Spherically symmetric systems of fields and black holes. II. Apparent horizon in canonical formalism

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We study the action of a two-dimensional model of gravity found in the preceding paper. We transform the action to the first-order Arnowitt-Deser-Misner form, and work out the generalized momenta and super-Hamiltonians. We propose to foliate the spacetime in such a way that the inside of the apparent horizon will be cut away. In the classical theory, no loss of information for the development of states from $\mathscr{I}^-$ to $\mathscr{I}^+$ can result, but in the corresponding quantum theory, some such losses could occur if a black hole evaporates. We study the boundary conditions for the fields at the apparent horizon which are implied by such a foliation, and calculate the corresponding surface correction to the Hamiltonian by the method of Regge and Teitelboim. We generalize the so-called Berger-Chitre-Moncrief-Nutku gauge in such a way that the fields cannot violate the boundary conditions. In this gauge, we perform an explicit total reduction of the canonical formalism so that only the true dynamical variables appear in the Hamiltonian. The reduced Hamiltonian splits into a black hole and a field part.

I. INTRODUCTION

In this paper, we are going to construct a canonical formalism which would be suitable for a description of black holes, their formation and their metamorphoses. We will limit ourselves to spherically symmetric systems of fields. The simplifying requirement of spherical symmetry leads to a field model on a two-dimensional curved spacetime, the so-called Berger-Chitre-Moncrief-Nutku (BCMN) model.[1] A black hole in a Cauchy surface will be defined as an intersection of the outermost apparent horizon with the surface (for the definition of an apparent horizon see Ref. 2); this is a (slight) modification of the usual definition. The construction of the BCMN-model action from different four-dimensional systems has been performed in Ref. 3. A more general definition of the apparent horizon than that of Ref. 2 has been given there and its basic properties have been derived. Hereafter, we will denote Ref. 3 as paper I. Our notation and conventions will be taken over from I.

We are constructing a canonical formalism for black holes with the final aim in mind to use it as a basis for a quantum theory of black holes (see I). Indeed, even the path-integral method of field quantization is (normally) based on a canonical formalism (see, e.g., Ref. 4).

The extension of the canonical formalism to systems with black holes meets, of course, difficulties at the level of first principles. The first step toward a canonical formalism is a foliation of the spacetime under consideration by a family of Cauchy surfaces (see, e.g., Ref. 5). The spacetime must, therefore, be totally hyperbolic and nonsingular. However, the spacetimes with black holes mostly do not satisfy these requirements. In I, we proposed to foliate only a part of the spacetime; the boundary of the part coincided with an outermost apparent horizon. Technically, any foliation is determined by some analytic gauge condition which becomes a part of the effective dynamics. We must, therefore, choose the gauge condition in such a way as to achieve the proper cut of any Cauchy surface of the foliation corresponding to the gauge and this at any dynamical situation. Surprisingly enough, it is not difficult: most of the quite natural coordinate systems automatically break down at the apparent horizon taking the laps and shift functions α and β to zero there, and this is nothing but the desired cut.

There is another problem, which is more subtle. If we quantize canonically a time development along a family of Cauchy surfaces, then from any pure state at any time, again only a pure state at any other time can result; there is no point where information can be lost. The semiclassical analysis of the black-hole evaporation suggests, however, that the process leads to a mixed state, even if the hole came into being during the time development of a pure state.[6] Does, therefore, the possibility to foliate the spacetimes with black holes in the above way mean that a "no-mixing theorem" has been proved?

This is not the case for the following reason. If the black hole evaporates, then there are effective negative-energy currents near the apparent horizon.[7] Hence, the mean motion of the apparent horizon need not be nontimelike (see I). There is, therefore, a possibility that the foliated part of the spacetime does not cover the whole of $\mathscr{I}^+$, and that part of $\mathscr{I}^+$ which will be covered, will not be sharply determined (it will be quantum fuzzy). Then, even if one will be able to calculate the pure state at $t=+\infty$, one will not have complete information of the state of the field at the whole of $\mathscr{I}^+$. This state could, therefore, be mixed. This is, of course, only a very rough, qualitative argument.

The plan of the paper is as follows. In Sec. II we use the second-order action of the generalized BCMN model and transform it to the usual, Arnowitt-Deser-Misner (ADM), first-order form. The super-Hamiltonians and the canonical momenta are obtained in this way. In Sec.

III, the method of Regge and Teitelboim[8] is employed to calculate the surface corrections to the Hamiltonian. Here, we must use the boundary conditions as given in I, as well as the "cut conditions" $\alpha=\beta=0$ at the apparent horizon. In Sec. IV we generalize the BCMN gauge so that it becomes compatible with all of the boundary conditions and so that an explicit reduction of the system remains possible. We are able to deal with the apparent horizon and with the regular center in a unified way. The reduced Hamiltonian splits into two terms; one is the energy of the apparent horizon and depends only on its surface area A and its charges, electrical one Q, and magnetic one P. The second part is an integral from the horizon to the infinity, which vanishes, if the true dynamical variables take on the "vacuum" value.

Many of the derived properties are, of course, valid only in the spherically symmetric case; with certainty, it is the explicit reduction, and, may be, also the splitting of the Hamiltonian. Nevertheless, using these two properties, we can derive theorems, which will be plausible starting points and working hypotheses for the full-fledged four-dimensional case.

II. FIRST-ORDER ACTION

The two-dimensional analog of the (3+1)-splitting of the metric in ADM method[5] is given by

$$g_{ab}=\begin{bmatrix}-\alpha^2+\frac{\beta^2}{\gamma} & \beta\\ \beta & \gamma\end{bmatrix},\quad g^{ab}=\begin{bmatrix}-\frac{1}{\alpha^2} & \frac{\beta}{\alpha^2\gamma}\\ \frac{\beta}{\alpha^2\gamma} & \frac{1}{\gamma}-\frac{\beta^2}{\alpha^2\gamma^2}\end{bmatrix}.$$

Thus,

$$g=-\alpha^2\gamma\ ,$$

and the normal unit vector n^a to the Cauchy surfaces $t=$const reads

$$n^a=\left[\frac{1}{\alpha},-\frac{\beta}{\alpha\gamma}\right]. \tag{1}$$

The second fundamental form, K_{11}, of the Cauchy surfaces is determined by its trace part K,

$$K=\gamma^{-1}K_{11}\ ,$$

where

$$K=\frac{1}{\alpha\sqrt{\gamma}}\partial_a(\alpha\sqrt{\gamma}\,n^a)\ .$$

Denoting $\partial/\partial t$ by an overdot and $\partial/\partial x$ by a prime, we have

$$K=\frac{1}{2\alpha\gamma}\dot{\gamma}-\frac{\beta'}{\alpha\gamma}+\frac{\beta\gamma'}{2\alpha\gamma^2}\ .$$

A straightforward calculation yields the following expression of R by means of the Lagrange multipliers α, β and the two fundamental forms, γ, K, of the Cauchy surfaces:

$$R=2n^a\partial_aK+2K^2-\frac{2}{\alpha\sqrt{\gamma}}\left[\frac{\alpha'}{\sqrt{\gamma}}\right]'\ .$$

Consider the action given by Eq. (21) of I. The term $\frac{1}{2}\,|g|^{1/2}f\varphi^2R$ in it becomes

$$-\alpha\sqrt{\gamma}\,Kn^a\partial_a(f\varphi^2)+\frac{\alpha'}{\sqrt{\gamma}}(f\varphi^2)'+\text{derivatives}\ .$$

Thus, the action itself can be written as

$$I=\int_{t_i}^{t_f}dt\int_{b(t)}^{\infty}dx\left\{\frac{1}{2}\alpha\sqrt{\gamma}\left[\frac{1}{G}f-f(n^a\varphi_a)^2+\frac{f}{\gamma}(\varphi')^2-Kn^a\partial_a(f\varphi^2)+\frac{\alpha'}{\alpha\gamma}(f\varphi^2)'\right]\right.$$
$$\left.+\alpha\sqrt{\gamma}\left[h\,|n^aD_a\psi|^2-\frac{h}{\gamma}|D_1\psi|^2-V-\frac{1}{2}(n^a\partial_a\varphi^2)(n^af_a)+\frac{1}{2\gamma}(\varphi^2)'f'\right]+\frac{1}{2}\alpha\sqrt{\gamma}\,\varphi^2E^2\right\}. \tag{2}$$

Here, we write out explicitly the bounds of integration; $b(t)$ denotes either the x coordinates of an apparent horizon at the time t, if there is one, or the x coordinate of the regular center at the time t. The foliation by t-constant surfaces is, so far, arbitrary. The velocities are contained in the expressions $K, n^a\partial_aq$ for any quantity q and in E [see I, Eq. (24)]. The canonical momenta as obtained from (2) by variation with respect to the velocities are given by

$$\pi_\psi=-\tfrac{1}{2}\sqrt{\gamma}\,\varphi^2Kf_1\psi^\dagger+\sqrt{\gamma}\,h(n^aD_a\psi)^\dagger-\varphi\sqrt{\gamma}\,f_1\psi^\dagger n^a\varphi_a\ , \tag{3}$$

$$\pi_\varphi=-\sqrt{\gamma}\,fn^a\varphi_a-\sqrt{\gamma}\,Kf\varphi-\sqrt{\gamma}\,\varphi n^af_a\ , \tag{4}$$

$$\pi_\gamma=-\frac{1}{4\sqrt{\gamma}}n^a(f\varphi^2)_a\ , \tag{5}$$

$$\pi_A=\varphi^2E\ . \tag{6}$$

To calculate the Hamiltonian, we go over to the real representation of ψ:$\psi=\psi_1+i\psi_2$, so that we have five real fields in (2).

The Lagrange function in (2) has the form

$$L=L_{kl}(\dot{q}^k-q_0^k)(\dot{q}^l-q_0^l)-W(q)\ ,$$

where the independent entries of the symmetric 5×5 matrix L_{kl} are

$$L_{11}=L_{22}=\frac{\sqrt{\gamma}\,h}{\alpha},\quad L_{13}=-\frac{\sqrt{\gamma}\,\varphi f_1\psi_1}{\alpha},\quad L_{23}=-\frac{\sqrt{\gamma}\,\varphi f_1\psi_2}{\alpha},\quad L_{14}=-\frac{\varphi^2 f_1\psi_1}{4\alpha\sqrt{\gamma}},$$

$$L_{24}=-\frac{\varphi^2 f_1\psi_2}{4\alpha\sqrt{\gamma}},\quad L_{33}=-\frac{\sqrt{\gamma}f}{2\alpha},\quad L_{34}=-\frac{\varphi f}{4\alpha\sqrt{\gamma}},\quad L_{55}=\frac{\varphi^2}{2\alpha\sqrt{\gamma}},$$

and where

$$q^1=\psi_1,\quad q^2=\psi_2,\quad q^3=\varphi,\quad q^4=\gamma,\quad q^5=A_1,$$

$$W=\frac{\alpha h}{\sqrt{\gamma}}|D_1\psi|^2+\alpha\sqrt{\gamma}\,V-\frac{\alpha\sqrt{\gamma}f}{2G}-\frac{\alpha f\varphi'^2}{2\sqrt{\gamma}}-\frac{\alpha'(f\varphi^2)'}{2\sqrt{\gamma}}-\frac{\alpha\varphi\varphi' f'}{\sqrt{\gamma}},$$

$$f_1(|\psi|^2)=\left.\frac{df(y)}{dy}\right|_{y=|\psi|^2}$$

and

$$q_0^1=eA_0\psi_2+\frac{\beta}{\gamma}(D_1\psi),\quad q_0^2=-eA_0\psi_1+\frac{\beta}{\gamma}(D_1\psi)_2,\quad q_0^3=\frac{\beta}{\gamma}\varphi',\quad q_0^4=2\beta'-\frac{\beta}{\gamma}\gamma',\quad q_0^5=A_0',$$

$$(D_a\psi)_1=\partial_a\psi_1-eA_a\psi_2,\quad (D_a\psi)_2=\partial_a\psi_2+eA_a\psi_1.$$

The Hamiltonian $\bar{H}$, as defined by

$$\bar{H}=\int dx(p_k\dot{q}^k-L),$$

is given by the formula

$$\bar{H}=\int dx(\tfrac{1}{4}L^{kl}p_kp_l+q_0^kp_k+W),$$

where L^{kl} is the inverse to L_{kl}. The matrix L^{kl} is readily obtained, if we first calculate the inverse to a similar matrix, $(S^{-1}LS)_{kl}$, where the nonzero entries of the orthogonal matrix S are given by

$$S_{11}=S_{22}=\frac{\psi_2}{(\psi_1{}^2+\psi_2{}^2)^{1/2}},\quad S_{12}=-S_{21}=\frac{\psi_1}{(\psi_1{}^2+\psi_2{}^2)^{1/2}},\quad S_{33}=S_{44}=S_{55}=1.$$

The result is

$$\bar{H}=\int dx(\alpha\mathscr{H}_0+\beta\mathscr{H}_1+A_0\mathscr{H}_2), \tag{7}$$

where the super-Hamiltonians $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_2$ are given by

$$\begin{aligned}\mathscr{H}_0=&\frac{f}{2\sqrt{\gamma}|\psi|^2\Delta}(\psi\pi_\psi+\psi^\dagger\pi_\psi^\dagger)^2-\frac{1}{4h\sqrt{\gamma}|\psi|^2}(\psi\pi_\psi-\psi^\dagger\pi_\psi^\dagger)^2-\frac{\varphi f_1}{\sqrt{\gamma}\Delta}\pi_\varphi(\psi\pi_\psi+\psi^\dagger\pi_\psi^\dagger)\\&-\frac{2\sqrt{\gamma}f_1}{\Delta}\pi_\gamma(\psi\pi_\psi+\psi^\dagger\pi_\psi^\dagger)+\frac{\varphi^2f_1{}^2|\psi|^2}{2f\sqrt{\gamma}\Delta}\pi_\varphi{}^2-\frac{4\sqrt{\gamma}}{\varphi f\Delta}(hf+\varphi^2f_1{}^2|\psi|^2)\pi_\varphi\pi_\gamma+\frac{4\gamma^{3/2}}{\varphi^2f\Delta}(hf+2\varphi^2f_1{}^2|\psi|^2)\pi_\gamma{}^2\\&+\frac{\sqrt{\gamma}}{2\varphi^2}\pi_A{}^2+\sqrt{\gamma}\,V-\frac{\sqrt{\gamma}f}{2G}-\frac{f\varphi'^2}{2\sqrt{\gamma}}+\left[\frac{(f\varphi^2)'}{2\sqrt{\gamma}}\right]'-\frac{\varphi\varphi'f'}{\sqrt{\gamma}}+\frac{h}{\sqrt{\gamma}}|D_1\psi|^2,\end{aligned} \tag{8}$$

$$\mathscr{H}_1=\frac{1}{\gamma}\pi_\psi D_1\psi+\frac{1}{\gamma}\pi_\psi^\dagger(D_1\psi)^\dagger+\frac{1}{\gamma}\varphi'\pi_\varphi-2\pi_\gamma'-\frac{1}{\gamma}\pi_\gamma\gamma', \tag{9}$$

$$\mathscr{H}_2=-ie(\psi\pi_\psi-\psi^\dagger\pi_\psi^\dagger)-\pi_A, \tag{10}$$

and

$$\Delta=2hf+3\varphi^2f_1{}^2|\psi|^2.$$

The secondary constraints read $\mathscr{H}_0=\mathscr{H}_1=\mathscr{H}_2=0$, and there will be no more constraints.[9]

III. THE SURFACE TERMS

The set of equations that are obtained by variation of the first-order action

$$I=\int_{t_i}^{t_f}dt\left[\int_{b(t)}^{\infty}dx(\dot{\psi}\pi_\psi+\dot{\psi}^\dagger\pi_\psi^\dagger+\dot{\varphi}\pi_\varphi+\dot{\gamma}\pi_\gamma+\dot{A}_1\pi_A)-\bar{H}\right]$$

contains, as it is usual in gauge theories, not only the field equations, i.e., here the Einstein-Maxwell-scalar equations, but also some boundary equations at $x=b$ (center or horizon) and $x=\infty$ (i^0). The latter are due to the total x-derivative terms which appear during the variation of I. In order to remove these disturbing equations we use the method invented by Regge and Teitelboim.[8] First, we have to vary $\overline{H}$ and to find the form of the surface term $(\delta\overline{H})_s$; second, $\overline{H}$ must be corrected by a surface term H_s, whose variation cancels $(\delta\overline{H})_s$: $\delta(H_s)=-(\delta\overline{H})_s$. A straightforward calculation yields

$$(\delta\overline{H})_s=\left[\frac{\alpha h}{\sqrt{\gamma}}(D_1\psi)^\dagger\delta\psi+\frac{\alpha h}{\sqrt{\gamma}}(D_1\psi)\delta\psi^\dagger+\alpha\delta\left[\frac{(f\varphi^2)'}{2\sqrt{\gamma}}\right]-\frac{\alpha'}{2\sqrt{\gamma}}\delta(f\varphi^2)-\frac{\alpha\varphi\varphi'\delta f}{\sqrt{\gamma}}-\frac{\alpha(f\varphi)'}{\sqrt{\gamma}}\delta\varphi\right.$$
$$\left.+\frac{\beta}{\gamma}\pi_\psi\delta\psi+\frac{\beta}{\gamma}\pi_\psi{}^\dagger\delta\psi^\dagger+\frac{\beta}{\gamma}\pi_\varphi\delta\varphi-2\beta\delta\pi_\gamma-\frac{\beta}{\gamma}\pi_\gamma\delta\gamma-A_0\delta\pi_A\right]_{b(t)}^{\infty}. \tag{11}$$

The boundary conditions at $b(t)$ and at ∞ determine, which of the terms in (11) are zero, and influence the form of the rest. Consider the different pieces of the boundary.

Let $x=b(t)$ be a regular center. Then it follows from the equation $\varphi(b(t),t)=0$, from the regularity of the metric g_{ab}, of the potential A_a, and of the scalar field ψ, and from relations (3)—(6) [$h=\varphi^2$ in cases (a) and (c), Eq. (70) of I holds in case (b)]:

$$\pi_\varphi=\pi_\psi=\pi_\gamma=\pi_A=0 \tag{12}$$

at $x=b(t)$. The boundary terms for $x=b(t)$ in (11) must, therefore, vanish: either is some finite expression multiplied by a power of φ or of a momentum, or we have a variation of φ or of a momentum, which also is zero at the center. Thus, no correction at the center is needed.

Let $x=b(t)$ be an apparent horizon. Then, the form of (11) suggests that we choose the boundary gauge at $x=b(t)$ as

$$\alpha|_{b(t)}=0,\quad \beta|_{b(t)}=0 \tag{13}$$

(see Ref. 10). This means that the time is standing still at the apparent horizons, so the final (initial) Cauchy surface, $t=t_f(t=t_i)$, intersects all future (past) apparent horizons [see the relation (56) and the discussion that follows it in I]. However, the variations of the fields are all zero on these surfaces, in particular

$$\delta(\varphi^2)|_{b(t)}=0,\ \delta\pi_A|_{b(t)}=0\ ,$$

and the whole $b(t)$ part of (11) again vanishes.

The boundary gauge (13) leads to a particular foliation of the spacetime. Only a totally hyperbolic part of it is covered and those apparent horizons which move in a timelike way remain outside of it. This leads to a loss of information as described in the Introduction.

Finally, at i^0, the boundary conditions given by Eqs. (61)—(64) of I lead to

$$(\delta\overline{H})_s=\lim_{x\to\infty}\frac{\varphi f}{\sqrt{G}}\delta\left[\frac{1}{\sqrt{\gamma}}\right]=\delta\lim_{x\to\infty}\left[\frac{f}{2G}x\left[\frac{1}{\gamma}-1\right]\right].$$

Hence, the corrected Hamiltonian H is given by

$$H=\overline{H}-\lim_{x\to\infty}\frac{f_\infty}{2G}x\left[\frac{1}{\gamma}-1\right]. \tag{14}$$

The Hamiltonian (14) is obtained from boundary conditions (13) as well as from Eqs. (61)—(64) of I. Any choice of gauge which one will subsequently do must, therefore, be in agreement with these boundary conditions.

IV. THE REDUCTION IN THE BCMN GAUGE

In Ref. 1, the Hamiltonian constraint could be explicitly solved for γ after the gauge $\varphi=x/\sqrt{G}$ and $\pi_\gamma=0$ had been chosen. The following is a very slight (but efficient) modification of it; we shall still call this the "*BCMN gauge*":

$$x=\left[\frac{Gf}{f_\infty}\right]^{1/2}\varphi,\quad \pi_\gamma=0,\quad A_1=0\ . \tag{15}$$

The geometrical meaning of the second equation is immediate from (5): the ($\pi_\gamma=0$) foliation is orthogonal to the ($f\varphi^2$) equipotentials. Our modification makes x constant along these equipotentials, so $\beta=0$ everywhere, and we cannot violate (13).

If we exclude φ, φ', π_γ, and A_1 with the help of (15), constraints (8)—(10) become

$$\mathscr{H}_0=-\frac{f_\infty}{2G}x\gamma^{-3/2}\gamma'-\frac{f_\infty}{2G}x\left[T'+\frac{f'}{f}-\frac{1}{x}\right]\gamma^{-1/2}-\frac{f_\infty}{2G}fF\gamma^{1/2}\ , \tag{16}$$

$$\mathscr{H}_1=\frac{1}{\gamma}\pi_\psi\psi'+\frac{1}{\gamma}\pi_\psi^\dagger\psi^{\dagger\prime}+\frac{1}{\gamma}\left[\frac{f_\infty}{Gf}\right]^{1/2}\left[1-\frac{x}{2}\frac{f'}{f}\right]\pi_\varphi\ , \tag{17}$$

$$\mathscr{H}_2=-ie(\psi\pi_\psi-\psi^\dagger\pi_\psi^\dagger)-\pi_A\ , \tag{18}$$

where we have introduced the abbreviations

$$T(x)=\frac{2G}{f_\infty}\int_x^\infty\frac{dy}{y}\left\{\frac{1}{2\Delta f\,|\psi|^2}\left[f(\psi\pi_\psi+\psi^\dagger\pi_\psi^\dagger)-\left[\frac{f_\infty}{2G}\right]^{1/2}yf_1\,|\psi|^2\pi_\varphi\right]^2-\frac{1}{4h\,|\psi|^2}(\psi\pi_\psi-\psi^\dagger\pi_\psi^\dagger)^2+h\,|\psi'|^2+\frac{3f_\infty}{8G}y^2\frac{f'^2}{f^2}\right\}, \tag{19}$$

$$F(x)=\frac{2G}{f_\infty}\left[\frac{1}{2G}-\frac{V}{f}-\frac{G}{2f_\infty}\frac{{\pi_A}^2}{x^2}\right]. \tag{20}$$

Observe that $T(x)$ is a non-negative, nonincreasing function with $T(\infty)=0$. The equation $\mathscr{H}_2=0$ has the solution

$$\pi_A(x)=Q-ie\int_b^x dy(\psi\pi_\psi-\psi^\dagger\pi_\psi^\dagger)\ . \tag{21}$$

The choice of the integration constant corresponds to the definition of Q [Eq. (57) of I] and to that of π_A (6). The equation $\mathscr{H}_1=0$ yields

$$\pi_\varphi=\left[\frac{Gf}{f_\infty}\right]^{1/2}\frac{1}{1-(x/2)f'/f}(\pi_\psi\psi'+\pi_\psi^\dagger\psi^{\dagger\prime})\ . \tag{22}$$

The equation $\mathscr{H}_0=0$ is an ordinary differential equation for γ of Bernoulli's type; we can write it in the form

$$\left[\frac{x}{f\gamma}\right]'-T'\frac{x}{f\gamma}=F\ ,$$

and its general solution is given by (we suppress the dependence on t)

$$\frac{1}{\gamma(x)}=\frac{f(x)}{x}\int_{x_1}^x dy\,F(y)e^{T(x)-T(y)}\ . \tag{23}$$

We have to determine the constant x_1. However, in the BCMN gauge, we have

$$g^{ab}(\sqrt{f}\,\varphi)_a(\sqrt{f}\,\varphi)_b=\frac{f_\infty}{G\gamma}\ . \tag{24}$$

Thus, at an *apparent horizon,* $1/\gamma=0$ [e.g., (55) in I] which leads to $x_1=b$, and we obtain

$$\frac{1}{\gamma(x)}=\frac{f(x)}{x}\int_b^x dy\,F(y)e^{T(x)-T(y)}\ . \tag{25}$$

At a *regular center,* we have from (24)

$$\frac{f_\infty}{G\gamma}=f_0g^{ab}\varphi_a\varphi_b\ .$$

This, together with Eq. (65) of I, implies

$$\left.\frac{1}{\gamma}\right|_{\varphi=0}=\frac{f_0}{f_\infty}\ , \tag{26}$$

where

$$f_0=\lim_{\varphi\to 0}f\ .$$

The constant x_1 must be specified so that we obtain the value f_0/f_∞ for $x=0$. If this should work at all, then $x_1=0=b$ again. Let us calculate the corresponding limit. Consider the function $T(x)$ near $x=0$. From

$$n^a(\sqrt{f}\,\varphi)_a=0\ ,$$

we obtain

$$n^a\varphi_a=-\tfrac{1}{2}\frac{\phi}{f}n^af_a\ .$$

The relations (3)–(6) and Eq. (70) of I then yield

$$\pi_\psi\sim x^2,\ \ \pi_A\sim x^2,\ \ \pi_\varphi\sim x\ . \tag{27}$$

The boundary condition which is given by Eq. (67) of I implies

$$f'\sim x\ .$$

Then, formula (19) shows that $T(x)$ is regular at $x=0$, and

$$\lim_{x\to 0}\frac{1}{\gamma(x)}=\frac{2G}{f_\infty}\lim_{x\to 0}\frac{f(x)}{x}\int_0^x dy\left[\frac{1}{2G}-\frac{V}{f}-\frac{G}{2f_\infty}\frac{{\pi_A}^2}{x^2}\right],$$

as $0<y<x$, we have $T(x)-T(y)\to 0$. We can see from formulas (22) and (42) of I and the above relation (27) that

$$\frac{1}{\gamma(0)}=\frac{f_0}{f_\infty}\ ,$$

as it should be. Hence, (25) again is valid.

The Lagrange multipliers α, β, A_0 are determined from the condition that the BCMN gauge holds for all times:

$$\dot\varphi=\frac{\delta H}{\delta\pi_\varphi},\ \ \dot\pi_\gamma=-\frac{\delta H}{\delta\gamma},\ \ \dot A_1=\frac{\delta H}{\delta\pi_A}\ .$$

Using (13), (14), (15), (3), and (4), we obtain again

$$n^a(f\varphi^2)_a=0\ ,$$

which is equivalent to [see (1)]

$$\beta=0\ . \tag{28}$$

The second equation yields, together with (7), (16), and (27)

$$2\frac{\alpha'}{\alpha}+T'+\frac{f'}{f}+\frac{1}{x}-\frac{fF\gamma}{x}=0\ ,$$

whereas the equation for γ is equivalent to

$$-\frac{\gamma'}{\gamma}-T'-\frac{f'}{f}+\frac{1}{x}-\frac{fF\gamma}{x}=0 .$$

Subtracting the two equations we obtain, as a general solution

$$\alpha(x)=\frac{C}{f[\gamma(x)]^{1/2}}e^{-T(x)} ,$$

where C is some constant. For any choice of C, we have at an apparent horizon

$$\alpha=0 ,$$

because $T(x)$ is regular and $1/\gamma=0$. Thus again, our gauge is compatible with (13). However, in order to determine the constant C, we have now to investigate the behavior of α at $x=\infty$. From the above formulas we obtain for $\alpha(\infty)$

$$\alpha(\infty)=\frac{C}{f_\infty[\gamma(\infty)]^{1/2}} .$$

It follows that $C=f_\infty$, because we should have

$$\gamma^{(\infty)}=1,\quad \alpha(\infty)=1$$

[see relation (62) of I]. Hence,

$$\alpha(x)=\frac{f_\infty}{f[\gamma(x)]^{1/2}}e^{-T(x)} . \tag{29}$$

To verify that $\gamma(\infty)=1$, we write

$$\frac{1}{\gamma(\infty)}=\lim_{x\to\infty}\frac{f(x)}{x}\int_b^x dy\, F(y)e^{-T(y)}$$
$$=\lim_{x\to\infty}\frac{f(x)}{x}\int_b^x dy\frac{1}{f_\infty}+\lim_{x\to\infty}\frac{f(x)}{x}\int_b^x dy\left[\frac{1}{f_\infty}(1-e^{T(y)})-\frac{2G}{f_\infty}\frac{V}{f}-\frac{G^2}{{f_\infty}^2}\frac{{\pi_A}^2}{x^2}\right]e^{-T(y)} .$$

The integral in the second term converges, so indeed, $\gamma(\infty)=1$.

Finally, $\delta H/\delta\pi_A=0$ leads to the following equation for A_0:

$$A_0'+\frac{G}{f_\infty}\frac{\alpha\sqrt{\gamma}f\pi_A}{x^2} .$$

Equations (21), (29), and the boundary condition (64) of I then imply

$$A_0(x)=\int_x^\infty\frac{dy}{y^2}e^{-T(y)}[Q-ie\int_b^y dz(\psi\pi_\psi-\psi^\dagger\pi_\psi^\dagger)] . \tag{30}$$

Equations (21), (22), (25), (15), (28), (29), and (30) express the quantities π_A, π_φ, γ, φ, π_γ, A_1, α, β, and A_0 as functionals of ψ and π_ψ. These are the only true dynamical variables in the system.

To finish the reduction, we have to replace all dependent and gauge variables in the Hamiltonian (14) by the corresponding expressions in ψ and π_ψ. To this aim, let us introduce the function $F_0(x)$ by

$$F_0(x)=1-\frac{1}{f_\infty}\frac{G^2(Q^2+P^2)}{x^2} . \tag{31}$$

Here, Q and P is the electric and magnetic charge, respectively, of an apparent horizon or a regular center [in the latter case, $Q=P=0$, see the boundary condition (68) of I]. We have

$$\int_b^x dy\, F_0(y)=x-2GM+\frac{1}{f_\infty}\frac{G^2(Q^2+P^2)}{x} ,$$

where M is given by formula (60) in I. Indeed,

$$M=\frac{1}{2G}\left[b+\frac{1}{f_\infty}\frac{G^2(Q^2+P^2)}{b}\right] ,$$

and Eqs. (15), together with the definition of A given by the relation (57) of I, implies

$$b=\left[\frac{f_b}{f_\infty}\right]^{1/2}\left[\frac{A}{4\pi}\right]^{1/2} .$$

Using this and (25), we can write

$$\frac{x}{\gamma}-x=f(x)\int_b^x dy\, F(y)e^{T(x)-T(y)}-x$$
$$=f(x)e^{T(x)}\int_b^x dy\left[F(y)e^{-T(y)}-\frac{1}{f_\infty}F_0(y)\right]+\frac{f(x)}{f_\infty}e^{T(x)}\int_b^\infty dy\, F_0(y)$$
$$=x\left[\frac{f(x)e^{T(x)}}{f_\infty}-1\right]+\frac{f(x)e^{T(x)}}{{f_\infty}^2}\frac{G^2(Q^2+P^2)}{x}-2GM\frac{f(x)e^{T(x)}}{f_\infty}-\frac{f(x)e^{T(x)}}{f_\infty}\int_b^x dy[F_0(y)-f_\infty F(y)e^{-T(y)}] .$$

The formula (19) together with the asymptotic properties of ψ and π_ψ shows that

$$\lim_{x\to\infty}x^2T'=\lim_{x\to\infty}x^2f'=0 .$$

Then, the first term vanishes in the limit $x\to\infty$. The second term vanishes obviously. Hence, using the above relation in (14), we obtain, finally

$$H = f_\infty M + \frac{f_\infty}{2G} \int_b^\infty dy [F_0(y) - f_\infty F(y) e^{-T(y)}] \; . \quad (32)$$

One can check that the canonical equations of the Hamiltonian (32) coincide with the equations of motion for ψ and π_ψ which we would obtain, if we expressed the gauge variables, the dependent variables, and the Lagrange multipliers by means of ψ and π_ψ in Eqs. (27)—(30) of I.

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