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Spherically symmetric systems of fields and black holes. III. Positivity of energy and of a new type Euclidean action

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Within the scope of the two-dimensional model of gravity which was defined and studied in the two preceding papers, we investigate the three famous positivity problems of general relativity: (1) energy, (2) Euclidean action, and (3) divergence identities. We show that the energy can be split in a unique way into a black-hole mass and a field energy. If there are no fields in the model which could discharge the hole, then the field energy itself is non-negative and has a zero minimum for the vacuum value of the fields. In the opposite case, the greatest lower bound for the total energy is the irreducible mass of the hole. We define a new type of Euclidean action for gravity theories, which is different from the action used currently in Euclidean quantum gravity. The new Euclidean action is obtained by a true analytical continuation of a reduced Lorentzian action so that the relation between the Euclidean and Lorentzian regimes is well defined. We prove that the new Euclidean action is positive definite without any additional "complexification." We show that the possibility of gravitational collapse leads to an unusual, saturation-curve-like form, of the Hamiltonian and of the new Euclidean action.

I. INTRODUCTION

In the Einstein theory of gravity, one encounters at least three important positivity problems: (1) the positivity of the total energy of an isolated, gravitating system, (2) the positivity of the Euclidean action of an asymptotically Euclidean space, and (3) the positivity of the right-hand side of the so-called divergence identities, which are used to show the uniqueness (no hair) theorems for stationary black holes.

Recently, new results concerning the first problem have been established (see, e.g., Refs. 1—4). The expression for the energy is a surface integral at infinity, which can be arbitrary large negative for some values of the fields. Only if the fields satisfy a part of the field equations, namely, the so-called constraints, can the positivity be shown (for details, see Refs. 1 and 2). This is a unique situation in the field theory. For example, the Yang-Mills fields also are constrained, but the expression for their total energy is manifestly positive for all values of the fields, irrespectively of whether they satisfy the constraints or not. The positivity proofs could be extended also to situations with black holes.[4] Here, the result is even stronger than positivity: the square of the total energy must be larger than the sum of the squared charges of the hole. Penrose[5] has formulated the following, still stronger, conjecture: the total energy is larger than the irreducible mass of the hole.

The second problem is met, if one attempts to construct a quantum theory of gravity. In the flat-spacetime quantum field theory, the so-called Euclidean method is very powerful. The quantum fields are obtained by first constructing the imaginary-time correlation functions (so-called Schwinger functions) as functional integrals and then applying certain general reconstruction theorems to recover the real-time theory (see, e.g., Refs. 6—8). For the whole construction, two simple features are of crucial importance: (a) the Euclidean action is positive definite (or, at least bounded from below) and (b) an analytic continuation provides a transition between the Euclidean and the Lorentzian regimes. In the so-called Euclidean quantum gravity,[9] one must proceed in a different way. First, one gets into the Euclidean regime by setting the spacetime metric to be positive definite; this has in general nothing to do with analytic continuation from the Lorentzian regime (there need not be any real Lorentzian section of the analytic continuation of the positive-definite metric). Thus, the transition between the Euclidean and Lorentzian regimes becomes problematical. Moreover, the Einstein-Hilbert action for the Euclidean metric is not bounded from below.[10] To overcome this obstacle, one splits the metric in the conformal structure and conformal factor, and performs an *ad hoc* analytical continuation of the conformal factor.[10] The Euclidean action prepared in this way is positive definite.[11]

The actual statement of the positive-action theorem[10,11] has features which are strongly analogous to the "positive-energy theorem": again, there is an expression, which can be arbitrary large negative for some values of the fields, but, if the fields satisfy an additional condition, it becomes positive definite. This additional condition is, for the energy, the constraints and, for the Euclidean action, the vanishing of the scalar curvature (this defines the representants of the conformal classes). In both cases, the additional condition is a part of the equations of motion.

The difficulties of the Euclidean quantum gravity, as well as the above analogy, suggest that one should try to define the Euclidean action in a different way. As in all gauge theories, the variables in the action can be split up into three groups: the true dynamical variables, the dependent variables, and the gauge variables. One can, at least in principle, exclude the gauge variables by choosing a gauge (including a time coordinate) and the dependent variables by solving the constraints. One obtains, in such a way, the reduced energy and the reduced action, which are functionals of only the true dynamical variables. From the "positive-energy theorem," it follows that the reduced energy is positive for all values of the true dynamical variables. Could one construct a new type of Euclidean action from the reduced Lorentzian action? Can such a Euclidean action be positive definite?

The answer to the first question is simplified by the fact that the process of reduction includes the choice of the spacetime foliation. Together with the reduced action, we are supplied by a particular time coordinate, so we can try whether or not just the usual Wick rotation will yield anything reasonable. If so, no transition problem between the Euclidean and Lorentzian regimes will plague the theory. There will be another difficulty, however: we are losing the manifest gauge invariance. Even worse, the Euclidean action corresponding to different foliations will not, in general, lead to equivalent quantum theories.[12] Still, it is quite plausible that the standard choice of time in asymptotically flat regions, where this time becomes, globally, a time coordinate of an inertial system, can lead to a unique scattering theory. There are interesting papers in this direction.[13]

As to the second question, the affirmative answer is again plausible. Indeed, the "bad" direction, in which the Euclidean action becomes negative, is a conformal deformation. The conformal factor at the space part of the metric is, however, a typical dependent variable, whose value is determined by the Hamiltonian constraint (see, e.g., Ref. 14). Hence, in the reduced theory, the freedom to make conformal deformations disappears.

We can, therefore, formulate the following *new positive-action conjecture.*

"If the foliation of the spacetime is chosen properly, and the system is totally reduced, then

(a) the corresponding Euclidean action is defined by the Wick rotation,

(b) it is positive definite,

(c) in asymptotically flat spacetimes, it defines a unique scattering theory."

The third positivity problem in general relativity, the divergence identities, is a difficult one. The question, what is the status of the uniqueness theorems from the standpoint of the field theory, for example, whether there is any relation to the lower bounds of energy in spacetimes with holes, have never been even posed (as far as I know). In fact, the origin of the divergence identities has been mostly considered as mysterious (for a review, see Ref. 15), and even after the very beautiful paper by Mazur,[16] it seems, at least to me, to remain a mystery.

In this paper, we will study the positivity problems in a simplified situation: we limit ourselves to spherically symmetric configurations. This leads to the so-called Berger-Chitre-Moncrief-Nutku (BCMN) model.[17–19] The BCMN model consists of a scalar field with a nonlocal self-coupling on a two-dimensional spacetime. Our analysis will be based on the preceding two papers[20,21] from which the notation and the starting formulas (1)—(9) are taken; we denote Ref. 20 by I and 21 by II. To understand this paper, reading of I and II is, however, not necessary.

In Sec. II we study the Hamiltonian of the model. To simplify the formulas, we choose a special case, namely, the minimal coupling [case (a) of I], but the results are valid for all other cases, too, and can be obtained for them by calculations which are completely analogous to those given here. We have to distinguish two subcases: (i) the scalar field is neutral ($e=0$) and (ii) the scalar field is charged ($e\neq 0$). The black hole is always charged ($Q\neq 0$). In case (i), the splitting of the Hamiltonian into the black-hole part and the field part in formula (1) is convex in the sense that (1) both parts are non-negative and (2) the second part vanishes only for the vacuum value of the field. Thus, the splitting is unique and gives a definition of energy of an apparent horizon (this is likely to be possible only for spherically symmetric configurations). In case (ii), we work out another splitting, where the black-hole part is the so-called irreducible mass; the field part is again bounded from below by zero, but there is no field configuration for which it would achieve this value. The static black-hole solution with zero scalar field around it is only a local minimum of the Hamiltonian. The greatest lower bound of the total energy is, in case (ii), lower than in case (i) because of the electromagnetic interaction of the hole with the scalar.

These results confirm and sharpen Penrose's conjecture: the greatest lower bound for the total energy in an asymptotically flat spacetime with a black hole is (i) the total mass of the hole, if there are no fields in the model which can discharge the hole, or (ii) the irreducible mass of the hole otherwise.

The convex splitting of the energy in case (i) could lead to a proof of a "uniqueness theorem," if it were supplied by the following statement: "The total mass of the static spacetime with a black hole is equal to the total mass of the hole itself." This statement would play the role of the so-called Smarr formula in the usual proofs of uniqueness. Hence, the staticity is needed only in the "Smarr formula" for this case. In case (ii), it is plausible that the static black-hole solution becomes even a global minimum of total energy, if we restrict the competing configurations to the static ones. This would again lead to a sort of positivity we need. These remarks could shed some light on the relation between the uniqueness theorems and the positivity of energy, but, of course, they could also turn up to be misleading.

In Sec. II we turn our attention to a sort of *upper* bounds of the energy. Indeed, the Hamiltonian approaches a finite limit, if the field grows to infinity somewhere. We show that this "saturation phenomenon" is a necessary consequence of gravitational collapse. We find that a new apparent horizon is forming in this case. We discuss all possible mechanisms of forming a new horizon. Our choice of gauge guarantees that our boundary conditions chosen in I and II will be automatically satisfied at the new as well as at the old horizon. This makes it possible either to keep the boundary conditions at the innermost apparent horizon fixed during all processes, or to jump always to the current outermost horizon. We perform the transformation between the Hamiltonians corresponding to these two versions.

In Sec. III we study the new positive-action conjecture within the spherically symmetric, uncharged scalar model. First, let us notice that the situation in this model is perfectly analogous to that in the full quantum gravity. The second-order action, which has been given in I, will not become bounded from below, if we substitute a positive-definite tensor for the metric in it and change its overall sign; the gravity part will then look as

$$I_g^E=-\tfrac{1}{2}\int d^2x\sqrt{|g|}\left[\frac{1}{G}+g^{ab}\varphi_{,a}\varphi_{,b}+\tfrac{1}{2}R\varphi^2\right].$$

The first two terms have a wrong sign, and the curvature scalar can be positive as well as negative.

For the proof of the new positive-action conjecture in the BCMN model, the explicit complete reduced form of the action as given in II is very helpful. This action is, however, in the Hamiltonian form. We can analytically continue in the velocity plane only after the generalized momenta are expressed by means of generalized velocities. The continuation must be done carefully, because the action as a function of complex velocities has branching points on the real axis. We find that these singularities are necessarily connected with the saturation phenomenon, i.e., with the gravitational collapse. We study a very simple system, the so-called exponentiated oscillator, to show that the branching points need not lead to any pathology in the classical or quantum dynamics. Then, finally, we prove the positivity.

We also observe that the action indeed is positive, but it does not grow sufficiently quickly for the path integral to converge. There is a similar saturation phenomenon as in the Hamiltonian function. This could, we hope, be corrected by an appropriate measure. We do not tackle this problem here.

II. THE HAMILTONIAN

The BCMN model has been generalized in Refs. 20 and 21 to include the electromagnetic field, a charged scalar field with some self-interaction and mass, and electrically and magnetically charged black holes. The constraints and gauge equations have been solved explicitly in a particular gauge and the system has been reduced to contain only the true dynamical variables. The form of the surface terms at all boundaries has been derived and the Hamiltonians have been calculated.

Let us consider the case with an electrically charged hole of arbitrary mass and a minimally coupled, electrically charged scalar field ψ with a self-interaction potential $V(|\psi|^2)$. The corresponding Hamiltonian reads[21]

$$H=M_0+\frac{1}{2G}\int_{x_0}^{\infty}dy[F_0(y)-F(y)e^{-T(y)}]\ , \quad (1)$$

where

$$M_0=\frac{1}{2G}\left[x_0+\frac{GQ_0{}^2}{x_0}\right] \quad (2)$$

is the total mass of the hole with radius x_0 and charge Q_0,

$$F(x)=1-Gx^2V-G\frac{\pi_A{}^2}{x^2}\ , \quad (3)$$

$$F_0(x)=1-\frac{GQ_0{}^2}{x^2}\ , \quad (4)$$

$$T(x)=G\int_x^{\infty}dy[4y^{-3}|\pi(y)|^2+y|\psi'(y)|^2]\ . \quad (5)$$

$\pi_A(x,t)$ is the total charge under the radius x and $\pi(x,t)$ is the canonical momentum of $\psi(x,t)$. We often suppress the variable t in the expressions, but we cannot suppress x because of the multiple space integrals.

The relevant solutions to the constraints are

$$\pi_A(x)=Q_0-ie\int_{x_0}^{x}dy[\psi(y)\pi(y)-\psi^\dagger(y)\pi^\dagger(y)]\ , \quad (6)$$

$$\frac{1}{\gamma(x)}=\frac{1}{x}\int_{x_0}^{x}dy\,F(y)e^{-T(y)+T(x)}\ , \quad (7)$$

and those to the corresponding gauge equations are

$$\alpha(x)=\frac{1}{\sqrt{\gamma(x)}}e^{-T(x)}\ , \quad (8)$$

$$A_t(x)=\int_x^{\infty}\frac{dy}{y^2}\pi_A(y)e^{-T(y)}\ . \quad (9)$$

Here, $\gamma(x,t)=g_{11}(x,t)$, $\alpha^2(x,t)=-g_{00}(x,t)$ are the only nonzero components of the two-dimensional metric of the model in our gauge and A_t is that of the electromagnetic potential; e is the electric charge of the field ψ. For the derivation of these results as well as more detailed interpretation of the equations, see Ref. 21. We are now going to study some properties of the Hamiltonian (1).

A. The positivity

Two relations are important for the positivity of the Hamiltonian (1):

$$x_0\geq\sqrt{G}\,|Q_0|\ , \quad (10)$$

$$V(|\psi|^2)\geq 0\ . \quad (11)$$

The first one holds for spherically symmetric apparent horizons,[20] the second one guarantees that the local energy density of the scalar field is non-negative.

We have to consider two cases, $e=0$ and $e\neq 0$.

Case $e=0$.

Here $\pi_A=Q_0$ and

$$F(x)=1-Gx^2V-G\frac{Q_0{}^2}{x^2}\ .$$

Then, (1) can be written as

$$H=M_0+\frac{1}{2G}\int_{x_0}^{\infty}dy[F_0(y)(1-e^{-T(y)})+Gy^2Ve^{-T(y)}]\ . \quad (12)$$

Equation (12) implies: For all configurations $\psi(x,t)$, $\pi(x,t)$, we have $H\geq M_0$, equality being reached only for such fields ψ,π which satisfy

$$\psi'(x)=0\ ,\quad \pi(x)=0\ ,\quad V(|\psi(x)|^2)=0\ ,\quad \forall x$$

(a true vacuum of ψ). This follows immediately from (4), (5), (10), and (11): $T(x)\geq 0$ and $F_0(x)>0$ for all $x\in(x_0,\infty)$.

Case $e\neq 0$.

Here, we have to write (2) in the form

$$M_0=M_{\rm irr}+\frac{1}{2G}\int_{x_0}^{\infty}dy\frac{GQ_0{}^2}{y^2}\ , \qquad (13)$$

where $M_{\rm irr}=(2G)^{-1}x_0$ is the irreducible mass of a spherically symmetric black hole with radius x_0 (Ref. 22, p. 889). Setting (13) into (1) and using (4) we obtain

$$H=M_{\rm irr}+\frac{1}{2G}\int_{x_0}^{\infty}dy[1-F(y)e^{-T(y)}]$$
$$=M_{\rm irr}+\frac{1}{2G}\int_{x_0}^{\infty}dy\{(1-e^{-T(y)}) +G[y^2V+y^{-2}\pi_A{}^2(y)]e^{-T(y)}\}\ .$$

Now, the positivity is again manifest; we have even the relation $H>M_{\rm irr}$. The equality $H=M_{\rm irr}$ cannot be achieved, because one has to satisfy the following incompatible relations:

$$T(x)=0\ ,\quad \pi_A(x)=0\ ,\quad \forall x\ .$$

We can, however, show the following.

Theorem. For any $\epsilon>0$, there is such a configuration $\psi'(x),\pi(x)\in C_0^\infty(x_0,\infty)$, $V(|\psi(x_0)|^2)=V(|\psi(\infty)|^2)=0$ that

$$H[\psi,\pi]-M_{\rm irr}<\epsilon\ .$$

Thus, $M_{\rm irr}$ is a true greatest lower bound of all possible H values.

Proof. Let $x_1>x_0, x_2>x_1$ be some radii and $f(x)$ be a function defined as

$$x<x_1:\ f(x)=0\ ,$$
$$x_1<x<x_2:\ f(x)=\left\{\int_{x_1}^{x}dy\exp\left[-\frac{1}{(x_2-x)(x-x_1)}\right]\right\}\left\{\int_{x_1}^{x_2}dy\exp\left[-\frac{1}{(x_2-x)(x-x_1)}\right]\right\}^{-1}\ ,$$
$$x_2<x:\ f(x)=1\ .$$

$f(x)$ is C^∞ and we have $f'(x)\geq 0, f'(x)\in C_0^\infty(x_0,\infty)$. Choose $\psi(x)$ and $\pi(x)$ to be

$$\psi(x)=\psi_1+(\psi_2-\psi_1)f(x)+\psi_0 f'(x)\ ,\quad \pi(x)=i\pi f'(x)\ ,$$

where π,ψ_1,ψ_2,ψ_0 are real constants such that $V(\psi_1{}^2)=V(\psi_2{}^2)=0$. We have, then, $V(|\psi(x)|^2)\leq V_M$, where $V_M>0$ is independent of x_1,x_2 (but dependent on ψ_1, ψ_2, and ψ_0). $T(x)$ satisfies

$$x_0<x\leq x_1:\ T(x)={\rm const}>0\ ,$$
$$x_1\leq x\leq x_2:\ T(x)>0\ ,\quad T'(x)<0\ ,$$
$$x_2\leq x:\ T(x)=0\ .$$

For $\pi_A(x)$, we obtain

$$x_0<x\leq x_1:\ \pi_A(x)=Q_0\ ,$$
$$x_1\leq x\leq x_2:\ \pi_A(x)=Q_0+2e\pi\psi_1 f(x)+e\pi(\psi_2-\psi_1)f^2(x)+2e\pi\psi_0\int_{x_1}^{x}dy\, f'^2(y)\ ,$$
$$x_2\leq x:\ \pi_A(x)=Q_0+e\pi(\psi_2+\psi_1)+2e\pi\psi_0\int_{x_1}^{x_2}dy\, f'^2(y)={\rm const.}$$

We can always choose π and ψ_0 such that $\pi_A(x_2)=0$ and $\pi_A(x)\leq Q_0$ for all $x>x_0$. Then, we have for $H-M_{\rm irr}$

$$H-M_{\rm irr}<\frac{1}{2G}(x_2-x_0)+\tfrac{1}{6}V_M(x_2{}^3-x_1{}^3)+\tfrac{1}{2}Q_0{}^2\left[\frac{1}{x_0}-\frac{1}{x_2}\right]$$
$$\leq\frac{1}{2G}(x_2-x_0)+\tfrac{1}{6}V_M(x_2{}^3-x_0{}^3)+\tfrac{1}{2}Q_0{}^2\left[\frac{1}{x_0}-\frac{1}{x_2}\right]\ .$$

From this, it is clear that there is a δ such that for $x_0<x_2<x_0+\delta, H-M_{\rm irr}<\epsilon$, Q.E.D.

The Reissner-Nordström solution is given by $\psi(x)=\psi_1,\pi(x)=0,V(|\psi_1|^2)=0$. Its energy is the total mass of the hole and it is not an absolute minimum of H. It must be an extremum, because the Reissner-Nordström solution is static. Varying H twice at this point, we obtain

$$\delta^2H=\frac{1}{2G}\int_{x_0}^{\infty}dy[\delta^2T(y)+G(y^2\delta^2V+2y^{-2}\delta\pi_A(y)\delta\pi_A(y))]\ .$$

We have also

$$\frac{1}{2G}\int_{x_0}^{\infty} dy\,\delta^2 T(y)=\int_{x_0}^{\infty} dy(y-x_0)[4y^{-3}\delta\pi^{\dagger}(y)\delta\pi(y)+y\delta\psi'^{\dagger}(y)\delta\psi'(y)]\ ,$$

and

$$\delta^2 V=V''(\psi_1)(\psi_1^{\dagger}\delta\psi+\psi_1\delta\psi^{\dagger})^2\ .$$

Thus, the Hessian is positive definite and the total mass of the wormhole is a local minimum.

As seen from the proof of the theorem, the initial data $\psi(x),\pi(x)$ whose energy lies sufficiently near to $M_{\rm irr}$ must form a shell of matter around the hole, which (1) is very near to the hole and (2) neutralizes it. Such initial data, of course, can never lead to a static solution. It is even conceivable that the restriction of the competing configurations to static fields could promote the total mass of the hole to an absolute minimum.

B. The formation of apparent horizons

The form of the Hamiltonian (1) is rather remarkable, it reminds us of the saturation curve $1-e^{-x}$. One can call this property of the Hamiltonian "saturation." Indeed, if $\pi(x)$ and/or $\psi'(x)$ will grow over all bounds at any given point x, H will not diverge, but approach some finite limit.

Let us study this property in more detail. We observe first that $T(x)$ is a positive, monotonic function, $T'(x)\le 0$ and $T(\infty)=0$. Thus, if there is a point, $\tilde{x}_1$, say, such that $x_0<\tilde{x}_1<\infty$ and $T(\tilde{x}_1)=\infty$, then $T(x)=\infty$ for all x which lie between x_0 and $\tilde{x}_1$. Let x_1 be the maximum of all points x where $T(x)=\infty$. Then,

$$x_0<x\le x_1:\ T(x)=\infty\ ,$$

$$x_1<x:\ T(x)<\infty\ .$$

At the point x_1, $\psi'(x)$, and/or $\pi(x)$ must diverge, but they can be regular at $x<x_1$. We can, therefore, write

$$H=M_0+\frac{1}{2G}\int_{x_0}^{\infty} dy\,F_0(y)-\frac{1}{2G}\int_{x_1}^{\infty} dy\,F(y)e^{-T(y)}\ .$$

Using (2), (3), and (4), this can be transformed as

$$H=\frac{1}{2G}\left[x_0+\frac{GQ_0{}^2}{x_0}\right]+\frac{1}{2G}\int_{x_0}^{\infty} dy[F_0(y)-F_1(y)]+\frac{1}{2G}\int_{x_0}^{x_1} dy\,F_1(y)+\frac{1}{2G}\int_{x_1}^{\infty} dy\,[F_1(y)-F(y)e^{-T(y)}]\ ,$$

where

$$F_1(x)=1-\frac{GQ_1{}^2}{x^2}$$

and

$$Q_1=\pi_A(x_1)\ .$$

However, we have

$$\frac{1}{2G}\left[x_0+\frac{GQ_0{}^2}{x_0}\right]+\frac{1}{2G}\int_{x_0}^{\infty} dt\,[F_0(y)-F_1(y)]+\frac{1}{2G}\int_{x_0}^{x_1} dy\,F_1(y)=\frac{1}{2G}\left[x_1+\frac{GQ_1{}^2}{x_1}\right]=M_1\ .$$

M_1 is the mass of a spherically symmetric black hole of radius x_1 and charge Q_1. Thus,

$$H=M_1+\frac{1}{2G}\int_{x_1}^{\infty} dy[F_1(y)-F(y)e^{-T(y)}]\ .$$

Of course, we could expect that a new (apparent) horizon has formed at x_1. This is confirmed by Eq. (7): setting $T(x_1)=\infty$ in it, we obtain

$$\frac{1}{\gamma(x_1)}=0\ .$$

This means that $x=x_1$ is a minimal surface.

The formation of a horizon should be an irreversible process. How can our Hamiltonian "organize" this? It could seem, e.g., that the fields $\psi'(x)$ and $\pi(x)$ could become finite again at some later time. Would then the horizon disappear? The answer is no and the reason is rather surprising: there will be no later time. For consider Eq. (8): $\alpha(x)$ becomes zero for all x in the interval $[x_0,x_1]$, and precisely at the moment, when $T(x_1)$ becomes singular.

Hence, the saturation has to do with the gravitational collapse of a field configuration outside of the hole. The total energy does not diverge, because the gravitational energy is negative and balances, or even overweights, the local ψ energy, even if this becomes infinite.

A more quiet possibility of how a new apparent horizon can form is that $F(x)$ becomes negative somewhere so that, for some x_1, we obtain

$$\int_{x_0}^{x_1} dy\,F(y)e^{-T(y)}=0\ .$$

If t_1 is the first time moment at which this happens, we have, at t_1, $\gamma^{-1}(x)>0$ for all $x \neq x_1$ in a neighborhood of x_1. $\alpha(x_1)$ becomes zero at t_1 and it stays zero for all $t>t_1$. Thus, the developments of the two Cauchy surfaces $x_0<x<x_1, t=t_1$ and $x_1<x<\infty, t=t_1$, become causally disconnected. As in the previous case, one can transform the Hamiltonian for the region $t>t_1, x>x_1$ to the shape

$$H=M_1+\int_{x_1}^{\infty} dy\,[F_1(y)-F(y)e^{-T(y)}] ,$$

where M_1 is the mass of the new hole.

III. THE EUCLIDEAN ACTION

In this section, we limit ourselves to the simple case $e=0$ and $V(|\psi|^2)=0$. Let us introduce the functionals A and B by the relations

$$A[x;f]=4G\int_x^{\infty} dy\, y^{-3}f^2(y) , \tag{14}$$

$$B[x;f]=G\int_x^{\infty} dy\, y\left[\frac{\partial f}{\partial y}\right]^2 , \tag{15}$$

where $f(x)$ is an arbitrary C_0^{∞} function. Then, $T(x)$ can be written as

$$T(x)=A[x;\pi(x)]+B[x;\psi(x)] .$$

The action corresponding to the Hamiltonian (1) in the present case reads

$$I=\int_{t'}^{t''} dt\left[\int_{x_0}^{\infty} dx\, \dot{\psi}\pi-H[\psi,\pi]\right] . \tag{16}$$

The Euclidean action I_E, can be obtained from it in the following two steps.

(1) Solving the velocity-momentum relation $\delta I/\delta\pi(x)=0$, for $\pi(x)$ and setting the solution into (16).

(2) Analytic continuation of $\dot{\psi}(x)$ to the imaginary axis defining

$$\dot{\psi}(x)=-i\dot{\psi}_E(x)$$

[$\dot{\psi}_E(x)$ is real]. The resulting functional of $\dot{\psi}_E(x)$ and $\psi(x)$ is iI_E.

The first step will lead to a double-valued functional (infinite-dimensional two-sheet Riemann surface). This is not an everyday situation, so we slightly digress at this point and study a related but much simpler problem first.

A. Exponentiated oscillator

Consider a one-dimensional system with coordinate $q(t)$ and momentum $p(t)$, whose Hamiltonian has the form

$$H=\frac{1}{\gamma}(1-e^{-\gamma T}) , \tag{17}$$

where

$$T=\frac{1}{2m}(p^2+m^2\omega^2q^2) , \tag{18}$$

m, ω, and γ are constants of the dimension $+1$, $+1$, and -1, respectively.

The classical equations of motion are

$$\dot{q}=\frac{1}{m}p\,e^{-\gamma T} ,\quad \dot{p}=-m\omega^2 q\,e^{-\gamma T} . \tag{19}$$

Clearly, T is a constant of motion and the general solution for q, therefore, is

$$q=A\cos(\omega_T t+\alpha) .$$

All three constants, A, α, and ω_T are, now, dependent on the initial data q_0,p_0. In particular,

$$\omega_T=\omega\exp\left[-\frac{\gamma}{2m}{p_0}^2-\gamma\frac{m\omega^2}{2}{q_0}^2\right] .$$

Thus, each classical trajectory of our system is just that of a harmonic oscillator. The quantum theory corresponding to the formal Hamiltonian (17) and (18) is readily constructed. With the substitution $p=-i\partial/\partial q$, T becomes an essentially self-adjoint operator on C_0^{∞} (see, e.g., Ref. 23, p. 175). There is a unique self-adjoint extension, $\overline{T}$, of it to $L^2(-\infty,\infty)$. We can define

$$H=\frac{1}{\gamma}(1-e^{-\gamma\overline{T}}) .$$

Such H is self-adjoint, bounded operator on $L^2(-\infty,\infty)$. It has the same eigenfunctions $\psi_n(q)$ as $\overline{T}$ and its spectrum is given by

$$\frac{1}{\gamma}[1-\exp(-\gamma\omega(n+\tfrac{1}{2}))] ,\quad n=1,\ldots,\infty .$$

Thus, the model does not seem to be pathological, either in its classical or in its quantum aspects.

It has, however, an interesting feature, which leads to complications, if one, e.g., is going to construct the Euclidean action: the relation between the velocity and momentum is not uniquely solvable. This relation is given by the first equation of (19) and can be written as

$$b(p)=a(\dot{q},q) , \tag{20}$$

where

$$b(x)=x\exp\left[-\frac{\gamma}{2m}x^2\right] ,$$

$$a(x,y)=mx\exp\left[\frac{\gamma m\omega^2}{2}y^2\right] .$$

The shape of the functions $b(p)$ and $H(p)$ is shown in Fig. 1. We see that the branching points of the function inverse to b coincide with the points of inflection of the Hamiltonian. Notice that such inflection points must always be present, if the Hamiltonian is approximately quadratic for small values, and has the saturation property for large values of the dynamical variables. Thus, the branching of the velocity-momentum relation seems to follow necessarily from the saturation. From Fig. 1, one also sees that the velocity $\dot{q}$ is bounded to the interval

$$-\frac{1}{\sqrt{\gamma m}}\exp(-\tfrac{1}{2}(1+\gamma m\omega^2q^2)) \le \dot{q} \le \frac{1}{\sqrt{\gamma m}}\exp(-\tfrac{1}{2}(1+\gamma m\omega^2q^2)) .$$

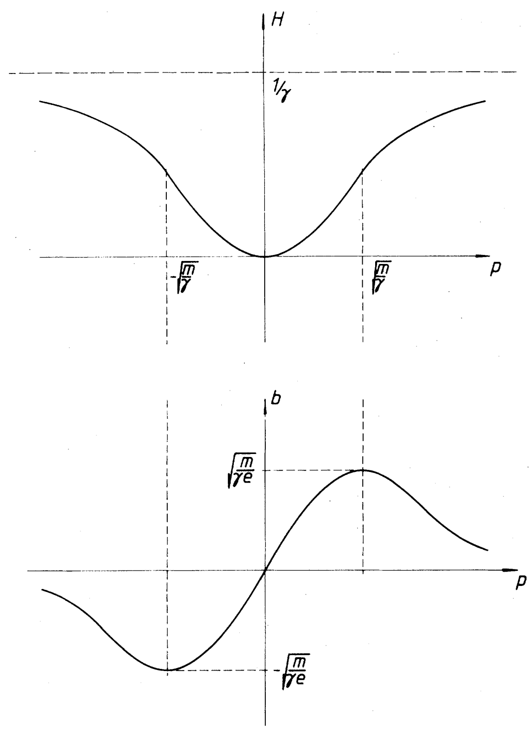

FIG. 1. H and b as functions of p.

In the gravitational case this has to do with the behavior of the time coordinate: higher proper-time velocity leads to speeding up of the coordinate time.

In a neighborhood of the branching points, we have

$$a=\pm\left[\frac{m}{\gamma}\right]^{1/2}e^{-1/2}\left[1-\frac{\gamma}{m}\delta p^2\right]+O(\delta p^3)\ ,$$

where $\delta p=p\mp(m/\gamma)^{1/2}$. Thus, the Riemann surface is two-sheeted, with singularities at $a=\pm(m/\gamma e)^{1/2}$ on the real axis. We can define the two sheets by cuts along the intervals $(-\infty,-(m/\gamma e)^{1/2}),((m/\gamma e)^{1/2},\infty)$ of the real axis. The first sheet defines p uniquely as a function of a, in particular at the imaginary a axis; this function is continuous along the whole of the axis. If we set $a=-ia_E,p=-ip_E$, then this function, if limited to the imaginary axis, satisfies identically the relation

$$a_E=p_E(a_E)\exp\left[\frac{\gamma}{2m}p_E{}^2(a_e)\right]\ .$$

The other possible function $p_E(a_E)$ on the imaginary axis, as defined by the second sheet, must be singular at the origin (see Fig. 1). It seems, therefore, that there is only one reasonable extension of $p(a)$ to the imaginary axis. This means that we can construct a unique Euclidean action for the model.

B. The infinite-dimensional case

Let us calculate the velocity-momentum relation by varying $H[\psi,\pi]$. We obtain from (16)

$$\delta_\pi H=\frac{1}{2G}\int_{x_0}^{\infty}dx\,F_0(x)e^{-T(x)}\delta_\pi T(x)\ ,$$

and from (14)

$$\delta_\pi T(x)=8G\int_x^\infty dy\,y^{-3}\pi(y)\delta\pi(y)\ .$$

Hence,

$$\delta_\pi H=4\int_{x_0}^{\infty}dx\,F_0(x)e^{-T(x)}\int_x^\infty dy\,y^3\pi(y)\delta\pi(y)$$
$$=4\int_{x_0}^{\infty}dy\,\delta\pi(y)\left[\frac{\pi(y)}{y^3}\int_{x_0}^{y}dx\,F_0(x)e^{-T(x)}\right]$$

and

$$\frac{\delta I}{\delta\pi(x)}=\dot\psi(x)-\frac{4\pi(x)}{x^3}\int_{x_0}^{x}dy\,F_0(y)e^{-T(y)}\ .$$

The resulting velocity-momentum relation reads

$$\dot\psi(x)=\frac{4\pi(x)}{x^3}\int_{x_0}^{x}dy\,F_0(y)e^{-T(y)}\ . \qquad (21)$$

An analogous calculation yields the second variation of the Hamiltonian:

$$\delta^2H(\pi)=\frac{\delta\dot\psi(x)}{\delta\pi(y)}$$
$$=[\delta(x-y)-2Gx^{-3}\pi(x)\pi(y)]$$
$$\times 4y^{-3}\int_{x_0}^{y}dz\,F_0(z)e^{-T(z)}\ . \qquad (22)$$

This can be considered as an operator: let us introduce the abbreviation

$$\xi(x)=\int_{x_0}^{x}dy\,F_0(y)e^{-T(y)}\ ,$$

then, for any C_0^∞ function $\delta\pi(x)$ on the interval (x_0,∞), we obtain the following function of x as the action of the operator:

$$(\delta^2H(\pi)\delta\pi)(x)=4x^{-3}\delta\pi(x)\xi(x)$$
$$-8Gx^{-3}\pi(x)\int_{x_0}^{\infty}dy\,\pi(y)y^{-3}\delta\pi(y)\xi(y)\ .$$

If we extend $C_0^\infty(x_0,\infty)$ to a Hilbert space Π by means of the auxiliary scalar product

$$(\delta\pi_1,\delta\pi_2)=\int_{x_0}^{\infty}dy\,y^{-3}\delta\pi_1(y)\delta\pi_2(y)\ ,$$

then we can apply the operator $\delta^2H(\pi)$ to all elements of it, obtaining again such elements, if only $\pi(x)\in\Pi$.

$\xi(x)$ defines a linear, invertible transformation ξ on Π by $\delta\pi(x)\to\xi(x)\delta\pi(x)$, because $\xi(x)\neq 0$ for all $x\in(x_0,\infty)$. Let $\Pi_{||}\subset\Pi$ be the one-dimensional subspace of Π spanned by $\pi(x)$ and let $\Pi_\perp\subset\Pi$ be the orthogonal complement of $\Pi_{||}$ in Π. The operator

$$D=4y^{-3}\delta(x-y)-8Gx^{-3}\pi(x)y^{-3}\pi(y)$$

maps each vector $\delta\pi\in\Pi_{||}$ into $4x^{-3}\delta\pi(x)(1-2G(\pi,\pi))$, whereas each vector $\delta\pi\in\Pi_\perp$ goes to $4x^{-3}\delta\pi(x)$. Hence, the product of D with ξ is a top linear isomorphism (see Ref. 24), if $1-2G(\pi,\pi)\neq 0$, and has a one-dimensional kernel $\xi^{-1}\Pi_{||}$, if $(\pi,\pi)=(2G)^{-1}$. The map (21) is, therefore, locally invertible, unless π lies on the sphere S_G of radius $(2G)^{-1/2}$ around the origin (see, e.g., Ref. 24, p. 13). The direction in which $\delta^2 H(\pi)$ changes signature is given by $\xi^{-1}\pi$; it is not tangential to the sphere S_G.

The situation is, therefore, completely analogous to that for the exponentiated oscillator. We have again a two-sheeted (but, this time, infinitely dimensional) Riemannian surface over the "plane" $\Pi\times\Pi$. The singularities lie along the surface given by the equations

$$(\pi_1,\pi_1)-(\pi_2,\pi_2)=\frac{1}{2G}\ ,$$

$$(\pi_1,\pi_2)=0\ ,$$

it crosses the real "axis" $(\pi_1,0),\pi_1\in\Pi$, at S_G and remains at a secure distance from the imaginary "axis" $(0,\pi_2),\pi_2\in\Pi$. There is a unique analytical continuation of (21) to the imaginary "axis" determined by the first sheet; if we define $\dot\psi_E(x)$ and $\pi_E(x)$ by

$$\dot\psi(x)=-i\dot\psi_E(x)\ ,\quad \pi(x)=-i\pi_E(x)\ ,$$

$$\psi_E(x)=\psi(x)\ ,$$

then we have

$$\dot\psi_E(x)=\frac{\pi_E(x)}{x^3}\int_{x_0}^{x} dy\, 4F_0(y)\exp(A[y;\pi_E]-B[y;\psi_E]) \tag{23}$$

along the imaginary "axis," and this relation has a unique global solution $\pi_E(x)$, for any given $\dot\psi_E(x)\in\Pi$: so (23) defines a real functional $\pi_E[\dot\psi_E,\psi_E]$.

After these preliminaries, we can, finally, turn to the proof of the main statement of this section. With the help of the functional π_E, the Euclidean action can be written as

$$I_E=\int_{\tau'}^{\tau''} d\tau\left[\int_{x_0}^{\infty} dx\,\dot\psi_E\pi_E[\dot\psi_E,\psi_E] + M_0+H_E[\psi_E,\pi_E[\dot\psi_E,\psi_E]]\right], \tag{24}$$

where

$$H_E[f_1,f_2] =\frac{1}{2G}\int_{x_0}^{\infty} dx\, F_0(x)[1-\exp(A[x;f_2]-B[x;f_1])]\ . \tag{25}$$

For the first term in the space integral in (24), we obtain

$$\int_{x_0}^{\infty} dx\,\dot\psi_E\pi_E=\int_{x_0}^{\infty} dx\frac{\pi_E{}^2(x)}{x^3}\times\int_{x_0}^{x} dy\, 4F_0(y)\exp(A[y;\pi_E]-B[y;\psi_E])\ .$$

However, from (14), we have

$$\frac{\pi_E{}^2(x)}{x^3}=-\frac{1}{4G}\frac{d}{dx}A[x;\pi_E]\ .$$

Integration by parts then yields

$$\int_{x_0}^{\infty} dx\,\dot\psi_E\pi_E=\frac{1}{G}\int_{x_0}^{\infty} dx\, F_0(x)A[x;\pi_E]\exp(A[x;\pi_E]-B[x;\psi_E])\ ,$$

because $A[\infty;\pi_E]=0, A[x_0;\pi_E]<\infty$. Using this in (24), we obtain

$$\begin{aligned}I_E&=\int_{\tau'}^{\tau''} d\tau\left\{M_0+\frac{1}{2G}\int_{x_0}^{\infty} dx\, F_0(x)\{2A[x;\pi_E]\exp(A[x;\pi_E]-B[x;\psi_E])+1-\exp(A[x;\pi_E]-B[x;\psi_E])\}\right\}\\ &=\int_{\tau'}^{\tau''} d\tau\left\{M_0+\frac{1}{2G}\int_{x_0}^{\infty} dx\, F_0(x)(\{1-\exp(-B[x;\psi_E])\}\right.\\ &\qquad\left.+\exp(-B[x;\psi_E])\{1+2A[x;\pi_E]\exp(A[x;\pi_E])-\exp(A[x;\pi_E])\})\right\}.\end{aligned}$$

For any x, $A[x;\pi_E]$ is a non-negative number; such numbers satisfy the inequalities

$$1-e^A+2Ae^A\le 2Ae^A\ , \tag{26}$$

$$Ae^A\le 1-e^A+2Ae^A\ , \tag{27}$$

where equality is only possible, if $A=0$. The first inequality is obvious. The second is obtained as follows. Define the function $h(A)$ by

$$h(A)=1-e^A+Ae^A\ .$$

We have $h(0)=0, h'(A)=Ae^A>0$, if $A>0$. Thus, $h(A)>0$, if $A>0$, and this is equivalent to (27).

We obtain, in this way, the following estimate for I_E:

$$I_E \geq \int_{\tau'}^{\tau''} d\tau \left[M_0 + \frac{1}{2G} \int_{x_0}^{\infty} dx\, F_0(x)(\{1-\exp(-B[x;\psi_E])\} + A[x;\pi_E]\exp(A[x;\pi_E]-B[x;\psi_E])) \right],$$

$$I_E \leq \int_{\tau'}^{\tau''} d\tau \left[M_0 + \frac{1}{2G} \int_{x_0}^{\infty} dx\, F_0(x)(\{1-\exp(-B[x;\psi_E])\} + 2A[x;\pi_E]\exp(A[x;\pi_E]-B[x;\psi_E])) \right].$$

In particular, we have shown the following.

Theorem. The Euclidean action I_E of the reduced theory is positive, reaching its minimal value $M_0(\tau''-\tau')$ for $\psi_E = \dot{\psi}_E = 0$.

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