Renormalized thermal stress tensor for arbitrary static space-times

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A method of constructing an analytical expression for the renormalized vacuum expectation value of the stress tensor $\langle T^\mu_\nu \rangle$ and the mean-square field $\langle \phi^2(x) \rangle$ for a conformally invariant scalar field propagating on static space-times is presented. Particular emphasis is given to the case where the background geometry corresponds to a general spherically symmetric black hole, and explicit results are given for an extremal Reissner-Nordström black hole. In the special case of a Schwarzschild black hole, Page's results are recovered. Possible extensions of the method to cover stationary black holes are briefly discussed.

I. INTRODUCTION

Hawking's[1] discovery of black-hole radiance following upon Parker's[2] earlier investigation of particle production by expanding universes have acted as a great stimulus for a detailed and systematic investigation of the theory of quantum fields propagating on curved space-times.[3]

Perhaps it is fair to claim that the biggest obstacle to a detailed understanding of quantum processes in the presence of gravity is the regularization and renormalization of expressions bilinear in the field operators. For example, the renormalized stress tensor $\langle T^\mu_\nu \rangle$ coupled to the Einstein tensor provides a scheme which, according to current belief, describes the effects of the quantum field on the background geometry, a process of great significance in cosmology and black-hole physics.[4] Further, $\langle T^\mu_\nu \rangle$ can serve as a means toward a detailed analysis of the Hawking radiation, while the mean-square field $\langle \phi^2(x) \rangle$ plays a role in the study of theories with spontaneous symmetry breaking.[5]

However, despite enormous effects, exact results for $\langle T^\mu_\nu \rangle$ or $\langle \phi^2(x) \rangle$ in four dimensions are very sparse. An exception is the class of $\langle T^\mu_\nu \rangle$ obtained by the method developed by Brown and Cassidy[6] which, however, is applicable only to conformally flat space-times. The absence of exact results has oriented researchers towards numerical estimations of $\langle T^\mu_\nu \rangle$ and $\langle \phi^2(x) \rangle$. The numerical work of Candelas[7] and Fawcett[8] provide a significant amount of information about $\langle T^\mu_\nu \rangle$ and $\langle \phi^2(x) \rangle$ for a Schwarzschild black hole in thermal equilibrium with its own radiation. In addition, Elster[9] (using Page's[10] results) has supplied approximate estimates for stresses, density, and the outgoing flux of radiation of an evaporating Schwarzschild black hole. Recently Page[10] has developed a technique of constructing approximate analytical expressions for the renormalized stress tensor and mean-square field for conformally invariant fields propagating on arbitrary static background geometries. By applying the results of Bekenstein and Parker[11] on the Gaussian approximation of the heat kernel, he has in the first place constructed an expression for the propagator for an arbitrary ultrastatic metric. A conformal transformation then gives the propagator in an arbitrary static metric. Moreover, as Page pointed out in the above procedure, the resulting $\langle T^\mu_\nu \rangle$ is conserved and possesses the right trace, provided the ultrastatic metric is conformally related to an Einstein static metric. (For this class of ultrastatic matrices the first and second Hamidew coefficients $a_1(x,x')$ and $a_2(x,x)$ [see Eqs. (4a) and (4b) below] vanish identically.) In particular, Page's approximations for $\langle T^\mu_\nu \rangle$ and $\langle \phi^2(x) \rangle$ are exact for the de Sitter and Nariai metrics.[12] For the Hartle-Hawking-Israel state[13] on a Schwarzschild background the comparison of Page's analytical expressions with the numerical estimates of Candelas[7] and Fawcett[8] indicate that they are very good approximations. More specifically, for points near and on the event horizon the comparison shows a very good agreement,[10] while for points far away the only significant difference occurs for $\langle \phi^2(x) \rangle$ and the tangential pressure in the vicinity of $r \cong 3M$ (Ref. 8).

However, for an arbitrary ultrastatic metric the method breaks down. The resulting stress tensor suffers from the fact that it does not possess the required trace, and therefore the Gaussian approximation of the heat kernel (which is identical to the first term in the DeWitt-Schwinger expression[14]) has to be appropriately modified. It is the main purpose of this paper to put forward an ansatz that accomplishes just this, although we are as yet unable to give a completely rigorous justification for it and in particular we are unable to give an estimate of the error. We propose an extension of Page's method which allows one to construct an expression for $\langle \phi^2(x) \rangle$ and a conserved stress tensor possessing the right trace for an arbitrary static metric. The effect of this ansatz is to add to Page's expression for $\langle T^\mu_\nu \rangle$ in the ultrastatic metric the term $(4\pi)^{-2} a_2(x,x) \xi^\mu \xi^\nu$, where $\xi^\mu = \delta^\mu_0$ is the timelike Killing vector field of the background geometry. The inclusion of this term does not destroy the conservation properties of $\langle T^\mu_\nu \rangle$ and naturally restores the required trace. Upon a conformal transformation of the metric an expression for $\langle \phi^2(x) \rangle$ and $\langle T^\mu_\nu \rangle$ can be obtained for an arbitrary static geometry.

As a first application of the above method an analytical expression for $\langle\phi^2(x)\rangle$ and $\langle T^\mu_\nu\rangle$ for a general spherically symmetric black hole in thermal equilibrium with its own radiation can be constructed. As is known, that picture would arise whenever a black hole is enclosed in a large box with perfectly reflecting walls. The theoretical description of the system is obtained by assuming the field $\phi(x)$ to be regular on the past and future horizon of the full time-symmetric analytical extension of the black-hole manifold. This condition defines the so-called Hartle-Hawking-Israel vacuum state.[13]

The plan of this paper is as follows. In Sec. II, we give the necessary steps leading to the modifications of Page's technique. In Sec. III we apply the method to the case of an extremal Reissner-Nordström black hole. We have found that $\langle T^\mu_\nu\rangle$ and $\langle\phi^2(x)\rangle$ are finite over the event horizon and $\langle T^\mu_\nu\rangle$ shows no thermal radiation at infinity. We also briefly discuss how the method can be extended in order to cover the stationary class of black holes. In Sec. IV we address ourselves to the delicate question of the accuracy of the results presented in Sec. III.

Encouraged with the successes of Page's approximation and taking into account that the results of this paper reduce to the one obtained in Ref. 10 whenever Page's restrictions are met, we hope that, despite the fact that $\langle T^\mu_\nu\rangle$ and $\langle\phi^2(x)\rangle$ are (perhaps crude) first approximations, they turn out to be close to the exact expressions. We also present some indications that the accuracy of the approximation might be better for points located on the event horizon. This appears plausible because the first and second Hamidew coefficients $a_1(x,x')$, $a_2(x,x)$ for any ultrastatic metric conformal to a general spherically symmetric black-hole metric vanish on the outer event horizon. Thus, if the vanishing of $a_1(x,x')$ and $a_2(x,x)$ is the criterion where the Gaussian approximation gives reliable results, then at least our results should be good on the event horizon.

II. RENORMALIZED STRESS TENSOR FOR ULTRASTATIC METRIC

Let $g_{\mu\nu}$ represent an ultrastatic positive-definite metric

$$ds^2=d\tau^2+ds^2_{(3)}\ , \tag{1}$$

where $ds^2_{(3)}=g_{ab}dx^a dx^b$. (Greek indices run from 0 to 3, latin indices run from 1 to 3. We perform all calculations in Euclidean signature and we use the sign convention of C. W. Misner, K. S. Thorne, and J. A. Wheeler [*Gravitation* (Freeman, San Francisco, 1973)]. For the metric (17) we choose $x^0=\tau$, $x^1=r$, $x^2=\theta$, $x^3=\Phi$.) The "time" coordinate $x^0=\tau$ is periodic with period $T^{-1}=2\pi/\kappa$ where κ is a real non-negative number. Let $\phi(x)$ be a Hermitian scalar field satisfying the conformally invariant Klein-Gordon equation, i.e.,

$$(-\nabla^\mu\nabla_\mu+\tfrac{1}{6}R)\Phi=0\ . \tag{2}$$

The renormalized stress tensor will be constructed using Wald's renormalization prescription.[15] Recall that this renormalization method begins with the assertion[16] that the singular part of the symmetrized product of the field operator in the vacuum state is the same one as that of a Hadamard elementary solution, i.e.,

$$\begin{aligned}G(x,x')&=\tfrac{1}{2}\langle\phi(x)\phi(x')+\phi(x')\phi(x)\rangle\\&=\frac{\Delta^{1/2}(x,x')}{(4\pi)^2}\left[\frac{2}{\sigma(x,x')}+(a_1(x,x')+a_2(x,x)\sigma(x,x'))\ln\sigma(x,x')\right.\\&\qquad\left.+W_0(x,x')-\tfrac{1}{4}[\Delta^{-1/2}\nabla^\mu\nabla_\nu(\Delta^{1/2}W_0)+\tfrac{1}{6}R]\sigma(x,x')-\tfrac{3}{4}a_2(x,x)\sigma(x,x')+O(\sigma^2)\right],\end{aligned} \tag{3}$$

where $\sigma(x,x')$ and $\Delta(x,x')$ stand for the familiar squared geodesic interval and van Vleck-Morette determinant, respectively, while the Hamidew coefficients $a_1(x,x')$ and $a_2(x,x)$ are given as[17,18]

$$\begin{aligned}a_1(x,x')&=\tfrac{1}{2}[\nabla_\mu\nabla_\nu a_1(x,x')]\sigma^\mu\sigma^\nu+O(\sigma^\mu\sigma^\nu\sigma^\kappa)\\&=\tfrac{1}{180}(2R_{\mu\kappa\lambda\rho}R_\nu{}^{\kappa\lambda\rho}+2R^\kappa_\lambda R^\lambda_{\mu\kappa\nu}-4R^\kappa_\mu R_{\kappa\nu}\\&\qquad+3\Box R_{\mu\nu}-\nabla_\mu\nabla_\nu R)\sigma^\mu\sigma^\nu+O(\sigma^\mu\sigma^\nu\sigma^\kappa)\ ,\end{aligned} \tag{4a}$$

$$a_2(x,x)=[a_2(x,x')]=\tfrac{1}{180}(R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}-R^{\mu\nu}R_{\mu\nu}+\Box R)\ . \tag{4b}$$

[In (3), (4a), (4b), and in what follows, outer square brackets denote coincidence limits.] In the Appendix we show that for the natural vacuum state associated with the timelike Killing vector field $\xi^\mu=\delta^\mu_0$, $W_0(x,x')$ is

$$W_0(x,x')=\frac{\kappa^2}{3}+\frac{\kappa^4}{90}(2(\tau-\tau')^2-\sigma(x,x'))+\widetilde{W}(x,x')\ . \tag{5}$$

However, since by definition $G(x,x')=G(x',x)$, $\widetilde{W}(x,x')$ is restricted to satisfying the following constraint equations:[19,20]

$$[\nabla_\mu\widetilde{W}(x,x')]=\tfrac{1}{2}\nabla_\mu[\widetilde{W}_0(x,x')] \tag{6a}$$

$$\nabla^\nu[\nabla_\nu\nabla_\mu\widetilde{W}_0(x,x')]-\tfrac{1}{4}\nabla_\mu\Box\widetilde{W}_0(x,x)-\tfrac{1}{4}\nabla_\mu[\Box\widetilde{W}_0(x,x')]-\tfrac{1}{3}R_{\mu\nu}\nabla^\nu\widetilde{W}_0(x,x)=\tfrac{1}{4}\nabla_\mu a_2\ . \tag{6b}$$

Using (5) in (3) and forming the divergence-free boundary-condition part

$$G^B(x,x')=G(x,x')-G^L(x,x') \tag{7}$$

with[31]

$$G^L(x,x')=\frac{\Delta^{1/2}(x,x')}{(4\pi)^2}\left[\frac{2}{\sigma(x,x')}+(-a_1(x,x')+a_2(x,x)\sigma(x,x'))\ln\sigma(x,x')-\tfrac{3}{4}a_2(x,x)\sigma(x,x')+O(\sigma^2)\right], \tag{8}$$

we obtain

$$\langle T^\mu_\nu\rangle=\frac{\pi^2T^4}{90}(\delta^\mu_\nu-4\delta^\mu_0\delta^0_\nu)+\tfrac{1}{3}\nabla^\mu\nabla_\nu\widetilde{W}(x,x)-\tfrac{1}{12}\Box\widetilde{W}(x,x)\delta^\mu_\nu-[\nabla^\mu\nabla_\nu\widetilde{W}(x,x')]+\tfrac{1}{4}\delta^\mu_\nu[\Box\widetilde{W}(x,x')]+\tfrac{1}{4}a_2(x,x)\delta^\mu_\nu\,. \tag{9}$$

[Note the last term in (9) has been put in by hand in order to restore conservation.[15]] Any particular $\widetilde{W}(x,x')$ satisfying (6a) and (6b) makes (9) conserved, with the proper trace. In Page's work the background geometry (1) satisfies

$$[a_{1\mu\nu}]=a_2(x,x)=0\,, \tag{10}$$

and in his approach $W(x,x')\equiv 0$ [which by virtue of (10) satisfies (6a) and (6b)]. However, it is clear in our case that $W(x,x')$ cannot be taken as equal to zero. On the contrary, a natural generalization of Page's work is succeeded by the following choice:

$$\widetilde{W}(x,x')=-\tfrac{1}{2}a_2(x,x)\xi^\mu\xi^\nu\sigma_\mu\sigma_\nu+A(x,x')\,. \tag{11}$$

[Although the term $-\tfrac{1}{2}a_2(x,x)\xi^\mu\xi^\nu\sigma_\mu\sigma_\nu$ satisfies the inhomogeneous constraint equation, for the moment it is not clear whether or not it is the unique solution of (6b) (and if so why). Currently that point is under investigation and we hope to come back to it in a future publication.] Using (11) in (9) we obtain

$$\langle T^\mu_\nu\rangle=\frac{\pi^2T^4}{90}(\delta^\mu_\nu-4\delta^\mu_0\delta^0_\nu)+\frac{a_2(x,x)}{(4\pi)^2}\delta^\mu_0\delta^0_\nu+\tfrac{1}{3}\nabla^\mu\nabla_\nu A(x,x)-\tfrac{1}{12}\Box A(x,x)\delta^\mu_\nu-[\nabla^\mu\nabla_\nu A(x,x')]+\tfrac{1}{4}[\Box A(x,x')]\delta^\mu_\nu\,. \tag{12}$$

It is obvious that (12) has the right trace, and is conserved since now $A(x,x')$ satisfies the homogeneous version of (6b). From (3), (5), and by definition, we also have

$$\langle\phi^2(x)\rangle=G^B(x,x)=\frac{\kappa^2}{3}+A(x,x)\,. \tag{13}$$

Equations (12) and (13) are the main results of this section. Unfortunately, we are unable to specify further $A(x,x')$, and from here on it will be taken as equal to zero (which is now allowed by the constraint equation).

III. STRESS TENSOR FOR SPHERICALLY SYMMETRIC BLACK HOLE

The importance of the results obtained in Sec. II stems from the fact that if $\bar{g}_{\mu\nu}=\Omega^2(x)g_{\mu\nu}$ is a conformal transformation of (1), with $\Omega^2(x)$ a time-independent but otherwise arbitrary conformal factor, then a Hadamard elementary solution $\bar{G}(x,x')$ in $\bar{g}_{\mu\nu}$ is given by

$$\bar{G}(x,x')=\Omega^{-1}(x)G(x,x')\Omega^{-1}(x')\,. \tag{14}$$

The corresponding functional relation between $\langle T^\mu_\nu\rangle$ in (1) and $\langle\bar{T}^\mu_\nu\rangle$ in $\bar{g}_{\mu\nu}$ resulting from (14) is described by the scale functional differential equation of Brown and Cassidy[6] and analytically has the form[10]

$$\langle\bar{T}^\mu_\nu\rangle=\Omega^{-4}\langle T^\mu_\nu\rangle-8\alpha\Omega^{-4}[\nabla_\alpha\nabla^\beta(C^{\alpha\mu}{}_{\beta\nu}\ln\Omega)+\tfrac{1}{2}R^\beta_\alpha C^{\alpha\mu}{}_{\beta\nu}\ln\Omega]+\beta[(4\bar{R}^\beta_\alpha\bar{C}^{\alpha\mu}{}_{\beta\nu}-2\bar{H}^\mu_\nu)-\Omega^{-4}(4R^\beta_\alpha C^{\alpha\mu}{}_{\beta\nu}-2H^\mu_\nu)]-\tfrac{1}{6}\gamma[\bar{I}^\mu_\nu-\Omega^{-4}I^\mu_\nu] \tag{15}$$

with

$$\alpha=(4\pi)^{-2}(120)^{-1},\quad \beta=-(4\pi)^{-2}(360)^{-1}\,,$$
$$\gamma=(4\pi)^2(180)^{-1}\,,$$

and

$$I_{\mu\nu}=-2g_{\mu\nu}\Box R+2\nabla_\mu\nabla_\nu R-2RR_{\mu\nu}+\tfrac{1}{2}R^2g_{\mu\nu}\,,$$
$$H_{\mu\nu}=-R^\alpha_\mu R_{\alpha\nu}+\tfrac{2}{3}RR_{\mu\nu}+(\tfrac{1}{2}R^\alpha_\beta R^\beta_\alpha-\tfrac{1}{4}R^2)g_{\mu\nu}\,,$$

while barred quantities are formed out of the metric $\bar{g}_{\mu\nu}$. Further use of (14) reveals the following representation of the renormalized $\langle\bar{\phi}^2(x)\rangle$ in $\bar{g}_{\mu\nu}$:

$$\langle\bar{\phi}^2(x)\rangle=\tfrac{1}{12}(T_{\rm loc}{}^2-T_{\rm acc}{}^2)-\frac{\bar{g}^{00}\bar{R}_{00}}{48\pi^2}\,, \tag{16}$$

where

$$T_{\rm loc}(x)=\frac{\kappa}{2\pi}\frac{1}{\Omega(x)}$$

and

$$T_{\rm acc}(x)=\frac{1}{2\pi}[\Omega^{-2}(x)\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\Omega\bar{\nabla}_\beta\Omega]^{1/2}$$

are the local and Unruh acceleration temperatures, respectively.[21,22] As a first application of the above-outlined procedure, let us assume the following particular form for (1):

$$ds^2=d\tau^2+\frac{dr^2}{f^2(r)}+\frac{r^2}{f(r)}(d\theta^2+\sin^2\theta\,d\phi^2)\,. \tag{17}$$

Choosing $\Omega^2=f(r)$ then,

$$d\bar{s}^2=f(r)d\tau^2+\frac{dr^2}{f(r)}+r^2(d\theta^2+\sin^2\theta\, d\phi^2) \tag{18}$$

is recognized as the Euclideanized version of a spherically symmetric black-hole metric, provided κ is identified with the surface gravity of the (outer) event horizon.[23] The case of a Schwarzschild black hole corresponds to the choice $f(r)=1-2m/r$ and $\langle \bar{T}^\mu_\nu\rangle$ and $\langle \bar{\phi}^2(x)\rangle$ are given by (12), (15), and (16) and they are identical to expressions obtained in Ref. 10. The choice $f(r)=(1-(m/r))^2$ characterizes a critically charged Reissner-Nordström black hole. $\langle \bar{T}^\mu_\nu\rangle$ is obtained by combining (12) and (15), and after an extremely lengthy calculation its components reduce to

$$\begin{aligned}2880\pi^2\langle T^\mu_\nu\rangle = {} & \frac{m^2}{r^6}(199x^2-344x+144)\delta^0_\nu\delta^\mu_0-\frac{m^2}{r^6}(-87x^2+144x-56)\delta^\mu_1\delta^1_\nu \\ & -\frac{m^2}{r^6}(49x^2-52x+4)(\delta^\mu_2\delta^2_\nu+\delta^\mu_3\delta^3_\nu)-\frac{m^2}{r^6}(34x^2-72x+36)\delta^\mu_\nu \\ & -\frac{24m}{r^5}(1-x)\ln(1-x)((-26x^2+26x-6)\delta^\mu_1\delta^1_\nu+(7x^2-7x+3)(\delta^\mu_2\delta^2_\nu+\delta^\mu_3\delta^3_\nu)-3x(1-x)\delta^\mu_\nu)\ ,\end{aligned} \tag{19}$$

where $x=m/r$. It can be easily checked that (19) is conserved, symmetric, and has the correct trace. Further, it is finite on the event horizon and shows (as expected for a zero-temperature extremal black hole) no Hawking thermal radiation at infinity. The value of the mean-square field $\langle \bar{\phi}^2(x)\rangle$, computed from (16), is given by

$$\langle \bar{\phi}^2(x)\rangle=0\ . \tag{20}$$

It is clear that $\langle \bar{T}^\mu_\nu\rangle$ and $\langle \bar{\phi}^2(x)\rangle$ for a Reissner-Nordström black hole are obtained from (12), (15), and (16) by choosing

$$f(r)=1-\frac{2m}{r}+\frac{e^2}{r^2}\ .$$

Their explicit form and properties will be discussed in a future publication. Perhaps it is worth mentioning that the causal structure of the Reissner-Nordström black hole makes $\langle \bar{\phi}^2(x)\rangle$ divergent on the inner event horizon (with similar behavior expected for $\langle \bar{T}^\mu_\nu\rangle$). The divergences of the quantum vacuum effects in combination with the divergences of the classical time-dependent perturbations on the inner horizon[24] lead to the conclusion that the interior geometry will be highly modified by back-reaction effects leading probably to the disappearance of the inner event horizon. Similar conclusions have also been drawn by Birrell and Davies and by Hiscock[25] by exploiting the conservation properties of $\langle T^\mu_\nu\rangle$ and the known form of $\langle T^\mu_\mu\rangle$.

Although the above formalism accommodates the spherically symmetric class of black holes, it can also give an approximate $\langle T^\mu_\nu\rangle$ for the stationary antisymmetric class of black holes. If

$$d\bar{s}^2=g_{00}dt^2+2g_{03}dtd\phi+g_{ab}dx^adx^b \tag{21}$$

represents a stationary axisymmetric black hole, then (21) can be rewritten as

$$d\bar{s}^2=\Omega^2ds^2$$

with

$$\Omega^2=\frac{{g_{30}}^2-g_{33}g_{00}}{g_{33}}$$

and

$$ds^2=-dt^2+\frac{g_{33}}{\Omega^2}(d\phi-\omega\phi t)^2+g_{AB}dx^Adx^B\ , \tag{22}$$

$A,B=1,2$ and $\omega=-g_{30}/g_{33}$ is the Bardeen angular velocity.[26] The slow variation of $\omega(r,\theta)$ permits us (at least around the event horizon) to approximate (22) by

$$\begin{aligned}&ds^2=-dt^2+\tilde{g}_{\phi\phi}d\tilde{\phi}^2+\tilde{g}_{AB}dx^Adx^B\ ,\\ &\tilde{\phi}=\phi-\omega t\ .\end{aligned} \tag{23}$$

Combining (23) and the results obtained in Sec. II, an approximate $\langle T^\mu_\nu\rangle$ and $\langle \phi^2(x)\rangle$ can be obtained. Further results on this approach will be given elsewhere.

IV. DISCUSSION

Perhaps the most obvious question concerning the results derived in Secs. II and III is their reliability. At the moment, as we have emphasized before, we cannot argue in one way or another. However, the properties shared by (19) are rather encouraging. It shares all the expected properties associated with the Hartle-Hawking-Israel vacuum state and it further satisfies $T^0_0(x=1)=T^r_r(x=1)$, which is the necessary condition for the regularity of T^μ_ν on the event horizon. Further, a careful examination of the coefficients $a_1(x,x')$ and $a_2(x,x)$ show the following.

For the background metric (17) the only nonzero components of $R^{\alpha\beta}{}_{\gamma\delta}$ are

$$R^{12}{}_{12}=R^{13}{}_{13}=\frac{\lambda}{2}\ ,$$

$$R^{23}{}_{23}=\mu$$

with

$$\lambda=f\left[f''-\frac{f'^2}{2f}\right]\ ,$$

$$\mu=\frac{f}{r^2}\left[1-f+rf'-\frac{r^2f'^2}{4f}\right]\ ,$$

and

$$f'=\frac{\partial f(r)}{\partial r},\quad f''=\frac{\partial^2 f(r)}{\partial r^2}\ .$$

If $r=r_0$ represents the location of the (outer) event horizon, then it is a straightforward task to show that

$$[\nabla^\mu\nabla_\nu a_1(x,x')]=\tfrac{1}{180}(2R^{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}+2R^{\beta}_{\alpha}R^{\alpha\mu}{}_{\beta\nu}-3R^{\mu}_{\alpha}R^{\alpha}_{\nu}+3\Box R^{\mu}_{\nu}-\nabla^\mu\nabla_\nu R)\,|_{r=r_0}=0\,,$$

$$[a_2(x,x')]=\tfrac{1}{180}(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-R^{\alpha\beta}R_{\alpha\beta}+\Box R)\,|_{r=r_0}=0\,.$$

It is perhaps worth pointing out that the vanishing of $[\nabla^\mu\nabla_\nu a_1(x,x')]$ and $a_2(x,x)$ at $r=r_0$ seems to be a general property for arbitrary black-hole metrics. Detailed calculations[27] show that this is true for the case when (1) is conformally related to a metric representing a tidally distorted black hole of the type studied by Israel[28] and more recently by Geroch and Hartle.[29] For the stationary axisymmetric class of black holes this is also true provided (1) is replaced by (23). The vanishing of these two coefficients implies our results are identical to those obtained by the Gaussian approximation of the heat kernel. The remarkable success of this approximation makes us believe that $\langle T^\mu_\nu\rangle$ and $\langle\phi^2(x)\rangle$ are reliable at least at the event horizon.

Note added in proof. After this paper was accepted for publication I became aware of recent numerical work of P. Candelas and K. W. Howard. They are reporting good qualitative agreement with Page's expression for all values of the radial coordinate. The disagreement reported in Ref. 8 is due to an error made by its author. The reference is K. W. Howard and P. Candelas, Center for Theoretical Physics, University of Texas at Austin, report, 1984 (unpublished).

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APPENDIX

In this section we would like to given an expression describing the short-distance behavior of the distribution

$$G(x,x')=\tfrac{1}{2}\langle\phi(x')\phi(x)+\phi(x')\phi(x)\rangle\,.$$

A convenient way of doing this is to look for a corresponding expression for the Euclidean Feynman propagator

$$G^E(x,x')=\langle T\phi(x)\phi(x')\rangle$$

(for the Euclidean signature the two functions have the same functional form for $x'\neq x$). By definition $G^E(x,x')$ is the unique Euclidean Green's function for the Klein-Gordon operator which approaches zero at large spatial distances and satisfies

$$(-\nabla^\mu\nabla_\nu+\tfrac{1}{6}R)G^E(x,x')=\delta^4(x,x')\,. \tag{A1}$$

Using the Gaussian approximation $G_1(x,x')$ obtained in Ref. 10 we factorize $G^E(x,x')$ as follows:

$$G^E(x,x')=G_{(1)}(x,x')F(x,x') \tag{A2}$$

with

$$G_1(\vec{x}\tau\vec{x}\,'0)=\frac{\kappa}{8\pi^2}\frac{\Delta^{1/2}(x,x')}{r}\frac{\sinh\kappa r}{\cosh\kappa r-\cos\kappa\tau}\,. \tag{A3}$$

$r=2\sigma(\vec{x},\vec{x}\,')$ is the geodesic distance formed out of the metric $g_{\alpha\beta}$. Substituting (A2) in (A1) we obtain the following differential equation satisfied by $F(x,x')$:

$$\nabla^\mu\nabla_\mu F+\lambda\nabla^0F+\mu_a\nabla^aF+\nu F=0 \tag{A4}$$

with

$$\lambda=-\frac{2\kappa\sin\kappa\tau}{\cosh\kappa r-\cos\kappa\tau}\,, \tag{A5}$$

$$\mu_a=2\nabla_a\ln\frac{\Delta^{1/2}(x,x')}{r}+2\frac{\kappa}{r}\frac{1-\cosh\kappa r\cos\kappa\tau}{\sinh\kappa r(\cosh\kappa r-\cos\kappa\tau)}\sigma_a\,, \tag{A6}$$

$$\nu=\Delta^{-1/2}\nabla^\mu\nabla_\mu\Delta^{1/2}-\tfrac{1}{6}R\,. \tag{A7}$$

Constructing the global solution of (A4) subjected to the boundary condition $F(x,x)=1$, and $\lim F(x,x')\to\text{const}$ for $|\vec{x}-\vec{x}\,'|\to\infty$ is a very difficult (if not impossible) task. However, we can look for the general solution of (A4) subjected to $F(x,x)=1$. In order to construct the general local solution of (A4) we follow a method similar to the one used by DeWitt and Brehme[30] or Adler *et al.*[31] in their construction of the Hadamard elementary solution for the wave operator. The most general form of the solution compatible with $F(x,x)=1$, in the limit of $x'\to x$, appears as follows:

$$F(x,x')=(V_1(x,x')\sigma(x,x')+V_2(x,x')\sigma(x,x')^2)\ln\sigma(x,x')+W_0(x,x')+W_1(x,x')\sigma(x,x')+W_2(x,x')\sigma(x,x')^2+O(\sigma(x,x')^3)\,, \tag{A8}$$

where the expansion coefficients V_1, V_2, W_0, W_1, and W_2 are smooth biscalars free of singularities. Substituting (A8) into (A4) and taking into account the following relations satisfied by λ and μ_a:

$$\sigma^\mu_\mu+\lambda\nabla^0\sigma+\mu_a\sigma^a=\tfrac{2}{3}\kappa^2\sigma+O(\sigma^2)\,, \tag{A9}$$

$$\mu_a\nabla^af+\lambda\nabla^0f=\frac{\nabla_a\Delta\nabla^af}{\Delta}-2\frac{\sigma_\mu\nabla^\mu f}{\sigma}+\frac{\kappa^2}{3}\sigma_\mu\nabla^\mu f+O(\kappa^2,\sigma)\,, \tag{A10}$$

where f is any sufficiently differentiable function, we arrive at the following recursion relation satisfied by V_1, V_2, W_0, W_1, W_2:

$$\sigma^\mu\nabla_\mu W_0=0\,, \tag{A11}$$

$$2\sigma^\mu\nabla_\mu V_1+2V_1=-\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2}W_0)+\tfrac{1}{6}RW_0\,, \tag{A12}$$

$$2\sigma^\mu\nabla_\mu V_2+4V_2=-\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2}V_1)+\tfrac{1}{6}RV_1-\frac{2\kappa^2}{3}(\tfrac{1}{2}\sigma^\mu\nabla_\mu V_1+V_1)\,, \tag{A13}$$

$$2\sigma^\mu \nabla_\mu W_2 + 4W_2 \equiv -\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2}W_2) + \tfrac{1}{6}RW_1 - \frac{2\kappa^2}{3}(\tfrac{1}{2}\sigma^\mu\nabla_\mu W_1 + W_1) - \tfrac{2}{3}\kappa^2 V_1 - (2\sigma^\mu\nabla_\mu V_2 + 6V_2) \ . \quad \text{(A14)}$$

Equation (A1) implies $W_0=1$, while the recursion relations (A2) and (A3) give

$$V_1(x,x') = -\tfrac{1}{2}a_1(x,x') \ , \quad \text{(A15)}$$

$$V_2(x,x) = \tfrac{1}{4}a_2(x,x) \ . \quad \text{(A16)}$$

Further, by taking the coincident limits of (A4) we obtain the following relation between W_2 and W_1:

$$4[W_2] = -[\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2}W_1) + \tfrac{1}{6}RW_1] - \tfrac{3}{2}a_2(x,x) - \frac{2\kappa^2}{3}[W_1] \ . \quad \text{(A17)}$$

Using (A15), (A16), and (A17), (A8) yields

$$F(x,x') = 1 + \tfrac{1}{2}[-a_1(x,x')\sigma(x,x') + \tfrac{1}{2}a_2(x,x)\sigma(x,x')^2]\ln\sigma(x,x') + W_1(x,x')\sigma(x,x') - \tfrac{1}{4}[\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2}W_1) - \tfrac{1}{6}RW_1]\sigma(x,x')^2 - \tfrac{3}{8}a_2(x,x)\sigma(x,x')^2 - \frac{\kappa^2}{6}[W_1]\sigma(x,x')^2 + O(\sigma(x,x')^3) \ . \quad \text{(A18)}$$

The above expression represents the general local solution of (A4) satisfying $F(x,x)=1$. $W_1(x,x')$ is so far quite arbitrary, it is uniquely determined provided the boundary condition $F(x,x')\to\text{const}$ for $|x-x'|\to\infty$ is taken into account. But since the expression (A8) is only valid for small $\sigma(x,x')$, we have no direct means of implementing this boundary condition. As a consequence of this fact the unspecified $W_1(x,x')$ will appear explicitly in $G(x,x')$. Using (A18), expanding (A3) in the limit of $x'\to x$, and substituting in (A2) we have the following expression for the propagator or $G(x,x')$:

$$G(x,x') = \frac{\Delta(x,x')}{(4\pi)^2}\left[\frac{2}{\sigma(x,x')} + (-a_1(x,x') + \tfrac{1}{2}a_2(x,x)\sigma(x,x'))\ln\sigma(x,x') + \frac{\kappa^2}{3} + \frac{\kappa^4}{90}(2(\tau-\tau')^2 - \sigma(x,x')) - \tfrac{3}{4}a_2(x,x)\sigma(x,x') + 2W_1(x,x') - \tfrac{1}{2}[\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2}W_1) - \tfrac{1}{6}RW_1]\sigma(x,x') + O(\sigma^\mu\sigma^\nu\sigma^\lambda)\right] , \quad \text{(A19)}$$

which are the results we want.

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