 5, the variable U_p is transformed as

$$\begin{aligned} &U'_p=U_p-\tfrac{1}{2}M^{-1}{}_{pq}B_q\ , \\ &U'_v=U_v\ , \end{aligned} \tag{3.18}$$

provided that the inverse matrix $M^{-1}{}_{p,q}$ exists. Then, the action functional S_{0B} is transformed to

$$\begin{aligned} &S'_{0B}=S'_{0C}+C\ , \\ &S'_{0C}=U'_pM_{pq}U'_q\ , \\ &C=-\tfrac{1}{4}B_pM^{-1}{}_{pq}B_q\ . \end{aligned} \tag{3.19}$$

The matrix $M_{p,q}$ is diagonalized by the transformation

$$\begin{aligned} &U''_p=R_{pq}U'_q\ , \\ &U''_v=U'_v\ , \end{aligned} \tag{3.20}$$

where $R_{p,q}$ satisfies

$$R^TR=RR^T=1\ . \tag{3.21}$$

Therefore, the action functional S'_{0C} is further transformed to

$$S''_{0C}=U''_pM^d_{pq}U''_q\ , \tag{3.22}$$

with the diagonal matrix $M^d_{p,q}$.

Thus, the transition amplitude in the Euclidean measure becomes, using Eqs. (3.15) and (3.19),

$$T_{\rm if}=\exp(-S_{0A}-C)\langle\exp(-S_I)\rangle\ . \tag{3.23}$$

Here, $\langle X\rangle$ means

$$\langle X\rangle=\frac{1}{Z}\int D[U_p]X\exp(-S_{0B}+C)\ . \tag{3.24}$$

This integral is rewritten using Eqs. (3.19) and (3.22),

$$\begin{aligned} \langle X\rangle&=\frac{1}{Z}\int D[U'_p]X\exp(-S'_{0C}) \\ &=\frac{1}{Z}\int D[U''_p]X\exp(-S''_{0C})\ , \end{aligned} \tag{3.25}$$

because the Jacobian is unity. In Eq. (3.23), the factor $\exp(-S_{0A}-C)$ corresponds to the normal product (external line) in the Feynman diagram.[10] The expectation value $\langle X\rangle$ for the background free field corresponds to the product of the Green's functions.

The transition amplitude is expanded as

$$T_{\rm if}=\exp(-S_{0A}-C)(\langle 1\rangle-\langle S_I\rangle+\tfrac{1}{2}\langle S_I{}^2\rangle+\cdots)\ . \tag{3.26}$$

Since S''_{0C} is expressed as

$$S''_{0C}=\sum_p\eta_pU_p{}^2\ , \tag{3.27}$$

with the diagonal element η_p of $M_{p,q}$ (in this case, the summation rule for the index p is not applied in repetition), all order terms with the multifold integral are reduced to the product of the following one-dimensional integral, using Eqs. (3.13), (3.18), and (3.20):

$$I=\int dU''_pU''^k_p\exp(-\eta_pU''^2_p)\ . \tag{3.28}$$

It is much simpler compared to the multifold integral.

When η_p is positive, this integral becomes

$$I=\begin{cases}\dfrac{(2j-1)!!}{2^j}\left[\dfrac{\pi}{\eta_p{}^{2j+1}}\right]^{1/2} & \text{for } k=2j \\ 0 \quad \text{for } k=2j+1\ . \end{cases} \tag{3.29}$$

The above calculational procedure can be used, when the inverse matrix $M^{-1}{}_{p,q}$ exists and the diagonal elements η_p are all positive. Here, a simple case is investigated where $N_x=3$ in Eq. (3.3). The matrix $M_{p,q}$ is given by

$$M_{p,q}=\begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}\ . \tag{3.30}$$

The indices p and q refer to the vertex $(x,t)=(i\xi_x, j\xi_t)$ and go over $p,q=(i,j)=(1,2),(2,2),(3,2)$.

The matrix elements become, using Eqs. (3.6) and (3.7), for the massless case

$$\begin{aligned} a&=\tfrac{4}{3}\left[d+\frac{1}{d}\right], \\ b&=-\tfrac{1}{3}\left[2d-\frac{1}{d}\right], \end{aligned} \tag{3.31}$$

where

$$d=\frac{\xi_t}{\xi_x}\ . \tag{3.32}$$

Then, the diagonal element η_p is obtained as

$$\eta_p=\begin{cases} a \\ a+\sqrt{2}b \\ a-\sqrt{2}b\ . \end{cases} \tag{3.33}$$

All these values are positive when

$$d>0\ . \tag{3.34}$$

In this case, the determinant of $M_{p,q}$ is given by

$$\mathrm{Det}[M_{p,q}]=\prod_p \eta_p>0\ , \tag{3.35}$$

so that the inverse matrix $M^{-1}{}_{p,q}$ exists.

Therefore, since the condition (3.34) is satisfied, all order of perturbation expansion terms can be obtained in analytic form with a finite value, independently of the coupling constant. Accordingly, the transition amplitude is obtained for the strongly coupled field by summing up the all order terms, as long as the perturbation series does not diverge. In addition, when N_x, $N_t\to\infty$, the transition amplitude for the conventional local field theory is obtained. This calculational procedure is much simpler compared to the Monte Carlo method, which needs much computational time.

IV. FURTHER OUTLOOK

The present paper has shown the effectiveness of the Ritz's method. In this paper, we restricted the basis function to the linear-interpolational type. In the above formulation, the derivative has the discontinuity on the boundary of the finite element. However, this problem does not occur if the higher-order polynomials are adopted as the basis function. Furthermore, there remain many possibilities for the selection of the basis functions. For example, the basis function can be chosen which has the form of the products of the linear-interpolational-type basis function in time coordinate and $\exp(ip_jx_j)$ in space coordinates. In this representation, the initial and final states can be described in momentum space. If this basis function is used in parameter t space, and the expansion terms for large momentum are truncated, this formalism gives the relativistic cutoff.

In the variation, it is possible to use the Lagrange multiplier theory for the initial and final conditions. Then, a more general basis function can be used in which the expansion coefficient does not coincide with the field value at the vertex. The same procedure can also be applied to the Lorentz condition.

In this paper, we restricted the discussion of the formulation to use it as the computational method. However, I think there might be some possibility that the real physical phenomena in a short-distance area can be described by the above or similar theories. However, it is beyond the scope of this paper, because no experiment has been performed for such phenomena in a short-distance area. More detailed discussions will be given elsewhere.

ACKNOWLEDGMENTS

The author is grateful to Professor H. Sato and Professor H. Adachi of Hyogo University of Teacher Education for their helpful discussions and advice. The author also would like to thank Professor T. Watanabe of the Institute of Physical and Chemical Research, Professor Y. Nagashima of Osaka University, and Dr. J. Mizuno of Japan Business Automation, Co., Ltd., for valuable discussions during the course of this work.

[1]K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[2]C. M. Bender, K. A. Milton, and D. H. Sharp, Phys. Rev. Lett. **51**, 1815 (1983).
[3]K. J. Bathe and E. L. Wilson, *Numerical Methods in Finite Element Analysis* (Prentice-Hall, Englewood Cliffs, New Jersey, 1976).
[4]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[5]P. Ramond, *Field Theory: A Modern Primer* (Benjamin, Reading, Mass., 1982).
[6]M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rep. **95**, 201 (1983).
[7]S. Tomonaga, Prog. Theor. Phys. **1**, 27 (1946).
[8]J. Schwinger, Phys. Rev. **74**, 1439 (1948).
[9]C. A. Brebbia and S. Walker, *Boundary Element Techniques in Engineering* (Butterworths, London, 1980).
[10]F. J. Dyson, Phys. Rev. **75**, 1736 (1949).