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Euclidean Schwarzschild negative mode

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The Euclidean (imaginary time) Schwarzschild solution of general relativity is known to possess a spin-2 metric perturbation which *decreases* its Euclidean action. This "negative mode" contributes an imaginary part to the effective action, and renders hot flat space unstable against the nucleation of black holes. In this paper, we enclose the black hole in a spherical "box" by imposing boundary conditions on the perturbations. Two conditions, which correspond to a fixed temperature (isothermal wall) and fixed energy (reflecting wall) are examined. The isothermal boundary condition eliminates the negative mode if the box is small enough, and stabilizes hot flat space.

I. INTRODUCTION

The precise analogy between black-hole mechanics and thermodynamics[1] led to the discovery of black-hole radiation[2] and has become a fruitful area for research. Black-hole radiation can be demonstrated by calculating the propagator for a quantum field propagating on the curved black-hole background spacetime. This propagator is periodic in imaginary time, with period $\beta=8\pi M$, where M is the black-hole mass. This is because the Euclidean Schwarzschild manifold has a Killing field $(\partial/\partial\tau)^a$ with closed (i.e., periodic) integral curves. Consequently, the fields have a temperature $1/\beta$.[3]

Because a black hole radiates energy, it will evaporate and gradually lose all of its mass. One way to prevent this evaporation is to place the black hole inside a special box.[4] The idea is that the box imposes a *reflecting* boundary condition on the fields, which keeps the total internal energy constant. The resulting system is stable. Depending upon the volume of the box and its total energy content, the maximum entropy (i.e., stable) configuration consists of either pure thermal radiation or a black hole in equilibrium with some thermal radiation.

We are interested in a different kind of boundary, called the *isothermal* wall. Unlike the reflecting wall, the isothermal wall holds the temperature constant, but not the energy. Physically it acts like a perfect absorber, which is held at a constant temperature by some imaginary external heat bath. The isothermal wall can act as a source or sink for energy, depending upon the temperature of its contents. Imagine surrounding a black hole with an isothermal wall of temperature $T=1/8\pi M$. Because a black hole has negative specific heat, any temperature fluctuation will cause it to evaporate completely, or grow until it swallows the box. The reverse of evaporation is also possible, i.e., a black hole can form spontaneously inside an isothermal box.

Recently, Gross, Perry, and Yaffe[5] studied this effect, by calculating the partition function $Z(\beta)$ for a large isothermal box. Their results pertain to a box of essentially infinite volume. In this paper, we see what happens if the isothermal box has finite volume. Both calculations rely on the fact that $\ln Z(\beta)$ has an imaginary part, which has the effect of making hot flat space unstable against the formation of black holes of the same temperature. The effect is analogous to the classic work of Langer[6] on the formation of condensed droplets in a supercooled gas, which shows that the rate of formation is proportional to the imaginary part of the free energy $F=(1/\beta)\ln Z$. The imaginary part of $\ln Z$ is due to the presence of a single "negative mode" of the Euclidean Schwarzschild solution. Related work by other authors[7-9] has confirmed this effect, although there is some question about the exact rate.

This paper is mostly about the negative mode. We begin by describing the Euclidean Schwarzschild solution, and express its one-loop effective action as a classical action plus a functional determinant. This determinant is the product of a great many positive eigenvalues. If one of those eigenvalues is negative, as it is for the negative mode, then the effective action ceases to be real, and acquires an imaginary part. In some situations, this imaginary part can destabilize a classically stable configuration.

Note. Throughout this paper, we use Planck units $\hbar=c=G=k=1$.

II. THE EUCLIDEAN SCHWARZSCHILD SOLUTION

The Euclidean Schwarzschild solution is obtained from the ordinary Lorentzian black-hole metric

$$ds^2=-\left[1-\frac{2M}{r}\right]dt^2+\left[1-\frac{2M}{r}\right]^{-1}dr^2$$
$$+r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \qquad (2.1)$$

by Wick rotating the time coordinate into $\tau=-it$. The metric is then of positive-definite signature for $r>2M$:

$$ds^2=\left[1-\frac{2M}{r}\right]d\tau^2+\left[1-\frac{2M}{r}\right]^{-1}dr^2+r^2d\Omega^2 . \qquad (2.2)$$

We shall see shortly that the two-sphere at $r=2M$ can also be included.

One can define a new radial coordinate by $\rho^2=(1-2M/r)$. Near $r=2M$, $\rho=0$, and the metric (2.2) is of the form

$$ds^2=16M^2\left[d\rho^2+\rho^2\left[\frac{d\tau}{4M}\right]^2\right]+r^2d\Omega^2 . \qquad (2.3)$$

Comparing this to the flat Euclidean metric on R^2, written in polar coordinates ($ds^2=dr^2+r^2d\theta^2$), we see that $\tau/4M$ must be identified with period 2π to keep the metric (2.3) regular at $\rho=0$. Any other identification is possible, but would produce a conical singularity at $r=2M$, and consequently a δ-function singularity in the curvature there.

Because τ has period $8\pi M$, any propagators which are calculated on the Lorentzian manifold are periodic in imaginary time. This means that any fields propagating around a black hole behave as if they are being held at a temperature $T=1/\beta$. For this reason, any field theory calculations carried out on the Euclidean background automatically reflect the thermal nature of the black hole. Furthermore, since the Euclidean manifold only includes the region $r\geq 2M$, the physical properties of the black hole are independent of any hidden behavior taking place inside the horizon or at the singularity.[3]

Left on its own, a black hole will slowly lose energy through its thermal radiation, and evaporate. One might hope to prevent this slow disappearance by immersing the hole in a thermal bath. Unfortunately, because the black-hole temperature $T=1/8\pi M$ is inversely proportional to its mass, the system has negative specific heat $\delta M/\delta T=-8\pi M^2$ and is unstable. The hole either evaporates entirely or else grows indefinitely as it swallows the energy stored in the heat bath.

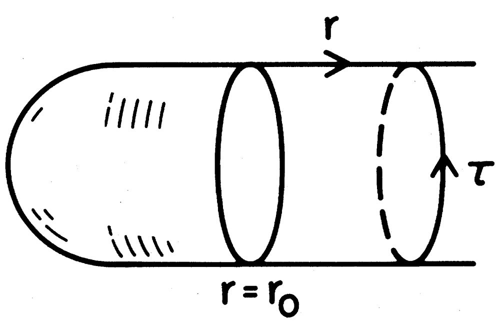

FIG. 1. The Euclidean Schwarzschild manifold has topology $\mathsf{R}^2\times S^2$. Here each point represents a two-sphere $S^2=(\theta,\varphi)$. The radial coordinate $r=2M$ at the leftmost point and increases to the right. The periodic time coordinate runs around the cylinder. To enclose the black hole in a spherical cavity, one imposes a boundary condition on the fields at $r=r_0$. The boundary has topology $S^1\times S^2$.

To investigate these issues, it is desirable to enclose the black hole within a finite-volume box. As discussed earlier, the isothermal-wall and reflecting-wall boxes correspond to different mathematical boundary conditions. To obtain a spherical "box" we can impose a boundary condition at $r=r_0$ on the Euclidean manifold with metric (2.2). As shown in Fig. 1, the complete manifold has topology $\mathsf{R}^2\times S^2$, and the boundary (box wall) has topology $S^1\times S^2$. By imposing different boundary conditions on the fields at $r=r_0$, we can mimic the effects of enclosing the black hole within different types of boxes.

III. THE ONE-LOOP EFFECTIVE ACTION AND THE NEGATIVE MODE

The effective Euclidean action I_E can be defined by a path integral over all positive-signature metrics g by

$$e^{-I_E}\equiv\int d[g]e^{-I[g]} , \qquad (3.1)$$

where $I[g]$ is the classical Euclidean action. We are going to investigate pure gravity and assume that no other fields are present. If we expand (3.1) to quadratic order around the Euclidean Schwarzschild solution, the one-loop effective action of the black hole is[3,5]

$$I_E=I+\tfrac{1}{2}\ln \mathrm{Det}(u^{-2}C)-\tfrac{1}{2}\ln \mathrm{Det}(u^{-2}F)$$
$$-\tfrac{1}{2}\ln \mathrm{Det}(u^{-2}G) , \qquad (3.2)$$

where I is its classical action and u is a regularization mass.

The classical action is the sum of a volume term and a boundary term. The volume term vanishes because a black hole is a vacuum solution, and has $R\equiv 0$. The boundary term yields[3]

$$I=4\pi M^2\left\{3+\frac{2r_0}{M}\left[\left[1-\frac{2M}{r_0}\right]^{1/2}-1\right]\right\} . \qquad (3.3)$$

The action has its minimum value of $-4\pi M^2$ when the boundary is at $r_0=2M$, passes through zero at $r_0=\frac{9}{4}M$, and approaches $4\pi M^2$ as $r_0/M\to\infty$.

The operators C, F, and G arise from the second variation of the action. F and C are scalar (spin 0) and vector (spin 1) operators arising from the gauge-fixing and ghost terms in the action. G is the physical, gauge-independent spin-2 operator

$$G_{abcd}=-g_{ac}g_{bd}\nabla_e\nabla^e-2R_{acbd} \qquad (3.4)$$

which acts on transverse, traceless, symmetric tensors h_{ab}. The determinant is formally defined as the product of the eigenvalues of its operator. In practice the rigorous definition of the functional determinants involves a process of analytic continuation and can be done using generalized ζ functions.[3]

The eigenvalues λ_n of G are defined by all solutions to the elliptic equation

$$G^{ab}{}_{cd}h_n^{cd}=\lambda_n h_n^{ab}\ , \tag{3.5}$$

where the eigenfunctions h_n^{ab} are real regular transverse traceless symmetric tensors which satisfy the boundary condition at r_0. If all the eigenvalues were positive, then $\mathrm{Det}(u^{-2}G)$ would be positive, and the effective action (3.2) would be real. However, if k of the eigenvalues are negative, then the effective action acquires an imaginary part $\mathrm{Im}(I_e)=\frac{1}{2}\pi k$. This imaginary part of the effective action can have important physical consequences.

The imaginary part of $\ln \mathrm{Det} G$ arises in the following way. Suppose that the eigenvalues $\lambda_n(z)$ are analytic functions of a complex parameter $z=x+iy$, and are real along the real z axis. If M of the eigenvalues change sign as z ranges (along the real axis) from 0 to x, this means $\mathrm{Det}[\lambda_n(z)]$ has M zeros between 0 and x. Hence if one analytically continues $\ln \mathrm{Det}[\lambda_n(z)]$ from 0 to x, it picks up an imaginary part $\pm k\pi i$ from the logarithmic branch cuts at those zeros. The C and F terms in (3.2) may also have an imaginary part, which is eliminated by an appropriate rotation of their path-integral contours. These terms are gauge-dependent, and are dealt with in detail elsewhere.[3,5]

Some recent work[5] has shown that the imaginary part of the effective action affects the stability of hot flat space. The partition function for hot flat space can be defined as a path integral over all fields which have a given, fixed periodicity ($\beta=1/T$) at the boundary:

$$Z(\beta)=\int d[g]e^{-I[g]}$$
$$\text{with } \beta=\int ds \text{ at } r=r_0,\ \theta=0,\ \varphi=0\ . \tag{3.6}$$

Using the method of steepest descents, its two stationary points correspond to hot flat space, and to a black hole with the given periodicity at r_0. The saddle-point contribution from the black hole yields its one-loop effective action, and consequently $\ln Z$, and the free energy $F=(1/\beta)\ln Z$, acquires an imaginary part.

One can show that if the free energy of some system has an imaginary part, then that system is metastable and can decay via the nucleation of droplets in a first-order phase transition. In our case, the thermal fluctuations of hot flat space will occasionally cause a black hole of mass $M>1/8\pi T$ to form. The hole will then grow in size, and eventually swallow the heat bath. The rate of black-hole formation (or nucleation) is proportional to $\mathrm{Im}(F)$.[5]

IV. THE NEGATIVE MODE AND BOUNDARY CONDITIONS

To search for negative modes of the operator G, we need to return to the eigenvalue equation (3.5). Its eigenfunctions h_n^{ab} can be expanded in a basis of spin-2 spherical harmonics, multiplied by radial and time-dependent functions. One can show that all of the eigenvalues are non-negative,[5] apart from the lowest-frequency, spherically symmetric static mode. This mode can be written as

$$h^a{}_b=\mathrm{Diag}[\,H_0(r),H_1(r),-\tfrac{1}{2}H_0(r)-\tfrac{1}{2}H_1(r)\ ,$$
$$-\tfrac{1}{2}H_0(r)-\tfrac{1}{2}H_1(r)] \tag{4.1}$$

in a coordinate basis (τ,r,θ,φ). Clearly this mode is traceless, and since it is transverse $\nabla_a h^a{}_b=0$ which implies that

$$H_0(r)=\left[\frac{-r(r-2M)}{r-3M}\frac{d}{dr}-\frac{3r-5M}{r-3M}\right]H_1(r)\ . \tag{4.2}$$

Hence the potential "negative mode" is determined entirely in terms of a single unknown function $H_1(r)$.

The eigenvalue equation (3.5) for this mode becomes an ordinary differential equation for the radial function $H_1(r)$:

$$\left[-\left[1-\frac{2M}{r}\right]\frac{d^2}{dr^2}-\frac{2(r-4M)(2r-3M)}{r^2(r-3M)}\frac{d}{dr}+\frac{8M}{r^2(r-3M)}\right]H_1(r)=\lambda H_1(r)\ . \tag{4.3}$$

The boundary conditions on the equation determine the spectrum of λ, and since the equation is second order, two such conditions are required. The original elliptic equation (3.5) required only a single condition on the boundary $S^1\times S^2$, and the spectrum was determined by the requirement that the solutions be regular everywhere. In terms of the radial equation, these conditions become (1) the boundary condition at $r=r_0$ and (2) regularity of the solution at $r=2M$.

The boundary condition at $r=2M$ is easy to understand. As we have shown, the manifold is regular there, since τ has been identified with period $8\pi M$. We can define a new radial coordinate $x=r/M-2$ and expand H_1 in a power series near $x=0$:

$$H_1(x)=x^s\sum_{k=0}^{\infty}a_k x^k,\ \ a_0\neq 0\ . \tag{4.4}$$

By substituting this power series into the differential equation (4.3) we can obtain relations satisfied by s and by the coefficients a_k.

There are two fundamental linearly independent solutions (4.4). One has $s=0$, and the other has $s=-1$. Since the $s=-1$ solution has a simple pole at $x=0$, it defines a function $H_1(x)$ which is not regular at $r=2M$. It is the $s=0$ solution which defines the unique regular solution to the elliptic boundary value problem (3.5). Putting $s=0$ in (4.4) one can obtain all the coefficients a_k in terms of a_0, which is the value of H_1 at $r=2M$:

$$\begin{aligned}&a_0=a_0\ ,\\&a_1=-(\lambda M^2+2)a_0\ ,\\&a_2=\tfrac{1}{6}(\lambda M^2+2)(2\lambda M^2+7)a_0\ ,\\&\ldots\ .\end{aligned} \tag{4.5}$$

Now it is clear how the spectrum of λ can be determined. It is only for particular values of λ that the solution defined by (4.1), (4.4), and (4.5) will satisfy the boundary condition at r_0.

It is easy to see that the metric perturbation (4.1) does not change the periodicity of τ, since the manifold with metric $g_{ab}+h_{ab}$ is still regular at $r=2M$. This is because (4.2) implies that $H_0(2M)=H_1(2M)$ so that the argument

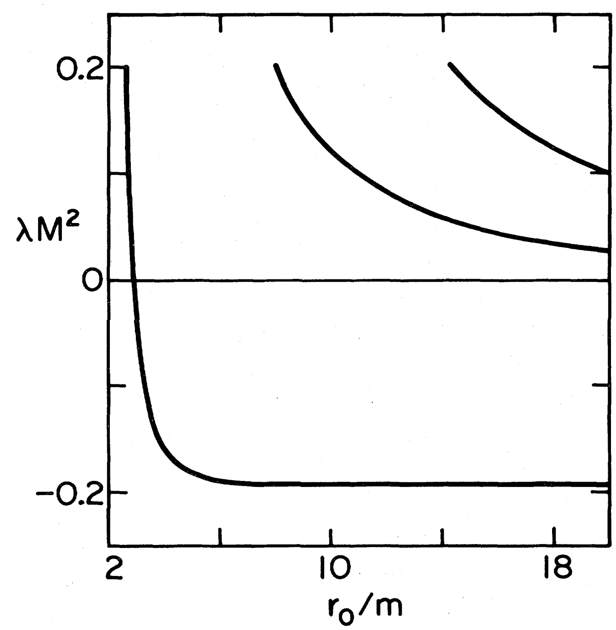

FIG. 2. The spectrum of eigenvalues λ is shown as a function of the radius r_0 of the isothermal box wall. At any value of r_0, there is an infinite number of positive eigenvalues, but at most a single negative one. The negative mode has eigenvalue $\lambda M^2 \cong -0.1919$ when the box is very big. When the wall reaches $r_c \sim 2.89M$, the negative eigenvalue becomes positive. In the limit as $r_0/M \to \infty$, all of the positive eigenvalues go to zero.

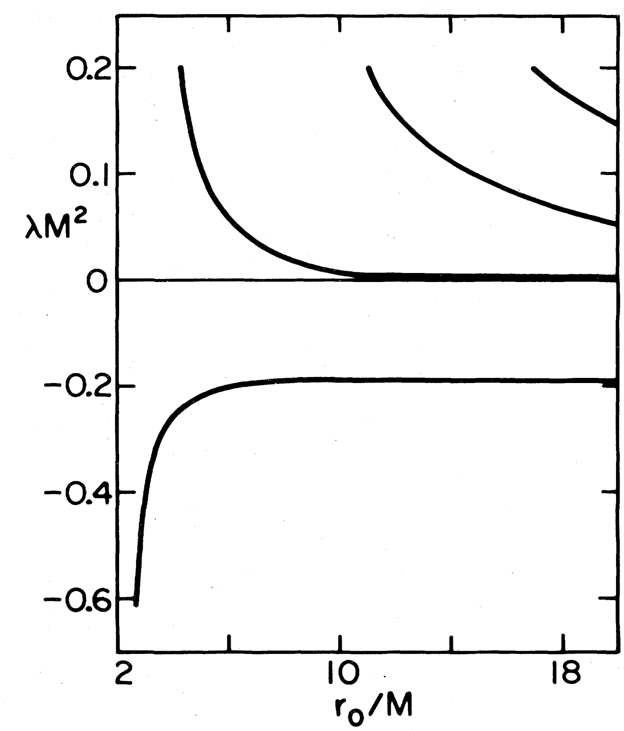

FIG. 3. The spectrum of eigenvalues λ is shown as a function of the radius r_0 of the reflecting box wall. As in the previous figure, this only shows the negative mode and the lowest-frequency positive modes. The negative eigenvalue becomes more negative as r_0 decreases, and approaches $-\infty$ as $r_0 \to 2M$.

of Sec. II still holds in the presence of the perturbation. In other words, the perturbation to the metric does not induce a conic singularity at $r=2M$. It is also easy to verify that the metric perturbation is regular everywhere on the manifold. In particular, one can show that H_0 and H_1 are regular at $r=3M$, by expanding them in power series about that point.

One boundary condition at r_0 which has been discussed at length by Hawking[3] is the isothermal condition. This requires that the proper length around the boundary in the S^1 direction (see Fig. 1) is unaffected by the perturbation. Physically it corresponds to a perfectly absorbing wall coupled to an infinite-energy, fixed-temperature heat bath. This wall does not behave in the same way as a perfectly reflecting boundary, because it does not conserve energy. A perturbation to the metric which changes the three-volume will also change the energy, since the wall emits radiation.

From the explicit form (4.1) of h_{ab}, no change in the size of the S^1 means that $H_0(r_0)=0$. Since H_0 is given in terms of H_1 by (4.2), this Dirichlet condition on H_0 defines a mixed boundary condition (BC) on H_1,

$$\text{Isothermal BC at } r_0\text{:}\quad \frac{H_1'(r_0)}{H_1(r_0)}+\frac{3r_0-5M}{r_0(r_0-2M)}=0\ . \tag{4.6}$$

The spectrum of λ determined by this isothermal boundary condition is shown in Fig. 2. For a given value of r_0 it displays the discrete (positive and negative) values of λ which satisfy (4.6). This graph was obtained by numerically integrating Eq. (4.3) outwards from $r=2M$ using initial conditions (4.5), and finding the values of λ which satisfied (4.6) at $r=r_0$. It can be seen that for $r_0<r_c \cong 2.89M$ the spectrum of G is positive.

It is also interesting to derive a boundary condition for a reflecting wall which conserves energy. There are at least two definitions of the energy within a spacelike (topological) two-sphere. Fortunately, the Hawking mass M_H (Ref. 10) and the Penrose mass M_P (Ref. 11) are equal for the spherically symmetric case. In the presence of the metric perturbation (4.1) the total mass $M_H(r)$ within a two-sphere of radius r, to first order in the perturbation, is

$$M_H(r)=M+\frac{1}{4}\left[r(r-2M)\frac{d}{dr}[H_0(r)+H_1(r)]+(r-3M)H_0(r)+(3r-7M)H_1(r)\right]\ . \tag{4.7}$$

The reflecting-wall boundary condition at r_0 is $M_H(r_0)=M$. Using the definition of H_0, (4.2), and the equation satisfied by H_1, (4.3), this can be expressed as

$$\text{Reflecting BC at } r_0: \ \frac{H_1'(r_0)}{H_1(r_0)}+\frac{2M(3r_0^2-10Mr_0+9M^2)-\lambda r_0^3(r_0-2M)(r_0-3M)}{Mr_0(r_0-2M)(2r_0-3M)}=0\ . \tag{4.8}$$

The spectrum of λ determined by this boundary condition is shown in Fig. 3. It can be seen that with this boundary condition the negative eigenvalue becomes more negative as the boundary is brought in from infinity.

V. CONCLUSIONS

Since the negative mode goes away for a small enough isothermal box whose wall area is less than

$$A<4\pi\left[\frac{r_c}{8\pi M}\right]^2\left[1-\frac{2M}{r_c}\right]^{-1}T^{-2}\cong 0.54T^{-2}\ , \tag{5.1}$$

hot flat space can be stabilized against decay. This is to be expected. If the box wall is at a temperature T, then for a black hole to form and grow (rather than to evaporate) it must have a temperature less than T, and hence a mass $M>1/8\pi T$. This means that its horizon area is greater than $A_{\rm BH}>16\pi M^2$. If this area is larger than the area of the spherical box wall, then the system is stable. Unfortunately this argument does not yield the correct factor of proportionality, perhaps because the quantum stress-energy tensor becomes thermal only far away from the horizon.

It is the nonthermal form of the quantum stress-energy tensor which causes this behavior. If the quantum stress energy were exactly thermal outside the hole, then the location of the isothermal wall would not matter. This is the case sufficiently far from the black hole, where the stress energy becomes asymptotically thermal. Here the negative eigenvalue tends to a constant value (see Figs. 2 and 3). However, near the black-hole horizon, the stress energy must be nonthermal. One can see this without detailed calculation. In order for a black hole to evaporate, its horizon area must decrease. The area theorems[12] then imply that the weak-energy condition must be violated. Since a thermal stress-energy tensor obeys the energy condition, the stress-energy tensor near the horizon must be nonthermal. So by putting an isothermal wall near the horizon the black hole is forced to act differently than it would with the wall at infinity.

The interpretation of the negative mode for a reflecting wall is not clear. It cannot indicate an instability like that of the isothermal box because the reflecting box has been shown to be stable.[4] With the isothermal boundary condition, the path integral yields the canonical ensemble $Z(\beta)$, and the negative mode indicates an instability to black-hole nucleation.[5] In the case of the reflecting wall, we do not understand the significance of the negative mode. If the microcanonical ensemble could be shown to be generated by a path integral

$$N(E)=\int_{H(g)=E}e^{-iS[g]}d[g]\ , \tag{5.2}$$

over all field configurations of a given fixed energy, then the negative mode would contribute an imaginary part to $\ln N(E)$. However, as things stand, we can only conclude that the effective action for a black hole in a reflecting box always has an imaginary part, regardless of the size of the box.

The significance of the negative mode for the canonical ensemble (the isothermal box) is reasonably well understood. It remains to be seen if the imaginary part of the effective Euclidean action for a reflecting box has some simple physical interpretation.

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[1] S. W. Hawking, Phys. Rev. D **13**, 191 (1976).
[2] S. W. Hawking, Nature **248**, 30 (1974).
[3] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, London, 1979), p. 746. (For isothermal boundary conditions, see Sec. 15.3, p. 756 and Sec. 15.8, p. 778.)
[4] G. W. Gibbons and M. J. Perry, Proc. R. Soc. London **A358**, 467 (1982).
[5] D. J. Gross, M. J. Perry, and L. G. Yaffe, Phys. Rev. D **25**, 330 (1982).
[6] J. S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967); **54**, 258 (1969).
[7] D. N. Page, Pennsylvania State University report (unpublished).
[8] T. Piran and R. M. Wald, University of Chicago report (unpublished).
[9] J. I. Kapusta, Phys. Rev. D **30**, 831 (1984).
[10] S. W. Hawking, J. Math. Phys. **9**, 598 (1968).
[11] K. P. Tod, Proc. R. Soc. London **A388**, 457 (1983).
[12] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, London, 1980).