Thermal effects of acceleration for a classical spinning magnetic dipole in classical electromagnetic zero-point radiation

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A classical spinning magnetic dipole undergoing uniform acceleration through classical electromagnetic zero-point radiation is shown to behave at equilibrium as though immersed in a thermal bath at the Unruh-Davies temperature $T=\hbar a/2\pi ck$. Although the electromagnetic field correlation functions seen by the uniformly accelerating dipole do not correspond to Planck's thermal radiation spectrum, the departure from Planckian form is canceled by additional terms arising in the relativistic radiation damping for the accelerating dipole. Thus the accelerating dipole behaves at equilibrium as though in an inertial frame bathed by exactly Planck's spectrum including zero-point radiation. An analogous result was reported recently for a classical harmonic oscillator undergoing uniform acceleration through classical electromagnetic zero-point radiation.

INTRODUCTION

It was shown recently[1] that a classical charged harmonic oscillator undergoing uniform acceleration through classical electromagnetic zero-point radiation behaves at equilibrium as though located in an inertial frame with exactly Planck's spectrum of radiation with the Unruh-Davies temperature[2] $T=\hbar a/2\pi ck$. This thermal behavior for the oscillator occurred despite the fact that the spectrum of random radiation seen by the accelerating oscillator was not Planckian. The terms in the relativistic radiation-reaction force associated with the accelerations altered the oscillator equation of motion in exactly such a way as to compensate the non-Planckian spectrum and to bring the equilibrium behavior of the oscillator into exactly Planckian form. It was conjectured that this compensation would occur for all physically allowed classical electromagnetic systems.[3] In the present article we show that exactly this same compensation arises in the case of a classical spinning magnetic dipole.

The context for our calculations is classical electron theory in which random classical radiation with a Lorentz-invariant spectrum has been included as the homogeneous boundary condition on Maxwell's equations. This classical electromagnetic theory is often termed stochastic or random electrodynamics, or, more descriptively, classical electrodynamics with classical electromagnetic zero-point radiation. The theory is described in some detail in several publications.[4]

The basic model for our present calculation was treated recently[5] within this classical theory. The model consists of a point classical magnetic dipole $\vec{\mu}$ spinning with an angular momentum $\vec{s}$ in a direction parallel to $\vec{\mu}$. The dipole is free to point in any direction. The mechanical moment of inertia of the system is assumed to vanish so that the torques go into changing the direction of the spin, and all of the system energy is electromagnetic. As a rough approximation the system can be pictured as a gyroscope with a permanent magnet mounted along the spin axis of the gyroscope.

If the spinning magnetic dipole is placed in a magnetic field, then it experiences a torque and so precesses. The precessional motion leads to the emission of electromagnetic radiation and hence produces a tendency for the dipole to align its magnetic moment along the direction of the magnetic field. However, in the presence of random classical radiation, the complete alignment never occurs because the random radiation produces torques which drive the spinning dipole away from the aligned position. The balance between the opposing tendencies of alignment and random motion leads to an equilibrium distribution for the orientation of the magnetic moment. In Ref. 5 we found this equilibrium distribution for the spinning dipole when located in an inertial frame. In the present work we find the equilibrium distribution when the dipole is accelerating uniformly through classical electromagnetic zero-point radiation. Our present calculation shows that under uniform acceleration through zero-point radiation, the spinning dipole arrives at the same equilibrium distribution as was found earlier for the dipole when at rest in an inertial frame in Planck's spectrum including zero-point radiation.

Although the conclusion of our calculation seems distinctly interesting, most of the intermediate calculations are not of general interest. Hence, in order to shorten the presentation while still making it possible for a critical reader to check our work, we will present our calculations as though the serious reader had Refs. 1, 5, and 6 before him for immediate reference. In order to prevent confusion between the numbered equations in the present calculations and in the references, we will include the reference number along with the reference's equation number. Thus, for example, Eq. (47) in Bhabha's work in Ref. 6 will be denoted as Eq. (6-47).

RELATIVISTIC SPIN EQUATION OF MOTION

The equation of motion we will use for the classical spinning dipole is that given by Bhabha[6] in 1940. Bhabha treated a point spinning magnetic dipole and found the covariant equation of motion by an extension of Dirac's technique[7] employing conservation of energy and angular momentum for the spinning particle and electromagnetic field, assuming Maxwell's equations to be valid everywhere in space, right up to the point particle. The equation of motion provided by Bhabha's work of 1940 seems quite complicated even though it corresponds to only a special case of the still more general analysis given by Bhabha and Corben[8] in 1941. However, we will find that for our assumption of uniform acceleration the equation simplifies enormously.

We are interested in the equation of motion of the spin in its own rest frame as this frame undergoes uniform proper acceleration. Thus, we denote by $\mathscr{S}^{\mu\nu}$ the antisymmetric spin tensor in its own rest frame, and by $S^{\mu\nu}$ the spin tensor in a given fixed inertial frame I_*. The simplifying restriction of Bhabha's work[6] of 1940 is that the spinning particle has only a magnetic dipole moment and no electric dipole moment in its own instantaneous rest frame. Thus in our notation we have

$$\mathscr{S}^{12}=s_z\ ,\quad \mathscr{S}^{13}=-s_y\ ,\quad \mathscr{S}^{23}=s_x \tag{1}$$

as the only nonzero elements above the diagonal in the antisymmetric spin tensor $\mathscr{S}^{\mu\nu}$ with

$$\vec{s}=\hat{i}s_x+\hat{j}s_y+\hat{k}s_z \tag{2}$$

giving the three-vector spin in an inertial frame I_τ instantaneously at rest with respect to the dipole at the dipole proper time τ. The simplifying restriction excluding an electric dipole moment in the particle's rest frame takes the covariant form in the inertial frame I_*

$$S^{\mu\nu}v_\nu=0\ , \tag{3}$$

where v^μ is the four-velocity of the point dipole as seen in I_*. The ratio between the components of the magnetic moment $\vec{\mu}$ and the spin $\vec{s}$ in the particle's rest frame will be denoted by g so that

$$\vec{\mu}=g\vec{s}\ . \tag{4}$$

Our interest is in the case of uniform proper acceleration (hyperbolic motion) along the z axis where the four-vector velocity of the particle takes the form[9] relative to the inertial frame I_*

$$v^\mu=\frac{dx^\mu}{d\tau}=\frac{dx^\mu}{dt}\frac{dt}{d\tau}=(\gamma,0,0,\beta\gamma) \tag{5}$$

and

$$\gamma=\cosh a\tau\ ,\quad \beta\gamma=\sinh a\tau\ . \tag{6}$$

Here $\vec{a}=\hat{k}a$ is the proper acceleration of the particle, τ is the proper time along the particle trajectory, and the velocity of light is taken as unity so that factors of c can be omitted. The Lorentz transformation $dx_\tau^\mu/dx_*^\nu=\Lambda^\mu{}_\nu$ which transforms a four-vector A_*^μ in I_* over to A_τ^μ in the frame I_τ has nonzero elements following from (5) as

$$\Lambda^0{}_0=\Lambda^3{}_3=\gamma\ ,\quad \Lambda^0{}_3=\Lambda^3{}_0=-\beta\gamma\ ,$$
$$\Lambda^1{}_1=\Lambda^2{}_2=1\ . \tag{7}$$

Then using the inverse Lorentz transformation $\Lambda^{-1\mu}{}_\nu$, we have $S^{\mu\nu}=\Lambda^{-1\mu}{}_\alpha\Lambda^{-1\nu}{}_\beta\mathscr{S}^{\alpha\beta}$, and from (1), the nonzero elements above the diagonal for the antisymmetric tensor $S^{\mu\nu}$ are

$$S^{01}=-\beta\gamma\mathscr{S}^{13}\ ,\quad S^{02}=-\beta\gamma\mathscr{S}^{23}\ ,\quad S^{12}=\mathscr{S}^{12}\ ,$$
$$S^{13}=\gamma\mathscr{S}^{13}\ ,\quad S^{23}=\gamma\mathscr{S}^{23}\ . \tag{8}$$

The time evolution of the spin tensor $S^{\mu\nu}$ with respect to the particle's proper time τ is given in Eqs. (6-46) and (6-47) of Bhabha's article[6] as

$$\begin{aligned}\dot{S}^{\mu\nu}+(v^\mu S'^{\nu}-v^\nu S'^{\mu})&\\ =\{S^\mu{}_\sigma[-g^2D^{\sigma\nu}+gF^{\mathrm{in}\sigma\nu}&\\ -g(F^{\mathrm{in}\sigma\alpha}v_\alpha v^\nu-F^{\mathrm{in}\nu\alpha}v_\alpha v^\sigma)]\}&-\{\mu\leftrightarrow\nu\}\ ,\end{aligned} \tag{9}$$

where the electromagnetic field tensor $F^{\mathrm{in}\mu\nu}$ is given by

$$F^{\mathrm{in}\mu\nu}=-(\partial^\mu A^{\mathrm{in}\nu}-\partial^\nu A^{\mathrm{in}\mu})\ . \tag{10}$$

The radiation damping tensor $D^{\mu\nu}$ is given by

$$\begin{aligned}D^{\mu\nu}=&d(S'^{\mu}\ddot{v}^{\nu}-S'^{\nu}\ddot{v}^{\mu})+(d-\tfrac{2}{3})(S''^{\mu}\dot{v}^{\nu}-S''^{\nu}\dot{v}^{\mu})\\ &+2(d+\tfrac{1}{3})(\dot{S}^{\mu\sigma}\dot{v}_\sigma\dot{v}^{\nu}-\dot{S}^{\nu\sigma}\dot{v}_\sigma\dot{v}^{\mu})\\ &-\tfrac{2}{3}(S'''^{\mu}v^\nu-S'''^{\nu}v^\mu)+\tfrac{2}{3}\dddot{S}^{\mu\nu}\\ &+(d-\tfrac{2}{3})(S'^{\mu}v^\nu-S'^{\nu}v^\mu)\dot{v}^2+\tfrac{2}{3}\dot{S}^{\mu\nu}\dot{v}^2\end{aligned} \tag{11}$$

with the abbreviations of Eq. (6-3)

$$S'^{\mu}\equiv\dot{S}^{\mu\nu}v_\nu\ ,\quad S''^{\mu}\equiv\ddot{S}^{\mu\nu}v_\nu\ ,\quad S'''^{\mu}\equiv\dddot{S}^{\mu\nu}v_\nu\ , \tag{12}$$

where the dots refer to differentiation with respect to the particle proper time τ and d is an unknown parameter which cannot be obtained from the conservation laws alone. Here again the velocity of light has been taken as unity, $c=1$.

SPIN EQUATION OF MOTION FOR CONSTANT ACCELERATION

We will rewrite Bhabha's relativistic equation of motion so as to restrict it to the special case of hyperbolic motion and to extract the equation of motion for the spin vector $\vec{s}(\tau)$ in the particle's rest frame.

Our first simplifying step makes use of the special form assumed by the particle four-vector (5) and (6). We notice that the higher derivatives of the velocity are proportional to lower derivatives. Thus, from (5) and (6) we find

$$\dot{v}^\mu=(a\beta\gamma,0,0,a\gamma) \tag{13}$$

and

$$\ddot{v}^\mu=a^2v^\mu\ ,\quad \dddot{v}^\mu=a^2\dot{v}^\mu\ . \tag{14}$$

Also because of the condition (3), we find that the higher derivatives of $S^{\mu\nu}$ are connected to combinations of lower derivatives of $S^{\mu\nu}$ and higher derivatives of v^μ as in Bhabha's Eq. (6-2). But then introducing the relations (14) into Eq. (6-2), we find

$$S'^{\mu}=-S^{\mu\nu}\dot{v}_{\nu}\ , \tag{15}$$

$$S''^{\mu}=-2\dot{S}^{\mu\nu}\dot{v}_{\nu}\ , \tag{16}$$

$$S'''^{\mu}=-3\ddot{S}^{\mu\nu}\dot{v}_{\nu}+2a^2S^{\mu\nu}\dot{v}_{\nu}\ . \tag{17}$$

Next we simplify Bhabha's radiation damping tensor $D^{\mu\nu}$ in our Eq. (11) by inserting the relations (14)–(17). This removes all the primed tensors and also shows that all terms depending upon the unknown parameter d vanish exactly, leaving for our case of uniform proper acceleration

$$D^{\mu\nu}=\tfrac{2}{3}(\ddot{S}^{\mu\nu}-a^2\dot{S}^{\mu\nu}) +2[(\ddot{S}^{\mu\sigma}-a^2S^{\mu\sigma})\dot{v}_{\sigma}v^{\nu}-(\ddot{S}^{\nu\sigma}-a^2S^{\nu\sigma})\dot{v}_{\sigma}v^{\mu}] +2(\dot{S}^{\mu\sigma}\dot{v}_{\sigma}\dot{v}^{\nu}-\dot{S}^{\nu\sigma}\dot{v}_{\sigma}\dot{v}^{\mu})\ . \tag{18}$$

At this point we insert the various tensors and collect terms in order to obtain the equation of motion for $\mathscr{S}^{\mu\nu}$. We list some intermediate steps for the convenience of any reader wishing to check our calculation.

We find $\dot{S}^{\mu\nu}$ by merely differentiating the terms listed in (8) with respect to the particle proper time τ where γ and $\beta\gamma$ are given in (6). Next S'^{μ} is evaluated from (15), (8), and (13) giving

$$S'^{\mu}=(0,a\mathscr{S}^{13},a\mathscr{S}^{23},0)\ . \tag{19}$$

Then using v^{μ} in (5) we can obtain the antisymmetric tensor

$$\dot{S}^{\mu\nu}+(v^{\mu}S'^{\nu}-v^{\nu}S'^{\mu})$$

with nonzero elements above the diagonal given by

$$(0,1)=-\beta\gamma\dot{\mathscr{S}}^{13}\ ,\quad (0,2)=-\beta\gamma\dot{\mathscr{S}}^{23}\ ,\quad (1,2)=\dot{\mathscr{S}}^{12}\ ,$$
$$(1,3)=\gamma\dot{\mathscr{S}}^{13}\ ,\quad (2,3)=\gamma\dot{\mathscr{S}}^{23}\ . \tag{20}$$

Next we work with the pieces needed for the tensor $D^{\mu\nu}$ in (18). We find

$$(\ddot{S}^{\mu\nu}-a^2S^{\mu\nu})\dot{v}_{\nu}=(0,-a\ddot{\mathscr{S}}^{13},-a\ddot{\mathscr{S}}^{23},0)\ . \tag{21}$$

The nonvanishing terms above the diagonal in the antisymmetric tensor $[(\ddot{S}^{\mu\sigma}-a^2S^{\mu\sigma})\dot{v}_{\sigma}v^{\nu}-(\ddot{S}^{\nu\sigma}-a^2S^{\nu\sigma})\dot{v}_{\sigma}v^{\mu}]$ are

$$(0,1)=a\gamma\ddot{\mathscr{S}}^{13}\ ,\quad (0,2)=a\gamma\ddot{\mathscr{S}}^{23}\ ,$$
$$(1,3)=-a\beta\gamma\ddot{\mathscr{S}}^{13}\ ,\quad (2,3)=-a\beta\gamma\ddot{\mathscr{S}}^{23}\ . \tag{22}$$

The term $\dot{S}^{\mu\nu}\dot{v}_{\nu}$ is

$$\dot{S}^{\mu\nu}\dot{v}_{\nu}=(0,-a\dot{\mathscr{S}}^{13},-a\dot{\mathscr{S}}^{23},0)\ , \tag{23}$$

and the antisymmetric tensor

$$(\dot{S}^{\mu\sigma}\dot{v}_{\sigma}\dot{v}^{\nu}-\dot{S}^{\nu\sigma}\dot{v}_{\sigma}\dot{v}^{\mu})$$

has nonvanishing above-diagonal elements

$$(0,1)=a^2\beta\gamma\dot{\mathscr{S}}^{13}\ ,\quad (0,2)=a^2\beta\gamma\dot{\mathscr{S}}^{23}\ ,$$
$$(1,3)=-a^2\gamma\dot{\mathscr{S}}^{13}\ ,\quad (2,3)=-a^2\gamma\dot{\mathscr{S}}^{23}\ . \tag{24}$$

Combining these pieces gives the antisymmetric tensor $D^{\mu\nu}$ with nonvanishing above-diagonal elements

$$(0,1)=-\tfrac{2}{3}\beta\gamma(\dddot{\mathscr{S}}^{13}-a^2\dot{\mathscr{S}}^{13})\ ,$$
$$(0,2)=-\tfrac{2}{3}\beta\gamma(\dddot{\mathscr{S}}^{23}-a^2\dot{\mathscr{S}}^{23})\ ,$$
$$(1,2)=\tfrac{2}{3}\dddot{\mathscr{S}}^{12}\ , \tag{25}$$
$$(1,3)=\tfrac{2}{3}\gamma(\dddot{\mathscr{S}}^{13}-a^2\dot{\mathscr{S}}^{13})\ ,$$
$$(2,3)=\tfrac{2}{3}\gamma(\dddot{\mathscr{S}}^{23}-a^2\dot{\mathscr{S}}^{23})\ .$$

Then contracting $D^{\sigma\nu}$ with $S^{\mu}{}_{\sigma}$ and antisymmetrizing we find the antisymmetric tensor $\{S^{\mu}{}_{\sigma}D^{\sigma\nu}\}-\{\mu\leftrightarrow\nu\}$ with nonvanishing above-diagonal elements

$$(0,1)=\tfrac{2}{3}\beta\gamma[\mathscr{S}^{12}(\dddot{\mathscr{S}}^{23}-a^2\dot{\mathscr{S}}^{23})-\mathscr{S}^{23}\dddot{\mathscr{S}}^{12}]\ ,$$
$$(0,2)=\tfrac{2}{3}\beta\gamma[\mathscr{S}^{13}\dddot{\mathscr{S}}^{12}-\mathscr{S}^{12}(\dddot{\mathscr{S}}^{13}-a^2\dot{\mathscr{S}}^{13})]\ ,$$
$$(1,2)=\tfrac{2}{3}\mathscr{S}^{13}(\dddot{\mathscr{S}}^{23}-a^2\dot{\mathscr{S}}^{23})-\mathscr{S}^{23}(\dddot{\mathscr{S}}^{13}-a^2\dot{\mathscr{S}}^{13})\ , \tag{26}$$
$$(1,3)=\tfrac{2}{3}\gamma[\mathscr{S}^{23}\dddot{\mathscr{S}}^{12}-\mathscr{S}^{12}(\dddot{\mathscr{S}}^{23}-a^2\dot{\mathscr{S}}^{23})]\ ,$$
$$(2,3)=\tfrac{2}{3}\gamma[\mathscr{S}^{12}(\dddot{\mathscr{S}}^{13}-a^2\dot{\mathscr{S}}^{13})-\mathscr{S}^{13}\dddot{\mathscr{S}}^{12}]\ .$$

The remaining terms on the right-hand side of Bhabha's equation of motion given in (9) involve the antisymmetric electromagnetic field tensor $F^{\mu\nu}$ in (10) with above-diagonal elements

$$F^{01}=E^1\ ,\quad F^{02}=E^2\ ,\quad F^{03}=E^3\ ,\quad F^{12}=B^3\ ,$$
$$F^{13}=-B^2\ ,\quad F^{23}=B^1\ . \tag{27}$$

Then combining this tensor with the four-vector v^{μ} we find the antisymmetric tensor $F^{\mu\nu}-(F^{\mu\alpha}v_{\alpha}v^{\nu}-F^{\nu\alpha}v_{\alpha}v^{\mu})$. The terms in the calculation of this tensor appear in forms such as $(\gamma^2-1)E^1-\beta\gamma^2B^2$. These can be rewritten in terms of the electromagnetic fields $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ seen in the I_{τ} frame instantaneously at rest with respect to the accelerating particle. Thus, for example,

$$(\gamma^2-1)E^1-\beta\gamma^2B^2=\beta^2\gamma^2E^1-\beta\gamma^2B^2 =-\beta\gamma(\gamma B^2-\beta\gamma E^1)=-\beta\gamma\mathscr{B}^2\ , \tag{28}$$

where we have used the standard Lorentz transformation[10] $\mathscr{B}^2=\gamma(B^2-\beta E^1)$ connecting the electromagnetic fields in I_* and I_{τ}. Transforming all the fields in an analogous fashion, we find the antisymmetric tensor

$$F^{\mu\nu}-(F^{\mu\alpha}v_{\alpha}v^{\nu}-F^{\nu\alpha}v_{\alpha}v^{\mu})$$

has nonzero above-diagonal elements

$$(0,1)=\beta\gamma\mathscr{B}^2\ ,\quad (0,2)=-\beta\gamma\mathscr{B}^1\ ,\quad (1,2)=\mathscr{B}^3\ ,$$
$$(1,3)=-\gamma\mathscr{B}^2\ ,\quad (2,3)=\gamma\mathscr{B}^1\ , \tag{29}$$

where, appropriately, only the magnetic fields in the rest frame of the particle appear. Now contracting this tensor with $S^{\mu}{}_{\sigma}$, we find the antisymmetric tensor $\{S^{\mu}{}_{\sigma}[F^{\sigma\nu}-(F^{\sigma\alpha}v_{\alpha}v^{\nu}-F^{\nu\alpha}v_{\alpha}v^{\sigma})]\}-\{\mu\leftrightarrow\nu\}$ with nonvanishing above-diagonal elements

$$(0,1)=\beta\gamma(\mathcal{S}^{12}\mathcal{B}^{1}-\mathcal{S}^{23}\mathcal{B}^{3}),$$
$$(0,2)=\beta\gamma(\mathcal{S}^{13}\mathcal{B}^{3}+\mathcal{S}^{12}\mathcal{B}^{2}),$$
$$(1,2)=\mathcal{S}^{13}\mathcal{B}^{1}+\mathcal{S}^{23}\mathcal{B}^{2}, \tag{30}$$
$$(1,3)=\gamma(\mathcal{S}^{23}\mathcal{B}^{3}-\mathcal{S}^{12}\mathcal{B}^{1}),$$
$$(2,3)=-\gamma(\mathcal{S}^{12}\mathcal{B}^{2}+\mathcal{S}^{13}\mathcal{B}^{3}).$$

The equations of motion now follow from Eq. (9) with the tensors given in (20), (26), and (30). We find that the same equations for $\dot{\mathcal{S}}^{13}$ and $\dot{\mathcal{S}}^{23}$ appear twice while that for $\dot{\mathcal{S}}^{12}$ occurs only once as

$$-\dot{\mathcal{S}}^{13}=g(\mathcal{S}^{12}\mathcal{B}^{1}-\mathcal{S}^{23}\mathcal{B}^{3}) -\tfrac{2}{3}g^2[\mathcal{S}^{12}(\dddot{\mathcal{S}}^{23}-a^2\dot{\mathcal{S}}^{23})-\mathcal{S}^{23}\dddot{\mathcal{S}}^{12}], \tag{31}$$

$$-\dot{\mathcal{S}}^{23}=g(\mathcal{S}^{13}\mathcal{B}^{3}+\mathcal{S}^{12}\mathcal{B}^{2}) -\tfrac{2}{3}g^2[\mathcal{S}^{13}\dddot{\mathcal{S}}^{12}-\mathcal{S}^{12}(\dddot{\mathcal{S}}^{13}-a^2\dot{\mathcal{S}}^{13})], \tag{32}$$

$$\dot{\mathcal{S}}^{12}=g(\mathcal{S}^{13}\mathcal{B}^{1}+\mathcal{S}^{23}\mathcal{B}^{2}) -\tfrac{2}{3}g^2[\mathcal{S}^{13}(\dddot{\mathcal{S}}^{23}-a^2\dot{\mathcal{S}}^{23}) -\mathcal{S}^{23}(\dddot{\mathcal{S}}^{13}-a^2\dot{\mathcal{S}}^{13})]. \tag{33}$$

Rewriting the equations in vector form from (1) and (4), and restoring the factors of c, we find (31)—(33) become

$$\dot{\vec{s}}=\vec{\mu}\times\vec{\mathcal{B}}^{\,\text{in}}-\frac{2}{3c^2}\vec{\mu}\times\left[\dddot{\vec{\mu}}-\frac{a^2}{c^2}(\dot{\vec{\mu}}-\hat{k}\hat{k}\cdot\dot{\vec{\mu}})\right]. \tag{34}$$

This, finally, is the equation of motion of a spinning magnetic dipole which undergoes a uniform acceleration $\vec{a}=\hat{k}a$. We notice that for zero acceleration, $\vec{a}=0$, this reduces to Bhabha's Eq. (6-51) for a spinning magnetic dipole fixed at rest in an inertial frame. The noninertial character of the particle's motion modifies only the radiation-reaction term.

THERMAL BEHAVIOR OF THE SPIN

In this paper we wish to evaluate the equilibrium behavior of a classical spinning magnetic dipole when it undergoes uniform acceleration through classical electromagnetic zero-point radiation. In order to find this behavior we follow a recent calculation[5] of the behavior of a spinning magnetic dipole at rest in an inertial frame in a magnetic field $\vec{B}_0=\hat{k}B_0$ along the z axis. Using the approximation that the magnetic moment was sufficiently small compared to the spin so that the dipole made many precessional revolutions during a small change of the orientation angle θ, we obtained a Fokker-Planck equation for the orientation angle between the spin direction and the z axis. The calculation for the present situation follows so closely that of the earlier analysis that we will simply note the few changes from the previous calculation.

The equation of motion for our accelerating dipole is Eq. (34) where we separate the magnetic field $\vec{\mathcal{B}}^{\,\text{in}}$ into the fixed magnetic field $\vec{\mathcal{B}}_0$ along the z axis and the random magnetic field $\vec{\mathcal{B}}_{\text{ZP}}(0,\tau)$. Since the particle is moving along the z axis in I_*, the magnetic field $\vec{\mathcal{B}}_0=\hat{k}\mathcal{B}_0$ seen in the rest frame of the particle is the same as the field $\vec{B}_0$ seen in the fixed inertial frame I_*, $\vec{\mathcal{B}}_0=\vec{B}_0$. The random magnetic field $\vec{\mathcal{B}}_{\text{ZP}}(0,\tau)$ seen at the rest position of the spinning dipole was evaluated in earlier work.[11] We showed that the accelerating particle finds a Gaussian random field with a spectrum

$$\rho_a(\omega)=\frac{\omega^2}{\pi^2c^3}\left[1+\left[\frac{a}{c\omega}\right]^2\right]\tfrac{1}{2}\hbar\omega\coth\left[\frac{\pi c\omega}{a}\right]. \tag{35}$$

Thus for our present work we must modify the analysis for a spinning magnetic dipole in an inertial frame so that the equation (5-1) becomes from (34)

$$\dot{\vec{S}}=\vec{\mu}\times\vec{B}_0-\tfrac{2}{3}c^{-3}\vec{\mu}\times[\dddot{\vec{\mu}}-(a/c)^2(\dot{\vec{\mu}}-\hat{k}\hat{k}\cdot\dot{\vec{\mu}})] +\vec{\mu}\times\vec{B}_R(0,\tau), \tag{36}$$

where the random field $\vec{B}_R(0,\tau)$ has a spectral function $\mathcal{H}(\vec{k},\lambda)$ with

$$\pi^2\mathcal{H}^2(\vec{k},\lambda)=[1+(a/c\omega)^2]\tfrac{1}{2}\hbar\omega\coth(\pi c\omega/a). \tag{37}$$

Note that in Eqs. (36) and (37) we have adopted the notation of Ref. 5, using $\vec{S}$ as the spin vector we had denoted by $\vec{s}$ above in the earlier part of this paper and also using $\vec{B}_R(0,\tau)$ for what we had denoted by $\vec{\mathcal{B}}_{\text{ZP}}(0,\tau)$ above.

Now we can follow Ref. 5 except for the following changes. When treating the radiation damping, we do not use the earlier energy conservation arguments but rather use the equation of motion as was done in the second half of the discussion of radiation damping in Ref. 5. In our approximation which computes $\dddot{\vec{\mu}}$ and $\dot{\vec{\mu}}$ from the unperturbed precession, we have from (5-5)

$$\dot{\vec{\mu}}=\hat{\phi}\mu\sin\theta(\dot{\phi}), \tag{38}$$

$$\dddot{\vec{\mu}}=-\hat{\phi}\mu\sin\theta(\dot{\phi})^3, \tag{39}$$

where (39) corresponds to (5-12), so that the quantity needed in the radiation damping is

$$[\dddot{\vec{\mu}}-(a/c)^2(\dot{\vec{\mu}}-\hat{k}\hat{k}\cdot\dot{\vec{\mu}})]=-\hat{\phi}\mu\sin\theta(\dot{\phi})^3[1+(a/c\dot{\phi})^2]. \tag{40}$$

Hence the radiation damping for the present accelerating situation follows when Eqs. (5-13) and (5-11) are modified by a factor of $[1+(a/c\dot{\phi})^2]$. Thus Eq. (5-11) is replaced here by

$$\left[\frac{d\theta}{d\tau}\right]_{\text{rr}}=-\frac{2}{3}\frac{\mu}{c^3B_0}\sin\theta(\dot{\phi})^4\left[1+\left[\frac{a}{c\dot{\phi}}\right]^2\right]. \tag{41}$$

All of the calculations of Ref. 5 now proceed as before until we come to Eq. (5-35). The additional factor in (41) for the radiation reaction changes (5-35) into

$$\frac{dP(\theta)}{d\theta}+\left[\frac{S^2}{\mu B_0}\eta^2\left[\frac{[1+(a/c\eta)^2]}{\pi^2\mathcal{H}^2(|\eta|)}\right]\sin\theta-\cot\theta\right]P(\theta)=0. \tag{42}$$

But then we see that the alteration in the radiation damping has the effect of replacing the spectral density $\pi^2\mathscr{H}^2(|\eta|)$ of Ref. 5 by an effective spectral density

$$\{\pi^2\mathscr{H}^2(|\eta|)/[1+(a/c\eta)^2]\} .$$

Substituting for $\mathscr{H}^2(|\eta|)$ from Eq. (37) and then introducing the Unruh-Davies temperature[2]

$$T=\hbar a/2\pi ck , \tag{43}$$

we see that the effective spectral density becomes

$$\frac{\pi^2\mathscr{H}^2(|\eta|)}{[1+(a/c\eta)^2]}=\tfrac{1}{2}\hbar\omega\coth(\pi c\omega/a)$$

$$=\left[\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}+\tfrac{1}{2}\hbar\omega\right] \tag{44}$$

which looks just like Planck's spectrum including zero-point radiation. Thus the spinning magnetic dipole in responding to the effective spectral density responds just as it did in an inertial frame with exactly Planck's spectrum as given earlier in (5-39). We have proved what we set out to show.

CLOSING SUMMARY

In order to treat a classical point spinning magnetic moment undergoing uniform proper acceleration, we turn to Bhabha's relativistic equation of motion and then specialize it to the case of uniform acceleration. We find that the equation of motion for the classical spin in its own rest frame takes the familiar form corresponding to a spinning magnetic moment in an inertial frame except for an alteration in the radiation damping term. Now the equilibrium distribution for the orientation of the spin relative to the direction of acceleration and of the external constant magnetic field depends upon the balance of the radiation damping effects and the random torques due to the random magnetic field seen by the accelerating dipole. The spectrum of the random magnetic field seen by the accelerating dipole departs from Planckian form at the Unruh-Davies temperature by a factor of $[1+(a/c\omega)^2]$. However, the change in the radiation damping term produced by the acceleration is precisely such as to cancel this departure from Planckian form. In its own rest frame the spinning magnetic dipole takes the same equilibrium distribution as it would in Planck's spectrum in an inertial frame.

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