Spherically symmetric systems of fields and black holes. I. Definition and properties of apparent horizon

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We investigate three model field theories: a minimally coupled charged scalar field together with gravity and electromagnetism, a minimally coupled SO(3) Yang-Mills field and gravity, and the Callan-Coleman-Jackiw scalar field. We restrict ourselves to spherically symmetric configurations; the corresponding dimensional reduction leads to an action functional on a two-dimensional spacetime which contains a metric, a neutral scalar, a charged scalar, and an electromagnetic field. The action is written in the second-order, covariant and gauge-invariant form. We generalize the definition of the future and past apparent horizon so that it will not be visible from the future and past null infinity, respectively, and will form a nontimelike surface, both also in the case of the Callan-Coleman-Jackiw model. We prove an inequality relating the surface area and the charges of the apparent horizon. We study the boundary conditions for the fields at the horizon, at the regular center, and at infinity. Finally, we speculate on the existence of static spherically symmetric solutions, where a black hole is surrounded by a matter shell; in two-dimensional spacetime, this looks like a kink.

I. INTRODUCTION

An outstanding problem in quantum gravity concerns the properties and a possible role of black holes. Already the theory of linear fields on the classical black-hole backgrounds gives very interesting results: the Hawking effect,[1] super-radiance,[2] solitonic character of some holes,[3] and many more. It is, however, necessary to surpass the linearized theory on a fixed background in order to really understand these effects.

If we attempt to quantize gravity and, in particular, to include strong fields and black holes, then it need not be just the theory of gravity which will change: the quantum theory as such could be modified as well. Quite concrete hints in this direction have been discussed in the literature.[4]

There is still another speculation which motivates the study of quantum properties of black holes. Consider two arbitrary particles with center-of-mass energy of order E scattering off each other. The chance that they come as close to each other as to find themselves under their common gravitational radius $R=GE$ is very small for $E \ll G^{-1/2}$. Indeed, the particles will be localizable within the radius $\sim E^{-1}$, which is much larger than R. However, if $E \gtrsim G^{-1/2}$, then the particles are localizable under R, and to get them there, one must just aim them properly. Hence, the cross section σ_c for the collapse will be

$$\sigma_c \sim 4\pi R^2 \sim 4\pi G^2 E^2 .$$

Should one be inclined to accept such a crude argument, then one would agree that the black-hole production could even dominate all very-high-energy processes, and maybe provide, in such a way, an effective cutoff at the Planck energy (e.g., Ref. 5).

There are more attempts to quantize black holes: some try to consider them as a sort of instanton,[6] others as a sort of soliton.[3] Our solitonic approach has been based on the perturbation expansion of the general quantum theory of solitons (e.g., Ref. 7) and we were able to calculate the first few terms in the series corresponding to the powers $(1/G, 1/\sqrt{G}, 1)$ of the Planck length $\sqrt{G}$.[8] Within this accuracy, the theory is practically equivalent to the linearized quantum field theory on the soliton background, so we could use the wealth of results obtained recently in this field. In particular, we have derived the Schrödinger equation for the motion of free black holes.

If we attempt to proceed in the perturbation expansion, then several problems appear. The first one we mention is present even within the $(\sqrt{G})^0$ order: it is the quantum instability of the black-hole solutions.[3] As yet, we have avoided this difficulty by dealing with those field-theoretical models which possess quantum stable black-hole solutions.

However, one does not need a stable "black-hole particle" in order that, for example, the black-hole production at very high energies works (with unstable black holes, there will be less pollution). It is conceivable that those classically stable, static, localized solutions to the Lorentzian field equations that are not quantum stable could play a role of unstable particles in the quantum theory and represent some intermediate form between solitons and instantons (e.g., Ref. 9). Then, of course, the technical point of the quantum soliton methods, the calculation of soliton-soliton and soliton-meson scattering amplitudes,[7] would not make sense for such particles, because the corresponding solutions would not appear in the asymptotic expansion of the fields.[10]

The second problem is the nonrenormalizability of the $\sqrt{G}$-power expansion of quantum gravity: One cannot calculate loop corrections in higher orders of the expansion. Here, one can try some more finite model, invoking, e.g., supergravity. Or, as in this paper, one can go over to two-dimensional models, and hope that one will succeed in an exact construction of the quantum fields.

The third problem is a calculational one: the higher-order expressions become very messy. We attempt to meet this difficulty by reducing the spacetime dimension of the model.

The fourth problem is the singularity or, more generally, the nontrivial causal structure of spacetimes with black holes. The solution of this problem within the accuracy $(1/G, 1/\sqrt{G}, 1)$ was based on the simple fact that this nontrivial structure was rigid, given by the background spacetime. We have cut away the singular part of the background in such a way that the rest (1) was globally hyperbolic and (2) contained all information we needed for the scattering theory—indeed, it was $J^+(\mathscr{I}^-)\cap J^-(\mathscr{I}^+)$. At the boundary (the surface of the hole), some boundary conditions on all fields have been imposed; these conditions are natural in the sense that they do nothing but to guarantee that the hole is a black hole of a certain type. In this way also the black-hole parameters appear in the calculations. We have studied the admissibility of the boundary conditions in Ref. 11.

In higher orders of the expansion the problem, however, reappears in a more severe form. We have obtained first hints in this direction, when we constructed the whole perturbation series for a two-dimensional model.[12] We have managed to expand in powers of a small, dimensionless parameter, to bring the interaction to a local, polynomial form, and to give the full formula for the S matrix, including the propagator and Feynman graphs. We have obtained a general formula for the superscattering operator and performed explicitly the sum over excited states of the hole in it. However, it seemed that the superscattering operator factorizes after the perturbation series is summed up. This strange result can be due to a false interpretation. Indeed, the *exact* classical theory leads, e.g., to black-hole and singularity formation resulting from regular data; the spacetime, which itself is a result of the dynamics, need not, as a whole, be then globally hyperbolic. A cut would again help, but this time its position had to be determined by the dynamics. This can be done: there are gauge conditions which lead automatically to a foliation by Cauchy surfaces of a part of the spacetime. The corresponding canonical quantization guarantees the quantum coherence (pure states develop into pure states). In Ref. 12, the cut has been performed in a way *independent* of the actual dynamics, whereas the foliation has been governed *by* the dynamics through a gauge condition.

To repair this defect, we could try to cut the spacetime along the actual future and past event horizon (for the definition, see Ref. 13). However, the position of, e.g., the future event horizon at a time t is only determined, if the whole development from t to $+\infty$ is known.[13] Such a cut would, therefore, lead to a badly nonlocal theory. Another possibility is to cut along the *apparent horizon* (defined in Ref. 13); this can be determined locally, at any Cauchy surface, from the Cauchy data. Recently, apparent horizons turned out to be natural surfaces, at which simple boundary conditions can be imposed so that a generalization of Witten's proof of positivity of energy[14] to a situation with black holes is possible.[15] The main idea in the present series of papers is to use the apparent horizon in the role of the black-hole boundary and to cut the inside of it away. We shall see, in the canonical formalism part (next paper of the series), how the gauge condition which determines the foliation and the cut along the apparent horizon are compatible.

The analysis of the problems listed above suggests that we should try to work first with some simplified models and then apply the results as working hypotheses for the full complicated case. In 1976, Unruh proposed to use the so-called Berger-Chitre-Moncrief-Nutku (BCMN) model[17] for the study of quantum black holes. This is a field system on a two-dimensional spacetime, whose field equations are identical with the four-dimensional Einstein-Maxwell-Klein-Gordon equations for spherically symmetric configurations. (The trick is analogous to the recent one used by Rubakov[18] to calculate the fermion number breaking in the field of magnetic monopole—the full effect is present already in the s mode.) Berger, Chitre, Moncrief, and Nutku have observed that the model can be totally reduced in a particular gauge.[17] However, only after a careful study of boundary conditions at infinity by Unruh was the proper Hamiltonian of the model obtained.[16] We are going to extend this study to the other boundary, namely, to the boundary of a black hole.

In this paper, we perform the dimensional reduction from four dimensions to the BCMN model in a covariant way and we obtain a covariant, second-order action. The dimensional reduction studied here is rather similar to the reduction techniques employed in the Kaluza-Klein theories (e.g., Refs. 19), the only difference is that we retain only the s mode from the corresponding harmonics expansion. This form of the model can be a starting point of other quantization methods than canonical, and it is also useful in deriving the properties of apparent horizons. In Refs. 15 and 16 one dealt only with the Arnowitt-Deser-Misner form.

We will reduce three quite different four-dimensional field-theoretical models: (a) the system consisting of gravity $g_{\mu\nu}$, electromagnetic field A_μ, and minimally coupled scalar field ψ (this is the original BCMN system); (b) gravity $g_{\mu\nu}$ and SO(3)-Yang-Mills field A^a_μ; (c) gravity $g_{\mu\nu}$ and scalar ψ in the so-called Callan-Coleman-Jackiw (CCJ) coupling.[20] In case (b), the dimensional reduction by Witten's ansatz[21] leads to a magnetically and electrically charged scalar field in two dimensions which is coupled to gravity in a more mild way than in case (a) (fewer derivatives in the coupling). Similarly, in case (c), the hope is to obtain a less divergent quantum theory. Another point which has lead us to use these three different models was to show that the apparent horizon idea can work quite generally. In this respect, the CCJ model is particularly nontrivial.

Indeed, the improved energy-momentum tensor,[20] which (up to a factor) becomes the source of gravity here, does not satisfy the weak energy condition:[13] the "usual" tensor is "improved" by subtracting a total divergence so that the total energy and momentum are still conserved and the total energy is still positive, but the energy density can be negative at places. However, the weak energy condition is indispensable to show that the apparent horizon is not visible from $\mathscr{I}^+$, as well as that its trajectory is nontimelike:[13] both are properties which we will need. We solve this problem by modifying Hawking's definition of the apparent horizon for the CCJ case. We prove that the general properties of an apparent horizon are preserved by our modification.

The plan of the paper is as follows. In Sec. II we calculate the dimensional reduction of the Hilbert-Einstein action. In Secs. III–V we reduce the matter Lagrangians for cases (a), (b), and (c), respectively, and find a form of the two-dimensional action that is valid for all three cases. In Sec. VI we transform the field equations into the double null coordinates, give the general definition of future and past apparent horizon, show the invisibility of them from $\mathscr{I}^+$ and $\mathscr{I}^-$, respectively, prove an important inequality relating the surface area A, the electric Q and the magnetic charge P of an apparent horizon, and, finally, derive some boundary conditions for the fields at the apparent horizon. In Sec. VII, we study the fields near i^0 and at the regular center, and review the boundary conditions there. In Sec. VIII, we summarize the well-known classical static solutions to the field equations and speculate about the existence of some other solutions.

II. THE GRAVITY PART OF THE ACTION

Consider a system of fields in a four-dimensional spacetime containing gravity. As we want to perform a dimensional reduction to two dimensions and retain only the s modes there must always be some matter fields, or else the dynamics would be trivial. Let the action have the form

$$\tilde{I}=\tilde{I}_g+\tilde{I}_m \ , \tag{1}$$

where

$$\tilde{I}_g=\frac{1}{16\pi G}\int d^4x\,|\tilde{g}|^{1/2}\tilde{R} \tag{2}$$

is the Hilbert action, $\tilde{g}_{\mu\nu}$ the metric of the four-dimensional spacetime of signature $+2$,

$$\tilde{g}=\mathrm{Det}(\tilde{g}_{\mu\nu}) \ ,$$

$\tilde{R}$ is the curvature scalar of $g_{\mu\nu}$ (our conventions

$$\tilde{R}^{\rho}{}_{\mu\nu\sigma}=\tilde{\Gamma}^{\rho}_{\mu\sigma,\nu}-\cdots \ ,$$

$$\tilde{R}_{\mu\nu}=\tilde{R}^{\rho}{}_{\mu\rho\nu}) \ ,$$

and $\tilde{I}_m$ is some not-yet-specified matter action $\tilde{I}_m=\tilde{I}_m(g_{\mu\nu},\varphi^A)$, where φ^A are the matter fields.

Varying $\tilde{I}$, we obtain the field equations

$$\tilde{G}_{\mu\nu}=8\pi G\tilde{T}_{\mu\nu},\quad \frac{\delta\tilde{I}}{\delta\varphi^A}=0 \ , \tag{3}$$

where

$$\tilde{G}_{\mu\nu}=\tilde{R}_{\mu\nu}-\tfrac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}$$

is the Einstein tensor and

$$\tilde{T}^{\mu\nu}=\frac{2}{|\tilde{g}|^{1/2}}\frac{\delta\tilde{I}_m}{\delta\tilde{g}_{\mu\nu}} \tag{4}$$

is the energy-momentum tensor of the matter. The most general spherically symmetric ansatz for the metric is

$$ds^2=g_{ab}dx^a dx^b+G\varphi^2(d\theta^2+\sin^2\theta\,d\phi^2) \ , \tag{5}$$

where x^0, x^1, θ, ϕ are some coordinates adapted to the spherical symmetry, $a,b=0,1$, $g_{ab}(x^0,x^1)$ is a metric on the surface $\theta=0$, $\phi=0$, and $\varphi(x^0,x^1)$ is the $1/\sqrt{G}$ times radius of the rotation group orbit through the point $(x^0,x^1,0,0)$. The rescaling by $1/\sqrt{G}$ yields a dimensionless field φ in two dimensions, which is in agreement with the status of bosons there. A simple calculation yields

$$\tilde{G}_{ab}=-2\varphi^{-1}\nabla_a\nabla_b\varphi+2g_{ab}\varphi^{-1}\nabla_c\nabla^c\varphi+g_{ab}\varphi^{-2}\varphi_c\varphi^c$$
$$-g_{ab}\frac{1}{G\varphi^2} \ , \tag{6}$$

$$\tilde{G}_{\theta\theta}=G\varphi\nabla_c\nabla^c\varphi-\tfrac{1}{2}G\varphi^2R \ , \tag{7}$$

where ∇_a is the covariant derivative corresponding to g_{ab} and R is the curvature scalar of g_{ab}. We abbreviate the simple derivative $\partial\varphi/\partial x^a$ as φ_a. The tensors (6) and (7) can be obtained as variations with respect to g_{ab} and φ of the two-dimensional action I_g defined by

$$I_g=\tfrac{1}{2}\int d^2x\,|g|^{1/2}\left[\frac{1}{G}+g^{ab}\varphi_a\varphi_b+\tfrac{1}{2}R\varphi^2\right] . \tag{8}$$

We have

$$\frac{\delta I_g}{\delta g_{ab}}=-\tfrac{1}{4}|g|^{1/2}\varphi^2\tilde{G}_{ab},\quad \frac{\delta I_g}{\delta\varphi}=-\frac{|g|^{1/2}}{G\varphi}\tilde{G}_{\theta\theta} \ , \tag{9}$$

where

$$g=\mathrm{Det}(g_{ab}) \ .$$

It is a well-known property of the dimensional reduction (e.g., Ref. 22) that the lower-dimensional theory acquires a cosmological term; here, this term is the only point at which the Newton constant comes into the theory. It cannot be canceled by any true cosmological Λ term in four dimensions: this would rather lead to a mass $\sqrt{\Lambda G}$, for the field φ in two dimensions. It is also amusing to notice that the action (8) can be directly obtained from (2) by the ansatz (5).

As for the matter Lagrangian $\tilde{I}_m$, we consider the following three different choices.

(a) Scalar electrodynamics with minimal coupling to gravity. This is the original BCMN model;[17] we add some mass and self-interaction, because we are also interested in the interaction of the black hole with solitonic shells.

(b) Yang-Mills field minimally coupled to gravity. In two dimensions, Yang-Mills, electromagnetic, and scalar fields can appear[23] whose Lagrangian has more symmetry and whose field equations are simpler in comparison with (a). We limit ourselves to the simplest possible case, SO(3), and use Witten's ansatz.[21]

(c) A system of scalar fields with conformal coupling to gravity.[20] The matter Lagrangian also has more symmetry, and the divergences of the quantum perturbation theory can be milder in this case.

III. THE MINIMAL COUPLING

The four-dimensional action will be of the form

$$\tilde{I}_m=-\frac{1}{4\pi}\int d^4x\,|\tilde{g}|^{1/2}[\tilde{g}^{\mu\nu}(\tilde{D}_\mu\tilde{\psi})^\dagger\tilde{D}_\nu\tilde{\psi}+\tilde{V}(|\tilde{\psi}|^2)]-\frac{1}{16\pi\tilde{e}^2}\int d^4x\,|\tilde{g}|^{1/2}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}\,. \tag{10}$$

Here, $\tilde{\psi}$ is a complex scalar field,

$$\tilde{D}_\mu\tilde{\psi}=\partial_\mu\tilde{\psi}+i\tilde{A}_\mu\tilde{\psi}$$

is the covariant derivative, $\tilde{A}_\mu$ is the electromagnetic potential, $\tilde{F}_{\mu\nu}$ is the corresponding field strength, $\tilde{e}$ is the electric charge of $\tilde{\psi}$, and $\tilde{V}(|\tilde{\psi}|^2)$ is a function, which is assumed to be bounded from below by zero:

$$\tilde{V}(x)\geq 0\quad\forall x\,. \tag{11}$$

Varying $\tilde{I}_m$, we obtain

$$\frac{1}{\tilde{e}}|\tilde{g}|^{-1/2}\partial_\nu(|\tilde{g}|^{1/2}\tilde{F}^{\mu\nu})=4\pi\tilde{J}^\mu\,, \tag{12}$$

$$|\tilde{g}|^{-1/2}\tilde{D}_\mu(|\tilde{g}|^{1/2}\tilde{g}^{\mu\nu}\tilde{D}_\nu\tilde{\psi})-\tilde{V}'\tilde{\psi}=0\,, \tag{13}$$

$$\tilde{T}^{\psi}_{\mu\nu}=\frac{1}{4\pi}\{(\tilde{D}_\mu\tilde{\psi})^\dagger\tilde{D}_\nu\tilde{\psi}+(\tilde{D}_\nu\tilde{\psi})^\dagger\tilde{D}_\mu\tilde{\psi}-\tilde{g}_{\mu\nu}[\tilde{g}^{\rho\sigma}(\tilde{D}_\rho\tilde{\psi})^\dagger\tilde{D}_\sigma\tilde{\psi}+\tilde{V}]\}\,, \tag{14}$$

$$\tilde{T}^{\mathrm{EM}}_{\mu\nu}=\frac{1}{4\pi\tilde{e}^2}(\tilde{F}_{\mu\rho}\tilde{F}_\nu{}^\rho-\tfrac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma})\,, \tag{15}$$

$$\tilde{J}_\mu=\frac{i\tilde{e}}{4\pi}[\tilde{\psi}^\dagger\tilde{D}_\mu\tilde{\psi}-\tilde{\psi}(\tilde{D}_\mu\psi)^\dagger]\,. \tag{16}$$

The most general spherically symmetric ansatz for $\tilde{A}_\mu$ is

$$\tilde{A}^{N,S}_\mu dx^\mu=\frac{\tilde{e}}{\sqrt{G}}A_a dx^a+\tilde{P}(\pm 1-\cos\theta)d\phi\,, \tag{17}$$

where $\tilde{P}$ is the magnetic charge, which takes on only integer values in order that $\tilde{A}^N_\mu$ and $\tilde{A}^S_\mu$ can be smoothly joined by a gauge transformation on the equators of the $\phi=$const spheres. We rescale the potential so that it becomes dimensionless and so that the coupling constant appears in front of all terms of higher order; we introduce the charge e (in two dimensions, charge has dimension 1) by

$$e=\frac{\tilde{e}}{\sqrt{G}}\,. \tag{18}$$

The gauge has been fixed so that only the freedom

$$A_a\to A_a+\frac{1}{e}\Lambda_{,a}$$

remains, where $\Lambda=\Lambda(x^0,x^1)$.

If we now try the ansatz

$$\tilde{\psi}(x^0,x^1,\theta,\phi)=\frac{1}{\sqrt{G}}\psi(x^0,x^1)\,, \tag{19}$$

then we will not be able to satisfy Eqs. (12)—(16) because of the terms with $\tilde{D}_3\tilde{\psi}$:

$$\tilde{D}_2\tilde{\psi}=0,\quad \tilde{D}_3\tilde{\psi}=i\tilde{A}_3(\theta)\tilde{\psi}\,.$$

They will destroy the spherical symmetry, if $\tilde{e}\neq 0$ and $\tilde{P}\neq 0$ simultaneously. In fact, there is no spherically symmetric ansatz for an electrically charged scalar in the presence of a magnetic charge, for such a configuration will always have an angular momentum (e.g., Ref. 24; if ψ were a fermion, the situation would be different[18]). We can, however, decouple from the magnetic field by hand, redefining the covariant derivatives as

$$\tilde{D}_a\tilde{\psi}=\frac{1}{\sqrt{G}}(\partial_a\psi+ieA_a\psi)=\frac{1}{\sqrt{G}}D_a\psi\,,\qquad \tilde{D}_2\tilde{\psi}=0,\quad \tilde{D}_3\tilde{\psi}=0\,. \tag{20}$$

Setting the spherically symmetric ansatz (5), (17), (19), and (20) into (12), (13), (14), (15), and (16), we obtain a self-consistent system of equations for g_{ab}, φ, A_a, and ψ in two dimensions, which contains $e=0$ and/or $\tilde{P}=0$ as special cases (and it is these two special cases that have a reasonable four-dimensional interpretation). A lengthy but straightforward calculation shows that the same system of equations can be obtained by varying the following action in two dimensions:

$$I=\tfrac{1}{2}\int d^2x\,|g|^{1/2}f\left[\frac{1}{G}+g^{ab}\varphi_a\varphi_b+\tfrac{1}{2}R\varphi^2\right]-\tfrac{1}{4}\int d^2x\,|g|^{1/2}\varphi^2F_{ab}F^{ab}-\int d^2x\,|g|^{1/2}[h(D_a\psi)^\dagger(D^a\psi)+V-\tfrac{1}{2}(\varphi^2)_a f^a]\,, \tag{21}$$

where f, h, and V are the following functions:

$$f(|\psi|^2)=1,\quad h(\varphi)=\varphi^2\,,\qquad V(\varphi,|\psi|^2)=\varphi^2G\tilde{V}\left[\frac{|\psi|^2}{G}\right]+\frac{1}{2}\frac{P^2}{\varphi^2}\,, \tag{22}$$

and

$$P=\frac{\tilde{P}}{\tilde{e}\sqrt{G}}\,. \tag{23}$$

With the abbreviation

$$E=|g|^{-1/2}(\partial_0A_1-\partial_1A_0)\,, \tag{24}$$

which enables us to write

$$F_{ab}=\epsilon_{ab}|g|^{1/2}E \tag{25}$$

and

$$F_{ac}F^c_b-\tfrac{1}{4}g_{ab}F_{cd}F^{cd}=-\tfrac{1}{2}g_{ab}E^2\ , \tag{26}$$

the equations of motion can be brought to the form

$$-2\varphi^{-1}\nabla_a\nabla_b\varphi+2g_{ab}\varphi^{-1}\nabla_c\nabla^c\varphi+g_{ab}\varphi^{-2}\varphi_c\varphi^c -\frac{1}{G}\varphi^{-2}g_{ab}=\frac{8\pi G}{f}\widetilde{T}_{ab}\ , \tag{27}$$

$$\nabla_a\nabla^a\varphi-\tfrac{1}{2}R\varphi=\frac{8\pi}{\varphi f}\widetilde{T}_{\theta\theta}\ , \tag{28}$$

$$\nabla_a(\varphi^2E)=j_a\ , \tag{29}$$

$$\frac{1}{h\,|\,g\,|^{1/2}}D_a(h\,|\,g\,|^{1/2}D^a\psi)-\frac{1}{h}V_2\psi=\rho\ , \tag{30}$$

where

$$\widetilde{T}_{ab}=\frac{1}{8\pi G}\{-g_{ab}E^2-2g_{ab}\varphi^{-2}V +(\nabla_a\nabla_b-g_{ab}\varphi^{-2}\nabla_c\varphi^2\nabla^c)f +2h\varphi^{-2}[(D_a\psi)^\dagger(D_b\psi)+(D_b\psi)^\dagger(D_a\psi) -g_{ab}(D_c\psi)^\dagger(D^c\psi)]\}\ , \tag{31}$$

$$\widetilde{T}_{\theta\theta}=\frac{\varphi^2}{8\pi}[E^2-\varphi^{-1}V_1-\varphi^{-1}h'(D_a\psi)^\dagger(D^a\psi) +(\varphi^{-1}\varphi^a\nabla_a-\varphi^{-2}\nabla_a\varphi^2\nabla^a)f] \tag{32}$$

are the components of the four-dimensional energy-momentum tensor and

$$j_a=ieh\,|\,g\,|^{1/2}\epsilon_{ab}g^{bc}[\psi^\dagger(D_c\psi)-\psi(D_c\psi)^\dagger]\ , \tag{33}$$

$$\rho=-\tfrac{1}{4}h^{-1}\varphi^2f'\psi\left[R+\frac{2}{G}\varphi^{-2}-2\varphi^{-2}\varphi_a\varphi^a -4\varphi^{-1}\nabla_a\nabla^a\varphi\right]\ , \tag{34}$$

$$V_1(\varphi,|\,\psi\,|^2)=\left.\frac{\partial V(x,y)}{\partial x}\right|_{x=\varphi,y=|\psi|^2}\ ,$$

$$V_2(\varphi,|\,\psi\,|^2)=\left.\frac{\partial V(x,y)}{\partial y}\right|_{x=\varphi,y=|\psi|^2}\ ,$$

$$f'(|\,\psi\,|^2)=\left.\frac{\partial f(y)}{\partial y}\right|_{y=|\psi|^2}\ ,$$

$$h'(\varphi)=\left.\frac{\partial h(x)}{\partial x}\right|_{x=\varphi}\ .$$

IV. THE YANG-MILLS COUPLING

The four-dimensional action is of the form

$$\widetilde{I}_m=\frac{1}{16\pi\widetilde{e}^{\,2}}\int d^4x\,|\,\widetilde{g}\,|^{1/2}\mathrm{Tr}(\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu})\ , \tag{35}$$

where $\widetilde{e}$ is the Yang-Mills coupling constant,

$$\widetilde{F}_{\mu\nu}=\partial_\mu\widetilde{A}_\nu-\partial_\nu\widetilde{A}_\mu-[\widetilde{A}_\mu,\widetilde{A}_\nu]$$

are the field strengths, and

$$\widetilde{A}_\mu=\widetilde{A}^n_\mu T_n$$

are the Yang-Mills potentials. The SO (3) generators T_n satisfy

$$[T_n,T_m]=-\epsilon_{mnk}T_k\ ,$$

$$\mathrm{Tr}(T_mT_n)=-2\delta_{mn}\ .$$

The Witten ansatz will, in our case, be of the form

$$\widetilde{A}_a=\frac{\widetilde{e}}{\sqrt{G}}A_aT_3\ ,$$

$$\widetilde{A}_\theta=-\widetilde{e}\psi_1T_1-\widetilde{e}\psi_2T_2\ ,$$

$$\widetilde{A}_\phi=-\widetilde{e}\psi_2\sin\theta\,T_1-\widetilde{e}\psi_1\sin\theta\,T_2-\cos\theta\,T_3\ ,$$

where $A_a(x^0,x^1)$, $\psi_1(x^0,x^1)$, and $\psi_2(x^0,x^1)$ are the effective fields in two dimensions; the gauge is fixed so that only a local U(1) survives, generated by T_3. Thus, A_a transforms as a Maxwell field and $\psi=\psi_1+i\psi_2$ as a complex scalar.

The field equations that are obtained from (35) and from the spherically symmetric ansatz are again identical with (27)–(34), if the functions f, h, and V are chosen as

$$f=1,\quad h=1\ ,$$
$$V(\varphi,|\,\psi\,|^2)=\frac{(e\,|\,\psi\,|^2+P_0)^2}{2\varphi^2}\ , \tag{36}$$

and

$$P_0=-\frac{1}{\sqrt{G}\,\widetilde{e}}\ .$$

The matter action, I_W, in this case, is the so-called Witten action:

$$I_\mathrm{W}=-\int d^2x\,|\,g\,|^{1/2}\left[\tfrac{1}{4}\varphi^2F_{ab}F^{ab}+(D_a\psi)^\dagger(D^a\psi) +\frac{1}{2}\frac{(e\,|\,\psi\,|^2+P_0)^2}{\varphi^2}\right]\ .$$

We observe that I_W is conformally invariant: if we replace

$$g_{ab}\to e^{2\sigma}g_{ab}\ ,$$

$$\varphi\to e^{\sigma}\varphi\ ,$$

$$A_a\to A_a,\quad \psi\to\psi\ ,$$

where $\sigma(x^0,x^1)$ is an arbitrary function, then the action will not be changed. The scalars φ and ψ have different conformal transformation laws than the usual scalars, because they have a different meaning in four dimensions.

The main difference to the previous case (a) is the missing of a derivative coupling between φ and ψ and the nontrivial interaction of ψ with the magnetic field of the hole. For the interpretation, it is important to know that the radial magnetic field in four dimensions is given by

$$B=\frac{e\,|\psi|^2+P_0}{\varphi^2}\,, \tag{37}$$

which corresponds to the following magnetic charge current j_m^a of ψ:

$$j_m^a=-e\,|g|^{-1/2}\epsilon^{ab}[\psi^\dagger(D_b\psi)+\psi(D_b\psi)^\dagger]\,.$$

V. THE CONFORMAL COUPLING

The matter action as proposed by Callan, Coleman, and Jackiw reads in four dimensions

$$\tilde{I}_m=-\frac{1}{8\pi}\int d^4x\,|\tilde{g}|^{1/2}[\tilde{g}^{\mu\nu}\tilde{\psi}_\mu\tilde{\psi}_\nu+\tilde{V}(\tilde{\psi})+\tfrac{1}{6}\tilde{R}\tilde{\psi}^2]\,.$$

The field equations are

$$\tilde{G}_{\mu\nu}=\frac{8\pi G}{\tilde{f}(\tilde{\psi})}\tilde{T}_{\mu\nu}\,, \tag{38}$$

$$|\tilde{g}|^{-1/2}\partial_\mu(|\tilde{g}|^{1/2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\psi})-\tfrac{1}{2}\tilde{V}'(\tilde{\psi})-\tfrac{1}{6}\tilde{R}\tilde{\psi}=0\,, \tag{39}$$

where

$$\tilde{f}(x)=1-\frac{Gx^2}{3} \tag{40}$$

and

$$\tilde{T}_{\alpha\beta}=\frac{1}{4\pi}\{\tilde{\psi}_\alpha\tilde{\psi}_\beta-\tfrac{1}{2}\tilde{g}_{\alpha\beta}(\tilde{\psi}_\rho\tilde{\psi}^\rho+\tilde{V}) -\tfrac{1}{6}[\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\tilde{\psi}^2)-\tilde{g}_{\alpha\beta}\tilde{\nabla}_\rho\tilde{\nabla}^\rho(\tilde{\psi}^2)]\} \tag{41}$$

is the so-called improved energy-momentum tensor. $\tilde{T}^{\mu\nu}$ is not identical to $(2/|\tilde{g}|^{1/2})\delta\tilde{I}_m/\delta\tilde{g}_{\mu\nu}$, because the part

$$-\frac{1}{48\pi}\frac{\delta(|\tilde{g}|^{1/2}\tilde{g}^{\alpha\beta})}{\delta\tilde{g}_{\mu\nu}}\tilde{R}_{\alpha\beta}\tilde{\psi}^2$$

of $\delta\tilde{I}_m/\delta\tilde{g}_{\mu\nu}$ is transferred to the left-hand side of (38). The full source of gravity is $\tilde{f}^{-1}\tilde{T}_{\mu\nu}$, so the improved tensor can be considered as a source only in the first approximation; the theory seems to make sense only if

$$0\le|\tilde{\psi}|<\frac{\sqrt{3}}{\sqrt{G}}$$

holds everywhere. We notice that the metric $\bar{\tilde{g}}_{\mu\nu}$ defined by

$$\bar{\tilde{g}}_{\mu\nu}=\tilde{\psi}^2\tilde{g}_{\mu\nu}$$

satisfies the relation

$$|\bar{\tilde{g}}|\bar{\tilde{R}}=6|\tilde{g}|^{1/2}(\tilde{g}^{\mu\nu}\tilde{\psi}_\mu\tilde{\psi}_\nu+\tfrac{1}{6}\tilde{R}\tilde{\psi}^2) -\partial_\mu[3|\tilde{g}|^{1/2}\tilde{g}^{\mu\nu}(\tilde{\psi}^2)_\nu]\,,$$

so that the matter action can be written as

$$\tilde{I}_m=-\frac{1}{8\pi}\int d^4x\,|\bar{\tilde{g}}|^{1/2}[\tfrac{1}{6}\bar{\tilde{R}}+\tilde{\psi}^{-4}\tilde{V}(\tilde{\psi})]\,.$$

Thus, if

$$\tilde{V}(\tilde{\psi})=\lambda\tilde{\psi}^4\,,$$

then $\tilde{I}_m$ is conformally invariant.

The equation of motion for the spherically symmetric fields can be obtained from the matter action.

$$I_m=-\tfrac{1}{2}\int d^2x\,|\bar{g}|^{1/2}\left\{\frac{1}{3}\left[\frac{1}{G}+\bar{g}^{ab}(\psi\varphi)_a(\psi\varphi)_b +\frac{1}{2}\bar{R}\psi^2\varphi^2\right]+\psi^2\varphi^2\frac{V(\psi)}{\psi^4}\right\}\,,$$

where

$$\bar{g}_{ab}=\psi^2 g_{ab},\quad V(x)=G\tilde{V}\left[\frac{x}{\sqrt{G}}\right],\quad \psi=\sqrt{G}\,\tilde{\psi}\,.$$

In this form, invariance with respect to transformations

$$\varphi\to e^\sigma\varphi,\quad \psi\to e^{-\sigma}\psi,\quad g_{ab}\to e^{2\sigma}g_{ab}$$

is manifest.

If we write out $\bar{g}_{ab}$ and $\bar{R}$, we obtain, as the total two-dimensional action, again (21), this time with

$$f(|\psi|^2)=1-\tfrac{2}{3}|\psi|^2\,,$$
$$h(\varphi)=\varphi^2\,, \tag{42}$$
$$V(\varphi,|\psi|^2)=\tfrac{1}{2}\varphi^2 G\tilde{V}\left[\frac{|\psi|^2}{G}\right]\,,$$

where, for $\tilde{V}(x)$, we again assume (11), and

$$\psi=\frac{1}{\sqrt{2}}\psi_1+\frac{i}{\sqrt{2}}\psi_2,\quad e=0\,,$$

$$\psi_2=0,\quad A_a=0\,.$$

The action (21) defines what we shall call "generalized BCMN model."

VI. THE APPARENT HORIZON

In this section, we define and study the classical dynamics of the spherically symmetric apparent horizon. For this purpose, it is convenient to introduce the double null coordinates u and v. The line element in two dimensions then reads

$$ds^2=-2e^{2\sigma}du\,dv\,,$$

where σ is a function of u and v; the sign of the RHS above is uniquely determined by the requirement that both u and v increase in the future direction. In fact, the double null coordinates are, themselves, uniquely determined by the boundary conditions

$$u\,|_{\mathscr{I}^-}=-\infty,\quad v\,|_{\mathscr{I}^+}=+\infty\ ,$$

$$\lim_{u\to-\infty}\sigma(u,v)=\lim_{v\to+\infty}\sigma(u,v)=0\ ,$$

but we will not require this, we just consider u as the retarded and v as the advanced time.

Equations (27)—(34) as written in the double null coordinates become

$$e^{2\sigma}\partial_a(e^{-2\sigma}\varphi_a)=-\frac{4\pi G}{f}\tilde{T}_{aa}\varphi,\quad a=u,v \tag{43}$$

$$2\varphi^{-1}\varphi_{uv}+2\varphi^{-2}\varphi_u\varphi_v=-\frac{1}{G}\varphi^{-2}e^{2\sigma}+\frac{8\pi G}{f}\tilde{T}_{uv}\ , \tag{44}$$

$$2\varphi^{-1}\varphi_{uv}+2\sigma_{uv}=-\frac{8\pi e^{2\sigma}}{\varphi^2 f}\tilde{T}_{\theta\theta}\ , \tag{45}$$

$$\partial_a(\varphi^2 E)=j_a,\quad a=u,v \tag{46}$$

$$D_u(hD_v\psi)+D_v(hD_u\psi)=-e^{2\sigma}(V_2\psi+h\rho)\ , \tag{47}$$

$$\tilde{T}_{aa}=\frac{1}{8\pi G}[e^{2\sigma}\partial_a(e^{-2\sigma}f_a)+4h\varphi^{-2}\,|D_a\psi\,|^2],\quad a=u,v \tag{48}$$

$$\tilde{T}_{uv}=\frac{1}{8\pi G}[e^{2\sigma}(E^2+2\varphi^{-2}V)-(\partial_u\partial_v+2\varphi^{-1}\varphi_u\partial_v+2\varphi^{-1}\varphi_v\partial_u)f]\ , \tag{49}$$

$$\tilde{T}_{\theta\theta}=\frac{1}{8\pi}\{\varphi^2E-\varphi V_1+\varphi h'e^{-2\sigma}[(D_u\psi)^\dagger(D_v\psi)+(D_v\psi)^\dagger(D_u\psi)]+e^{-2\sigma}(2\varphi^2\partial_u\partial_v+\varphi\varphi_u\partial_v+\varphi\varphi_v\partial_u)f\}\ , \tag{50}$$

$$j_u=-ieh[\psi^\dagger(D_u\psi)-\psi(D_u\psi)^\dagger]\ , \tag{51}$$

$$j_v=ieh[\psi^\dagger(D_v\psi)-\psi(D_v\psi)^\dagger]\ , \tag{52}$$

$$\rho=-h^{-1}f'\psi e^{-2\sigma}\left[\varphi^2\sigma_{uv}+\frac{1}{2G}e^{2\sigma}+\varphi_u\varphi_v+2\varphi\varphi_{uv}\right]\ .$$

We observe that Eqs. (43) and (46) are the so-called null constraints: they contain only u, respectively, v derivatives, even if $E=e^{-2\sigma}(\partial_u A_v-\partial_v A_u)$, because the "wrong" derivative can always be removed by a gauge transformation.

The expression $\tilde{T}_{uu}$ ($\tilde{T}_{vv}$) represents the current of energy through the surfaces v=const (u=const) in four dimensions: they should not be negative, at least within the classical theory. This is indeed the case if $f=1$. However, for the improved tensor, we obtain

$$\tilde{T}_{aa}=\frac{1}{8\pi G}[\tfrac{8}{3}\,|D_a\psi\,|^2-\tfrac{2}{3}(D_a{}^2\psi)\psi^\dagger+\tfrac{4}{3}\sigma_a(D_a\psi)\psi^\dagger+\text{H.c.}]\ ,$$

which is negative for some ψ.

One usually defines the apparent horizon to be such a point (t,x), at which one has (Ref. 13)

$$\varphi_v=0,\quad \varphi_{uv}\le 0\ .$$

The above definition is sensible for theories, in which the energy currents are nonspacelike and future directed. In such models, the light rays that lost their divergence at some time can never become diverging again, so they never reach $\mathscr{I}^+$; this guarantees that there is a black hole (e.g., Ref. 13). For case (c), however, the energy density can be negative, so nondiverging rays can become diverging later. Such an "apparent horizon" will not, in general, signal that a black hole is present.

Still, it is possible to modify the definition of apparent horizon so that the properties of the apparent horizons from the theories with non-negative energy currents will be preserved. This is apparently due to the fact that the total energy current in case (c) is non-negative. We define, for all cases (a), (b), and (c): the *future (past) apparent horizon* is such a point (t,x), at which we have

$$\partial_v(f\varphi^2)=0 \quad (\partial_u(f\varphi^2)=0) \tag{53}$$

and

$$\partial_u\partial_v(f\varphi^2)\leq 0 . \tag{54}$$

If $f=$const, then the new definition is equivalent to the old one; in particular, (53) guarantees that the outgoing (ingoing) rays will be parallel at the horizon, and the inequality (54) guarantees that the spherically symmetric null hypersurfaces of constant retarded time u (advanced time v) diverge (converge) just outside of the apparent horizon. If the *outermost* apparent horizon forms a boundary of the spacetime, then we have even

$$\partial_v(f\varphi^2)>0, \quad \partial_u(f\varphi^2)<0$$

everywhere inside. We can express a part of these conditions in a coordinate independent manner as follows: Inside the spacetime

$$g^{ab}(f\varphi^2)_a(f\varphi^2)_b>0 .$$

At the apparent horizon:

$$g^{ab}(f\varphi^2)_a(f\varphi^2)_b=0 . \tag{55}$$

The behavior of the function $f\varphi^2$ along the retarded (advanced) time surface $u=$const ($v=$const) which, at $v=v_0$ ($u=u_0$), intersects a future (past) apparent horizon is given by the null constraints (43). They have the following form ($a=u$, or $a=v$), if we use (48):

$$e^{2\sigma}\partial_a(e^{-2\sigma}\varphi_a) = -\frac{\varphi}{2f}[e^{2\sigma}\partial_a(e^{-2\sigma}f_a)+4h\varphi^{-2}\,|D_a\psi|^2] ,$$

or, multiplying by $2f\varphi$, we obtain

$$2f\varphi e^{2\sigma}\partial_a(e^{-2\sigma}\varphi_a)+\varphi^2e^{2\sigma}\partial_a(e^{-2\sigma}f_a)=-4h\,|D_a\psi|^2,$$

which is equivalent to

$$f\varphi^2e^{2\sigma}\partial_a(f^{-1}\varphi^{-2}e^{-2\sigma}\partial_a(f\varphi^2)) = -4h\,|D_a\psi|^2-f^{-1}\varphi^2f_a{}^2-2f\varphi_a{}^2 .$$

Notice that $f,\varphi,e^{2\sigma}$ are all positive functions, and, at a_0, $\partial_a(f\varphi^2)=0$. Hence,

$$\partial_a(f\varphi^2)\,|_a=-f(a)\varphi^2(a)e^{2\sigma(a)}\int_{a_0}^{a}d\xi\,f^{-1}(\xi)\varphi^{-2}(\xi)e^{-2\sigma(\xi)}[4h(\xi)\,|D_\xi\psi(\xi)|^2+f^{-1}(\xi)\varphi^2(\xi)f_\xi{}^2(\xi)+2f(\xi)\varphi_\xi{}^2(\xi)] . \tag{56}$$

Within the classical (nonquantum) theory, the left-hand side of (56) is always non-negative for $a<a_0$, and nonpositive for $a>a_0$. Thus, e.g., (56) implies together with (54) that the outermost future (past) apparent horizon moves outward (inward) with respect to the surfaces of constant retarded (advanced) time, i.e., its retarded (advanced) time never increases (cf. Ref. 13).

Another important classical inequality concerning apparent horizons can be obtained as follows. Multiplying Eq. (44) by $f\varphi^2$, using (49) and rearranging terms, we have

$$\partial_u\partial_v(f\varphi^2)=e^{2\sigma}\varphi^2\left[-\frac{f}{G}\varphi^2+E^2+2\varphi^{-2}V\right] .$$

Thus, the inequality (54) implies

$$-\frac{f}{G}\varphi^{-2}+E^2+2\varphi^{-2}V\leq 0$$

at the apparent horizon. The state of an apparent horizon can be described by three parameters: its total electric charge Q, its magnetic charge P, and its surface area A. For spherically symmetric horizons (and our choice of dimensions) these parameters are defined by

$$Q=E_b\varphi_b{}^2,\quad P=B_b\varphi_b{}^2,\quad A=4\pi G\varphi_b{}^2 , \tag{57}$$

where E_b is the radial electric, B_b is the radial magnetic field, and φ_b is the value of φ at the horizon.

Consider cases (a) and (c). We have from formulas (22) and (42)

$$-\frac{f_b}{G}\varphi_b{}^{-2}+E_b{}^2+2G\widetilde{V}_b+\frac{P^2}{\varphi_b{}^4}\leq 0 ,$$

where, in case (c), $P=0$. This leads to the inequalities

$$0\leq\widetilde{V}_b\leq\frac{f_b}{8G^3(Q^2+P^2)} ,$$

$$\frac{8\pi G^2(Q^2+P^2)}{f_b+[f_b{}^2-8G^3(Q^2+P^2)\widetilde{V}_b]^{1/2}} \leq A\leq\frac{8\pi G^2(Q^2+P^2)}{f_b-[f_b{}^2-8G^3(Q^2+P^2)\widetilde{V}_b]^{1/2}} . \tag{58}$$

In case (b),

$$P=e\,|\psi|^2+P_0 ,$$

so we obtain, in a straightforward way,

$$f_bA\geq 4\pi G^2(Q^2+P^2) . \tag{59}$$

The last inequality follows from (58), as $V_b\geq 0$, so it is the common property in all three cases. It plays an important role in the physics of apparent horizons: for example, it is used in Ref. 25 to prove the positivity of energy. It is a generalization of the well-known condition for Killing horizons to be event horizons[26] to a dynamical situation for the case of spherical symmetry.

As we shall see in the next paper of this series,[25] one can define the mass M of an apparent horizon by

$$M=\frac{1}{2G}\left[\frac{1}{f_\infty}\right]^{1/2}\left[\left[\frac{f_bA}{4\pi}\right]^{1/2} +G^2(Q^2+P^2)\left[\frac{f_bA}{4\pi}\right]^{-1/2}\right] , \tag{60}$$

where

$$f_\infty = \lim_{x\to\infty} f, \quad f_b = \lim_{x\to b} f .$$

If Q and P are fixed, then M has a minimum as a function of A at

$$f_b A = 4\pi G^2(Q^2+P^2) ,$$

equal to

$$M_{\rm min} = \left[\frac{1}{f_\infty}\right]^{1/2} (Q^2+P^2)^{1/2} .$$

Hence the minimal surface area corresponds to the minimal energy compatible with given charges if $f_b=1$.

We attempt, in agreement with our general strategy (cf. Ref. 8), to cut away the interior of the apparent horizon. The spacetime which results in this way will have an internal boundary in addition to the usual "external" one ($\mathscr{I}$ and i^0). For the classical dynamics, no loss of information results, because the spacetime remains globally hyperbolic [see Eq. (56)]. In the quantum theory, a loss of information seems to be very plausible (cf. Ref. 4).

In any case, we have to be careful with boundary conditions for our model, because we have unusual boundaries. At i^0, or at a regular center, the conditions will be specified shortly. At the apparent horizon, Eq. (55) will suffice at the moment; more boundary conditions at the horizon will be chosen and discussed in the next paper of this series.

VII. BOUNDARY CONDITIONS

A. The spacelike infinity

Let the coordinates be chosen so that $t=$const, $x\to\infty$ approaches i^0. Then,

$$\varphi - \frac{x}{\sqrt{G}} \le O(\varphi^{-1}) \tag{61}$$

as $x\to\infty$. The asymptotic behavior of g_{ab} is as usual

$$g_{00} = -1+O(\varphi^{-1}), \quad g_{01} \sim O(\varphi^{-2}) , \qquad g_{11} = 1+O(\varphi^{-1}) . \tag{62}$$

For ψ, the asymptotic behavior depends on the function V and the vacuum chosen. If V contains a mass term, and ψ approaches some ψ_0 that minimizes V, then $|\psi-\psi_0|$ will fall off exponentially, otherwise as φ^{-1},

$$|\psi-\psi_0| \le O(\varphi^{-1}) . \tag{63}$$

We assume further that the electromagnetic gauge can be chosen such that

$$A_a = O(\varphi^{-1}) . \tag{64}$$

Finally, we will assume that the derivatives of all fields fall off always by one degree stronger than the fields themselves.

B. The regular center

The center is characterized by

$$\varphi = 0 ;$$

the coordinates there can always be chosen such that the following holds:

$$g^{ab}\varphi_a\varphi_b = \frac{1}{G} , \tag{65}$$

$$g<0 , \tag{66}$$

$$\psi\in C^\infty, \quad g^{ab}\psi_a\varphi_b = 0 , \tag{67}$$

$$P=0, \quad E=0 . \tag{68}$$

For case (b), we have from (68)

$$|\psi|^2 = \frac{1}{\tilde{e}^2} .$$

This, however, is not sufficient for the four-dimensional field strengths $F_{\theta a}$ and $F_{\phi a}$ to vanish, we must have, in addition

$$D_a\psi = 0 , \tag{69}$$

or "Higgs vacuum" at the center. Then, because of (67),

$$n^a D_a \psi \sim O(\varphi^2) \tag{70}$$

for all n^a that satisfy $n^a\varphi_a=0$.

VIII. THE STATIC SOLUTIONS

In the case of the regular center, we always have the flat spacetime solution:

$$\varphi = \frac{x}{\sqrt{G}}, \quad g_{00}=-1, \quad g_{01}=0, \quad g_{11}=1 ,$$

$$A_a=0, \quad \psi=\psi_0, \quad V(\psi_0)=0, \quad P=0 .$$

If there is an apparent horizon with the parameters A, Q, P, then we have always the Reissner-Nordström solution:

$$\varphi = \frac{x}{\sqrt{G}}, \quad g_{00} = -\frac{x^2-2GMx+G^2(Q^2+P^2)}{x^2} ,$$

$$g_{01}=0, \quad g_{11} = -\frac{1}{g_{00}} ,$$

$$A_0 = -\frac{Q}{x}, \quad A_1=0 ,$$

$$\psi=\psi_0 ,$$

where ψ_0 minimizes V, and M is given by formula (60).

The existence of other static solutions is quite plausible: a charged hole can be surrounded by a positively charged ψ shell which interpolates between two different minima of V. In two dimensions, such a solution looks like a kink. The shell will not, in general, fall into the hole, because the latter can be too small for the kink (which needs some typical space to "feel comfortable"). Numerical calculations confirm this hypothesis; more work, however, will be necessary to prove the existence.

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