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Method of finding axially symmetric stationary vacuum solutions of the equations of general relativity

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A method of finding axially symmetric stationary vacuum solutions of the equations of general relativity is presented. The method is based on the calculation of the elements of a matrix, from an arbitrary function of certain arguments or from an arbitrary solution of a second-order linear partial differential equation. From this matrix, solutions of the Ernst equation, in the form of which the equations of general relativity are written, are obtained explicitly.

The problem of finding solutions of the Ernst equation[1] for axially symmetric stationary gravitational fields has received much attention recently.[2-3] Several methods of doing that have been proposed based on group-theoretic techniques,[4-8] on integral equations,[9] on Bäcklund transformations,[10-12] etc. Also, it has been shown that in the static axially symmetric case Yang's equations in the R gauge for self-dual Yang-Mills fields[13] become identical to the Ernst equation.[14] Therefore, methods similar to those applied for self-dual fields can be used to get solutions of the Ernst equation.[15] In this work the last approach is followed.

From the Cartesian coordinates x_i, $i=1,2,3$ we construct the variables $y=x_1+ix_2$, $\bar{y}=x_1-ix_2$, $z'=2x_3=2z$, and we consider the system

$$\begin{aligned} &f(f_{y\bar{y}}+f_{z'z'})-f_yf_{\bar{y}}-f_{z'}f_{z'}-e_yg_{\bar{y}}-e_{z'}g_{z'}=0\ ,\\ &f(e_{y\bar{y}}+e_{z'z'})-2f_{\bar{y}}e_y-2f_{z'}e_{z'}=0\ ,\\ &f(g_{y\bar{y}}+g_{z'z'})-2f_yg_{\bar{y}}-2f_{z'}g_{z'}=0\ , \end{aligned} \tag{1}$$

where $f_y=\partial f/\partial y$, etc. We can show that if f', e', g' is a solution of the system (1) another solution f, e, g is obtained from the expressions

$$\begin{aligned} &\frac{f}{f^2-eg}=\frac{1}{f'},\quad \frac{\partial}{\partial z'}\left[\frac{g}{f^2-eg}\right]=\frac{g'_{\bar{y}}}{f'^2}\ ,\\ &\frac{\partial}{\partial y}\left[\frac{g}{f^2-eg}\right]=-\frac{g'_{z'}}{f'^2}\ ,\\ &\frac{\partial}{\partial z'}\left[\frac{e}{f^2-eg}\right]=-\frac{e'_y}{f'^2},\quad \frac{\partial}{\partial \bar{y}}\left[\frac{e}{f^2-eg}\right]=\frac{e'_{z'}}{f'^2}\ . \end{aligned} \tag{2}$$

A system analogous to (1) and a Bäcklund transformation analogous to (2) was considered by Corrigan, Fairly, Yates, and Goddard.[16] Using their approach we can find solutions of the system (1). Let Δ_s, $s=0,\pm1,\pm2,\ldots$ be a solution of the system

$$\frac{\partial\Delta_s}{\partial z'}=\frac{\partial\Delta_{s+1}}{\partial\bar{y}},\quad \frac{\partial\Delta_s}{\partial y}=-\frac{\partial\Delta_{s+1}}{\partial z'}\ , \tag{3}$$

which imply that all Δ_s satisfy the three-dimensional Laplace equation. Then, by the method of Corrigan *et al.*, we can show that for any integer $n\geq 2$ a solution $f^{(n)}$, $e^{(n)}$, $g^{(n)}$ of the system (1) is obtained from the matrix relation

$$\begin{aligned} &\begin{bmatrix} e^{(n)} & \cdots & f^{(n)}\\ \vdots & & \vdots\\ f^{(n)} & \cdots & g^{(n)} \end{bmatrix}\\ &=\begin{bmatrix} \Delta_{-n+1} & \Delta_{-n+2} & \cdots & \Delta_{-1} & \Delta_0\\ \Delta_{-n+2} & \Delta_{-n+3} & \cdots & \Delta_0 & \Delta_1\\ \vdots & \vdots & & \vdots & \vdots\\ \Delta_{-1} & \Delta_0 & \cdots & \Delta_{n-3} & \Delta_{n-2}\\ \Delta_0 & \Delta_1 & \cdots & \Delta_{n-2} & \Delta_{n-1} \end{bmatrix}^{-1}\\ &\equiv[\Delta^{(n)}]^{-1}\ , \end{aligned} \tag{4}$$

that is from the corner elements of the matrix $[\Delta^{(n)}]^{-1}$, which is assumed to be invertible. For $n=1$ we have $f^{(1)}=e^{(1)}=g^{(1)}=\Delta_0^{-1}$.

If $\Delta_0=\delta_0(\rho,z')$, where $\rho=(x_1{}^2+x_2{}^2)^{1/2}=(y\bar{y})^{1/2}$, from Eqs. (3) we find that the functions Δ_n and Δ_{-n} are of the form

$$\Delta_n=\bar{y}^n\delta_n(\rho,z'),\quad \Delta_{-n}=y^n\delta_{-n}(\rho,z')\ . \tag{5}$$

Furthermore, if $\delta_0(\rho,z')$ is real all $\delta_n(\rho,z')$ are real and we can prove by induction that we can take

$$\Delta_{-n}=(-1)^n\bar{\Delta}_n=(-1)^ny^n\delta_n(\rho,z')\ . \tag{6}$$

Using Eqs. (5a) and (6) we find that the matrix $\Delta^{(n)}$ can be written in the form

$$\Delta^{(n)}=\begin{matrix} & y^0 & y^{-1} & \cdots & y^{-(n-2)} & y^{-(n-1)} & \\ & \left[\begin{matrix}(-1)^{n-1}\delta_{n-1} \\ (-1)^{n-2}\delta_{n-2} \\ \vdots \\ -\delta_1 \\ \delta_0\end{matrix}\right. & \begin{matrix}(-1)^{n-2}\delta_{n-2} \\ (-1)^{n-3}\delta_{n-3} \\ \vdots \\ \delta_0 \\ \rho^2\delta_1\end{matrix} & \begin{matrix}\cdots \\ \cdots \\ \\ \cdots \\ \cdots\end{matrix} & \begin{matrix}-\delta_1 \\ \delta_0 \\ \vdots \\ \rho^{2(n-3)}\delta_{n-3} \\ \rho^{2(n-2)}\delta_{n-2}\end{matrix} & \left.\begin{matrix}\delta_0 \\ \rho^2\delta_1 \\ \vdots \\ \rho^{2(n-2)}\delta_{n-2} \\ \rho^{2(n-1)}\delta_{n-1}\end{matrix}\right] & \begin{matrix}y^{n-1} \\ y^{n-2} \\ \\ y \\ y^0,\end{matrix}\end{matrix} \quad , \tag{7}$$

where the above notation means that we must multiply all elements of the first row of the matrix by y^{n-1}, which is written on the right-hand side of the row, all elements of the second row by $y^{n-2}, \ldots,$ and also all elements of the first column by y^0, which is written above the column, all elements of the second column by $y^{-1}, \ldots$.

Let us call $D^{(n)}$ the matrix we obtain from the right-hand side of Eq. (7) if we omit the factors y^{n-1}, $y^{n-2}, \ldots$ and the factors $y^0, y^{-1}, \ldots,$ which multiply the rows and the columns, respectively. Also let us symbolize by $\tilde{A}_{ij}$ the cofactor which corresponds to the element of the i row and the j column of the matrix A. Then, for every n from the definition of $D^{(n)}$ and Eqs. (4) and (7), we get $\det\Delta^{(n)}=\det D^{(n)}$ and

$$f^{(n)}=(-1)^{n+1}\frac{\det D^{(n-1)}}{\det D^{(n)}}, \quad e^{(n)}=\frac{\bar{y}^{n-1}\tilde{D}_{11}^{(n)}}{\rho^{2(n-1)}\det D^{(n)}} ,$$
$$g^{(n)}=\frac{y^{n-1}\tilde{D}_{nn}^{(n)}}{\det D^{(n)}} . \tag{8}$$

From Eq. (8a) we see immediately that $f=f(\rho,z')$. To compare the expressions $\tilde{D}_{11}^{(n)}$ and $\tilde{D}_{nn}^{(n)}$, we recall that the determinant of a matrix does not change if we turn the matrix around one of its diagonals. Then by a trick analogous to that used in Eq. (7) we get

$$\tilde{D}_{nn}^{(n)}=(-1)^{n-1}\rho^{-2(n-1)}\tilde{D}_{11}^{(nn)} ,$$

which implies that Eq. (8b) becomes

$$e^{(n)}=(-\bar{y})^{n-1}\frac{\tilde{D}_{nn}^{(n)}}{\det D^{(n)}} . \tag{9}$$

The functions e and g of Eqs. (9) and (8c) depend not only on ρ and z', but also on $\bar{y}$ and y, respectively. To find solutions f, e, g of the system (1), which depend on ρ and z' only, we shall take into account that if f', e', g' is a solution of (1), then another solution f, e, g of this system is obtained from the expressions $f=c\rho^u f'$, $e=c\bar{y}^u e'$, and $g=cy^u g'$, where c and u are constants. This can be proved by substitution. For $c=(-1)^{n+1}$, $u=-n+1$, and f', e', g' the expressions (8a), (9), and (8c), respectively, we get

$$f^{(n)}=\rho^{-n+1}\frac{\det D^{(n-1)}}{\det D^{(n)}}, \quad e^{(n)}=\frac{\tilde{D}_{nn}^{(n)}}{\det D^{(n)}} ,$$
$$g^{(n)}=(-1)^{n-1}e^{(n)} . \tag{10}$$

The above expressions give a solution of the system (1), which depends only on ρ and z'.

To put the solution (10) in another, more convenient form, let us write

$$\gamma_m=\rho^m\delta_m , \tag{11}$$

and let us define the matrix $\Gamma^{(n)}$ by the relation

$$\Gamma^{(n)}=\begin{bmatrix}(-1)^{n-1}\gamma_{n-1} & (-1)^{n-2}\gamma_{n-2} & \cdots & -\gamma_1 & \gamma_0 \\ (-1)^{n-2}\gamma_{n-2} & (-1)^{n-3}\gamma_{n-3} & \cdots & \gamma_0 & \gamma_1 \\ \vdots & \vdots & & \vdots & \vdots \\ -\gamma_1 & \gamma_0 & \cdots & \gamma_{n-3} & \gamma_{n-2} \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} & \gamma_{n-1}\end{bmatrix} . \tag{12}$$

Using (11) we easily find that the matrix $D^{(n)}$ is obtained from the matrix $\Gamma^{(n)}$ if we multiply the first row of $\Gamma^{(n)}$ by ρ^0, the second row by $\rho^1, \ldots,$ and also the first column of $\Gamma^{(n)}$ by ρ^{-n+1}, the second column by $\rho^{-n+2}, \ldots$. Then using this fact we can show that $\det D^{(n)}=\det\Gamma^{(n)}$ and $\tilde{D}_{nn}^{(n)}=\rho^{-n+1}\tilde{\Gamma}_{nn}^{(n)}$. Therefore, Eqs. (10) become

$$f^{(n)}=\rho^{-n+1}\frac{\det\Gamma^{(n-1)}}{\det\Gamma^{(n)}}, \quad e^{(n)}=\rho^{-n+1}\frac{\tilde{\Gamma}_{nn}^{(n)}}{\det\Gamma^{(n)}} ,$$
$$g^{(n)}=(-1)^{n-1}e^{(n)} . \tag{13}$$

Thus, if we know the matrix $\Gamma^{(n)}$ we get from the above expressions a solution of the system (1), which depends only on ρ and z'.

If $n=2l$, Eq. (13c) gives $g^{(2l)}=-e^{(2l)}$. Then if we write $\mathscr{E}^{(2l)}=f^{(2l)}+ie^{(2l)}$ we find that the system (1) reduces to the relation

$$f^{(2l)}\nabla^2\mathscr{E}^{(2l)}-\mathscr{E}_\rho^{(2l)}\mathscr{E}_\rho^{(2l)}-\mathscr{E}_z^{(2l)}\mathscr{E}_z^{(2l)}=0 , \tag{14}$$

where

$$\nabla^2=\partial_\rho{}^2+\frac{1}{\rho}\partial_\rho+\partial_z{}^2 .$$

Equation (14) is the Ernst equation for the axially symmetric gravitational field. Therefore, the expressions $f^{(2l)}$ and $e^{(2l)}$ of Eqs. (13a) and (13b) give a solution of the Ernst equation.

To find the form of the matrix elements of $\Gamma^{(n)}$ let us define the function Δ by the relation

$$\Delta = \sum_{s=-\infty}^{\infty} \Delta_s e^{-is\psi} . \tag{15}$$

Then, if $\zeta = e^{i\psi}$, we get from the above relation

$$\Delta_s = \frac{1}{2\pi i} \oint \Delta \zeta^{s-1} d\zeta ,$$

where the contour integral is taken along the circumference of the unit circle with center at the origin. Substituting these expressions for Δ_s into Eqs. (3) we find that the function Δ must satisfy the relations $\partial\Delta/\partial z' - \zeta\partial\Delta/\partial\bar{y} = 0$ and $\partial\Delta/\partial y + \zeta\partial\Delta/\partial z' = 0$. The general solution of these equations is $\Delta = \Delta(\zeta y - \zeta^{-1}\bar{y} - z', \zeta)$, where Δ is an arbitrary function of its arguments which can be expanded as in Eq. (15). Then if we write $y = \rho e^{i\phi}$ and $\phi + \psi = \vartheta$ we get

$$\Delta_s = \frac{e^{-is\phi}}{2\pi i} \oint \Delta[2i\rho \sin\vartheta - z', e^{i(\vartheta-\phi)}] e^{i(s-1)\vartheta} d(e^{i\vartheta}) .$$

Since we want the function $\Delta_0 = \delta_0$ to be a real function of ρ and z', Δ must be a function of $\zeta y - \zeta^{-1}\bar{y} - z'$ only. Then, since according to (5a) and (11) we have $\Delta_s = \bar{y}^s \rho^{-s} \gamma_s$, if we put $e^{i\vartheta} = w$ and $z' = 2z$, from the above expression for Δ_s we get

$$\gamma_s(\rho,z) = \frac{1}{2\pi i} \oint \Delta[\rho(w - w^{-1}) - 2z] w^{s-1} dw . \tag{16}$$

Equation (16) in which Δ is an arbitrary function of $\rho(w - w^{-1}) - 2z$, $\Delta(v)$ is real for real v [since we want $\gamma_s(\rho,z)$ to be real], and the contour integral is taken on the circumference of the unit circle with center at the origin is the general expression for the $\gamma_s(\rho,z)$. The above relation can be written in the form

$$\gamma_s(\rho,z) = \frac{1}{2\pi} \int_0^{2\pi} \Delta(i\rho \sin\vartheta - z) e^{is\vartheta} d\vartheta . \tag{17}$$

The matrix elements $\gamma_s(\rho,z)$ can also be calculated by solving differential equations. Indeed, from Eqs. (3), (5a), and (11) we find that the functions γ_s satisfy the relations

$$\rho \frac{\partial \gamma_s}{\partial z} = (s+1)\gamma_{s+1} + \rho \frac{\partial \gamma_{s+1}}{\partial \rho} ,$$
$$s\gamma_s - \rho \frac{\partial \gamma_s}{\partial \rho} = \rho \frac{\partial \gamma_{s+1}}{\partial z} , \tag{18}$$

from which we get

$$\left[\partial_\rho{}^2 + \frac{1}{\rho}\partial_\rho + \partial_z{}^2 - \frac{s^2}{\rho^2} \right] \gamma_s = 0 . \tag{19}$$

Starting from a solution of Eq. (19), we can proceed to calculate the matrix elements γ_s we need, with the help of Eqs. (18).

As an example let us take

$$\Delta = [\rho(w - w^{-1}) - 2z]^\alpha = \left[\frac{\rho}{w} \right]^\alpha (w - \lambda_+)^\alpha (w - \lambda_-)^\alpha ,$$

where α is a constant, and if $r = (\rho^2 + z^2)^{1/2}$ it is $\lambda_\pm = (z \pm r)/\rho$. For this choice of Δ we get from Eq. (16)

$$\gamma_s = \lambda^{s-\alpha} \rho^\alpha \frac{\Gamma(-\alpha+s)}{s!\Gamma(-\alpha)} F(-\alpha+s, -\alpha; s+1; -\lambda^2) , \tag{20}$$

where $\lambda = \lambda_+$ for $z < 0$, $\lambda = \lambda_-$ for $z > 0$, and $F = {}_2F_1$ is a hypergeometric function. Then using Eqs. (12), (13a) and (13b) for $n = 2l$, and (20), we get explicitly a large class of solutions of the Ernst equation. We have not found an asymptotically flat solution, in the class we get in this way. Generally, to get asymptotically flat solutions we must make a proper choice of the function $\Delta[\rho(w - w^{-1}) - 2z]$, or in the case when we use Eqs. (18) and (19), of the solution of Eq. (19) that we start from. It is not clear how this can be done.

In conclusion it is shown that the expressions $f^{(2l)}$ and $e^{(2l)}$ of Eqs. (13a) and (13b), where the matrix $\Gamma^{(2l)}$ is given by Eq. (12) and its matrix elements are obtained from Eq. (16), or from Eqs. (18) and (19), give a solution of the Ernst equation.

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