Multiplicity moments with interacting Pomerons to order ϵ^2

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(Received 5 March 1984)

Multiplicity moments within the framework of Reggeon field theory (RFT) are considered. RFT and the renormalization group are reviewed, along with Feynman rules for interacting Pomerons as well as rules for generating unitarity cuts. A variety of prescriptions for calculating multiplicity moments to second order are given as well as explicit expressions for multiplicity moments to first order. As has been noted elsewhere, the first-order moments are in good agreement with suitably corrected experimental results. The second-order corrections are found to be of the same order of magnitude as the first-order corrections, indicating a lack of convergence of the perturbation series. We speculate that the RFT perturbation series for multiplicity moments may in fact be an asymptotic series where higher-order corrections lead to worsening agreement.

I. INTRODUCTION

High-energy hadron collisions are dominated by multiparticle production. Multiparticle production cross sections are related, through the optical theorem, to the imaginary part of the forward elastic scattering amplitude. Thus, to understand high-energy hadron-collision phenomena properly, one needs a framework which takes into account information on both elastic scattering and inelastic scattering or multiparticle production.

The bulk of the present paper will be devoted to the calculation of multiplicity moments in high-energy collisions based on Reggeon field theory. This theory is based on Pomeron-Pomeron interactions and it contains the ingredients for the description of both elastic scattering and multiparticle production. To put our present work in a proper context, we first would like to make some historical remarks on the applications of Regge theory and point out the distinction between Pomeron and non-Pomeron contributions.

The application of Regge theory to hadron physics has a long history.[1] The initial enthusiasm came about after the recognition of a momentum-transfer-dependent power behavior in the scattering amplitude. This led to the prediction of a certain universal shrinkage phenomenon at high energies. This enthusiasm was soon dampened by a series of careful measurements on elastic differential cross sections (EDCS's). One found that the shrinkage phenomena for different processes were significantly different. For instance, as energy increases, the *pp* EDCS's shrink noticeably while the $\bar{p}p$ EDCS's expand. These differences have taught us that it is important to distinguish the nondiffractive contribution from the diffractive contribution. The former is contributed by proper Regge poles and they have particle partners in the crossed channel. By contrast, the latter involves the Pomeron contribution. The Pomeron is crossing even, and at $t=0$, has cross-channel angular momentum $J=1$. The Pomeron does not have a clearly identifiable particle partner. In the energy region where both the Pomeron and the non-Pomeron contributions are important, the near-forward peaks in the EDCS's for various processes exhibit a complicated shrinkage pattern.

Historically,[2] the nondiffractive component, through the notion of duality, led to the discovery of the dual resonance model and later on to the discovery of the dual string, which still serves as a model for QCD confinement.

In the realm of the diffractive component, through high-energy data[3] we have also gained considerable insight into the nature of the Pomeron. From the rise of the total cross section, we learn that the Pomeron is not a simple pole. Its crossing-even property and the approximate universality of the shrinkage rate for various processes have also been confirmed, especially after the CERN ISR data became available. There is also clear experimental evidence for the triple-Pomeron interaction.

The theoretical framework which systematically takes into account the Pomeron and its interactions is the Reggeon field theory.[4,5] We recall that within this framework, by the introduction of a formal energy variable $E=1-j$, one maps the asymptotic behavior of the theory near $t=0$ to the infrared behavior of the theory near $E=0$. In the context of a field theory, this has been elegantly handled by the renormalization-group technique.

The critical-Pomeron solution is the fixed-point solution of the theory which predicts the following asymptotic behavior:[4–9]

Total cross section:

$$\sigma_T \cong [\ln(s)]^{\eta}[1+O(\ln(s)^{-\lambda})+O(\ln(s)^{-1-\eta})] ,$$

Differential cross section:

$$\frac{d\sigma}{dt} \cong [f(t)]^4[\ln(s)]^{2\eta}\ln(t\ln^2 s) ,$$

n-particle cross section:

$$\sigma_n \cong F(n/\langle n\rangle) ,$$

where η and λ are the critical indices of the fixed-point solution. We do not know *a priori* the energy scale where the asymptotic behavior sets in. Some crude estimates suggest that it should dominate at $\ln(s)\cong 10^2$ (s in GeV^2). But from CERN ISR and SPS collider energies, $\ln(s)$ ranges only from 8 to 12. Despite this large difference, some authors take a more phenomenological approach and ask whether the present data can be described by the scaling parametrization.[10] Fits to the total-cross-section data have been obtained. In the ISR energy region, the ratio of the leading contribution to the nonleading contribution is about 3 to -2. Although the fine-tuned fit to the ISR data does give an extrapolated value in the vicinity of the SPS data, the success of the model is not convincing. The differential-cross-section fits to the ISR data were also carried out previously. The present UA4 $\bar{p}p$ EDCS data at 540 GeV and $|t|\sim 0.8$ GeV^2 are a factor of 10 higher than the corresponding secondary maxima predicted in Ref. 9 based on the critical-Pomeron solution: Again, a discrepancy.

What is the situation on the multiplicity data? The use of the ϵ expansion for the critical-Pomeron solution to derive multiplicity moments has been considered by Caneschi and Jengo[6] and by Suranyi.[7] If one assumes some suitable model for cluster productions, their results are in excellent agreement with the SPS collider experiment.[10]

To further study the viability of the theory we asked the following question: Would this agreement hold up if higher-order diagrams and higher-order terms in the ϵ expansion are included? That agreement might be in doubt can be seen in the paper by Bronzan and Dash,[8] where they computed the critical exponents γ and ζ/α and found that the order-ϵ^2 contribution was of the same order of magnitude as the order-ϵ contribution originally calculated by Abarbanel and Bronzan.[5]

We find that the higher-order diagrams as well as the higher-order ϵ^2 terms make large contributions to the multiplicity moments. Indeed, we find ourselves echoing the same sentiments expressed by Bronzan and Dash,[8] namely, that the ϵ expansion is at best slowly convergent, and in fact, it appears to be a questionable means to obtain accurate predictions for both the critical exponents and now the multiplicity moments. Thus, it remains a challenge to construct a realistic model for the diffractive contribution, which contains contributions involving the Pomeron.

The outline of this paper is as follows. In Sec. II we review the generating function for the multiplicity moments as well as Reggeon field theory and the renormalization group. In Sec. III we list the cutting rules given in Ref. 6 and the Feynman rules given in Ref. 5. We then write down the Feynman integrals corresponding to the correction diagrams for the cut-Pomeron propagator and obtain a formula for multiplicity moments C_p. Integrals are renormalized and evaluated in the Appendix.

II. GENERATING FUNCTIONS, REGGEON FIELD THEORY, AND THE RENORMALIZATION GROUP

Consider the collisions of any two given hadrons at high energies, and denote the square of their center-of-mass energy by s. The final states can be labeled by n, the number of particles produced, where $n=0,1,2,\ldots$. To compute quantities related to the multiplicities of final states, it is convenient to work with the generating function defined by

$$\sigma(z,s)=\sum_{n=0}^{\infty} z^n\sigma_n(s)\ , \tag{1}$$

where $\sigma_n(s)$ is the cross section for the production of n particles. Note that $\sigma(1,s)=\sigma_T(s)$, the total cross section. The multiplicity moments

$$n_p=\langle n(n-1)\cdots(n-p+1)\rangle$$

are defined by

$$n_p(s)=\frac{1}{\sigma_T(s)}\frac{\partial^p}{\partial z^p}\sigma(z,s)\,|_{z=1}\ . \tag{2}$$

By letting $z=z-1+1$ in Eq. (1) in terms of the multiplicity moments, one may obtain

$$\sigma(z,s)=\sum_{p=0}^{\infty}\frac{1}{p!}(z-1)^p n_p(s)\sigma_T(s)\ . \tag{3}$$

We are going to compute $\sigma(z,s)$ based on Reggeon field theory. Our approach follows closely the techniques developed in Refs. 6 and 7. The Pomeron propagator is assumed to be built up by multiperipheral production or multiparticles. The multiplicity content of the Pomeron is revealed by cutting the Pomeron propagator. In fact, as shown in Refs. 6 and 7, for the critical-Pomeron solution, $\sigma(z,s)$ can be obtained by considering the interactions of both the uncut Pomeron (UCP) and the cut Pomeron (CP). Their intercepts are, respectively,

$$\begin{aligned}&\alpha_p=1\ ,\\ &\alpha_C=1-b_0(z-1)\ ,\end{aligned} \tag{4}$$

where b_0 is related to the strength of the multiperipheral production in the Pomeron propagator. Note that as z goes to 1 the CP reduces to the usual Pomeron. Both types have a linear trajectory

$$\alpha(k^2,z)=\alpha_0-\alpha_0'k^2 \tag{5}$$

with α_0 being 1 or $1-b_0(z-1)$ depending on whether we mean a UCP or a CP. This gives, for the formal energy variable mentioned in the Introduction, $E=1-j=1-\alpha(k^2,z)$,

$$E=\alpha_0'k^2+b_0(z-1)\ . \tag{6}$$

The Mellin transform of $n_p(s)\sigma_T(s)$ is given by[6]

$$M[n_p(s)\sigma_T(s)]=\frac{\partial^p}{\partial z^p}iG_R^{(1,1)}(E,k^2=0,z)\,|_{z=1}\ , \tag{7}$$

where G_R is the renormalized propagator of the CP computed to some desired order in perturbation theory. The field theory by which G_R will be computed will be given shortly.

Our program will be this. We will compute G_R perturbatively to fourth order in the coupling constant. Divergent integrals will be handled by dimensional regularization.[5] That is, we let D, the dimension of the transverse momentum $\vec{k}$, be a free parameter. We let $\epsilon=4-D$ and extract the singularities as poles in ϵ. Physics takes place at $\epsilon=2$, i.e., $D=2$. We will obtain a power series in $z-1$ for G_R. We then carry out the inverse Mellin transform from E space to s space and read off the values for $n_p(s)\sigma_T(s)$. The moments of interest will be

$$C_p=n_p(s)/[n_1(s)]^p\ , \tag{8}$$

which to leading order are independent of s. We now turn to a review of Reggeon field theory and the renormalization group.

The Reggeon field theory for the cut Pomeron is based on the following free Lagrangian:

$$\mathscr{L}_0=\tfrac{1}{2}i\psi^\dagger\frac{\partial}{\partial t}\psi-\alpha_0'\nabla\psi^\dagger\cdot\nabla\psi-b_0(z-1)\psi^\dagger\psi\ . \tag{9}$$

Here $\psi=\psi(\vec{x},t)$ is the unrenormalized CP field, written as a function of $\vec{x}$, a D-dimensional space vector conjugate to the D-dimensional transverse-momentum vector $\vec{k}$, and t, a variable conjugate to $E=1-j$. We note that the free Lagrangian for the UCP field is the same as the above except that the last term is absent.

The interaction chosen is the triple-Pomeron coupling with nonzero bare coupling ir_0; the factor i being dictated by signature factors of the even signature Pomeron. Our full Lagrangian is then

$$\mathscr{L}=\mathscr{L}_0-\tfrac{1}{2}ir_0(\psi^\dagger\psi^2+\text{H.c.})\ . \tag{10}$$

All possible interactions between cut and uncut Pomerons are symbolically included in Eq. (10). We define dimensions such that

$$[x]=k^{-1},\ \ [t]=E^{-1},\ \ \left[\int d^Dx\,dt\,\mathscr{L}\right]=1\ , \tag{11}$$

which implies

$$[\psi]=k^{D/2},\ \ [\alpha_0']=Ek^{-2}\ ,\qquad [b_0]=E,\ \ [r_0]=Ek^{-D/2}\ . \tag{12}$$

The Green's functions for n incoming and m outgoing Reggeons are defined as

$$G^{(n,m)}(\vec{x}_i,t_i;\vec{y}_j,\tau_j)=\prod_{i=1}^{n}\prod_{j=1}^{m}\langle 0\,|\,T\psi^\dagger(\vec{y}_j,\tau_j)\psi(\vec{x}_i,t_i)\,|\,0\rangle \tag{13}$$

with Fourier transform

$$\delta\left[\sum E\right]\delta^D\left[\sum \vec{k}\right]G^{(n,m)}(E_i,\vec{k}_i) =\int\prod_{i=1}^{n}d^Dx_i dt_i e^{i(E_it_i-\vec{k}_i\cdot\vec{x}_i)} \times\prod_{j=1}^{m}d^Dy_j d\tau_j e^{-i(E_j\tau_j-\vec{k}_j\cdot\vec{y}_j)} \times G^{(n,m)}(\vec{x}_i,t_i;\vec{y}_j,\tau_j)\ . \tag{14}$$

The unrenormalized connected proper vertex functions $\Gamma^{(n,m)}$ are defined by taking the external legs off the connected part of $G^{(n,m)}$. We write

$$\Gamma^{(n,m)}(E_i,k_i)=\prod_{i=1}^{n+m}[G^{(1,1)}(E_i,k_i)]^{-1}G_c^{(n,m)}(E_i,k_i)\ , \tag{15}$$

where G_c is the connected part of the Green's function. Renormalization is carried out at $z=1$, $k^2=0$, and $E=-E_N$ with $E_N>0$. The renormalized Green's functions depend on E_N and the renormalized quantities r, α', and b which are themselves functions of E_N and the unrenormalized parameters r_0, α_0', and b_0. The connection between Γ_R and Γ is

$$\Gamma_R^{(n,m)}(E_i,k_i,r,\alpha',b,E_N) =Z^{(n+m)/2}\Gamma^{(n,m)}(E_i,k_i,r_0,\alpha_0',b_0)\ , \tag{16}$$

where

$$Z^{-1}=\frac{\partial}{\partial E}i\Gamma^{(1,1)}\bigg|_{\substack{k^2=0\\E=-E_N\\z=1}}\ . \tag{17}$$

The normalization conditions we impose on $\Gamma_R^{(1,1)}$ are

$$\Gamma_R^{(1,1)}(E,k^2,z)\bigg|_{\substack{k^2=0\\E=0\\z=1}}=0\ , \tag{18}$$

$$\frac{\partial}{\partial E}i\Gamma_R^{(1,1)}(E,k^2,z)\bigg|_{\substack{k^2=0\\E=-E_N\\z=1}}=1\ , \tag{19}$$

$$\frac{\partial}{\partial k^2}i\Gamma_R^{(1,1)}(E,k^2,z)\bigg|_{\substack{k^2=0\\E=-E_N\\z=1}}=-\alpha'(E_N)\ , \tag{20}$$

$$\frac{\partial}{\partial z}i\Gamma_R^{(1,1)}(E,k^2,z)\bigg|_{\substack{k^2=0\\E=-E_N\\z=1}}=-b(E_N)\ . \tag{21}$$

Note that Eq. (18) corresponds to $\alpha_p=1$ and Eqs. (19) through (21) are the renormalization conditions which also serve as the definitions of the renormalized parameters.

It is convenient to define dimensionless couplings $g_0(E_N)$ and $g(E_N)$ by

$$g_0(E_N)=r_0E_N^{(D/4-1)}(\alpha_0')^{-D/4}\ , \tag{22}$$

$$g(E_N)=rE_N^{(D/4-1)}(\alpha')^{-D/4}\ . \tag{23}$$

The renormalization-group equation for $\Gamma_R^{(n,m)}(E_i,k_i,g,\alpha',b,E_N)$ is obtained by noting that in keeping g_0, α_0', and b_0 fixed, $\Gamma^{(n,m)}(E_i,k_i,g_0,\alpha_0',b_0)$ does not depend on E_N. Using Eq. (17) and the chain rule one obtains

$$\left[E_N\frac{\partial}{\partial E_N}+\beta(g)\frac{\partial}{\partial g}+\zeta(\alpha',g)\frac{\partial}{\partial \alpha'}+\eta(g)\frac{\partial}{\partial b}-\tfrac{1}{2}(n+m)\gamma(g)\right]\Gamma_R^{(n,m)}(E_i,k_i,g,\alpha',b,E_N)=0\,, \tag{24}$$

where we have inserted g, defined in Eq. (23), in place of r, and let

$$\gamma(g)=E_N\frac{\partial}{\partial E_N}\ln Z(E_N)\bigg|_{\alpha_0',r_0,b_0\ \text{fixed}}\,, \tag{25}$$

$$\zeta(g,\alpha')=E_N\frac{\partial}{\partial E_N}\alpha'(E_N)\bigg|_{\alpha_0',r_0,b_0\ \text{fixed}}\,, \tag{26}$$

$$\beta(g)=E_N\frac{\partial}{\partial E_N}g(E_N)\bigg|_{\alpha_0',r_0,b_0\ \text{fixed}}\,, \tag{27}$$

$$\eta(g)=E_N\frac{\partial}{\partial E_N}b(E_N)\bigg|_{\alpha_0',r_0,b_0\ \text{fixed}}\,. \tag{28}$$

Using the dimensional analysis given previously one may obtain

$$\left\{\xi\frac{\partial}{\partial \xi}-\beta(g)\frac{\partial}{\partial g}+[\alpha'-\zeta(\alpha',g)]\frac{\partial}{\partial \alpha'}+[b-\eta(g)]\frac{\partial}{\partial b}+\left[\frac{n+m}{2}\gamma(g)-1\right]\right\}\Gamma_R^{(n,m)}(\xi E_i,k_i,g,\alpha',b,E_N)=0\,, \tag{29}$$

where $\xi=e^t$ is a scaling parameter whose value we are at liberty to choose. This has the solution

$$\Gamma_R^{(n,m)}(\xi E_i,k_i,g,\alpha',b)=\xi^{1-(n+m)\gamma(b)/2}\,\Gamma_R^{(n,m)}(E_i,k_i,g(\xi),\alpha'(\xi),b(\xi))\,, \tag{30}$$

where the ξ dependent parameters satisfy the differential equations

$$\xi\frac{\partial}{\partial \xi}g(\xi)=\beta(g(\xi))\,, \tag{31}$$

$$\xi\frac{\partial}{\partial \xi}\alpha'(\xi)=-\alpha'(\xi)+\zeta(\alpha'(\xi),g(\xi))\,, \tag{32}$$

$$\xi\frac{\partial}{\partial \xi}b(\xi)=-b(\xi)+\eta(g(\xi))\,. \tag{33}$$

Fortunately, we do not have to compute γ, β, ζ, and η and then solve these equations. γ was computed in Ref. 3. Its corrected value is taken from Dash and Grandou:[6]

$$-\gamma=\frac{\epsilon}{12}+\frac{\epsilon^2}{(12)^2}\left(\tfrac{161}{12}\ln\tfrac{4}{3}+\tfrac{37}{24}\right)\,. \tag{34}$$

We also do not need to compute $\eta(g(\xi))$ defined in Eq. (28) due to an observation by Caneschi and Jengo.[1] They find that due to the cutting rules, the result (in their notation) $\gamma_4=\eta$ holds to all orders. Translated into our notation $\gamma_4=-\gamma$ and $\eta=-\eta(g(\xi))/b(\xi)$ so that Eq. (33) becomes

$$\xi\frac{\partial}{\partial \xi}b(\xi)=-b(\xi)[1-\gamma(g)]\,. \tag{35}$$

This has the solution

$$b(\xi)=\xi^{-[1-\gamma(g)]}b\,. \tag{36}$$

Notice that while both b and $\Gamma_R^{(1,1)}$ have dimension E, the scaling factor for b is inverse to that obtained for $\Gamma_R^{(1,1)}$ from Eq. (30).

Now suppose we have obtained $\Gamma_R^{(1,1)}(-E_N,k^2=0,g,\alpha',b)$. If we substitute for g and b their corresponding scaling solutions (note that α' does not enter here since $k^2=0$) $g(\xi)$ and $b(\xi)$, multiply by $\xi^{1-\gamma}$, and then let $\xi=-E/E_N$ we would have an expression for $\Gamma_R^{(1,1)}(E,k^2=0,g,\alpha',b)$ on the left-hand side of Eq. (30). The main effort of the Appendix is to get $\Gamma_R^{(1,1)}(-E_N,k^2=0,g,\alpha',b)$.

Before we close this section we wish to quote the value of g at which the Gell-Mann—Low function $\beta(g)$ has a zero. We denote this special value of g by g_1. From Ref. 6, which corrects the value obtained in Ref. 3, we have

$$\frac{{g_1}^2}{(8\pi)^2}=\frac{\epsilon}{6}+\frac{\epsilon^2}{12}\left[\gamma_{\text{EM}}-\ln\pi-\tfrac{1}{144}(28\ln 2+106\ln 3+23)\right]\,, \tag{37}$$

where $\gamma_{\text{EM}}=0.577\,216$ is the Euler-Mascheroni constant.

III. MULTIPLICITY MOMENTS TO ORDER ϵ^2

We consider the following diagrammatic perturbation series, shown in Fig. 1, for the CP propagator. Each diagram, save the zeroth-order one, may be cut in various ways. These ways of cutting ensure that all multiparticle production diagrams considered have the same leading asymptotic behavior. Further, with each type of cut there is an associated weight factor. Let ir_0 be the coupling constant of the three-Reggeon vertex with all Reggeons uncut and r_0 real. We record the following rules, given in Ref. 5.

(i) The CP propagator carries a factor of 2.

(ii) A vertex with at least one CP and an odd number of UCP lines is $\frac{1}{2}ir_0$, with an even number of UCP lines is $\frac{1}{2}r_0$.

(iii) For each diagram there is one and only one plane through which the cutting takes place: this plane cannot intersect UCP lines, which in our diagram can be thought of as rubber strings that can be pulled above or below the aforementioned plane. Whenever a vertex such as the

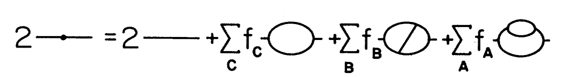

FIG. 1. A diagrammatic perturbation series for the complete cut-Pomeron propagator. The sums are sums over the various ways of cutting the diagrams. The factors f and 2 are the weights associated with the diagrams.

ones in Fig. 2 occurs, the cutting plane must pass between the two uncut lines. All possible positions of the UCP lines must be counted as independent. (Here, and in what follows, an X on a line denotes it as being cut. The dashed line shows the plane of the cut.)

(iv) The vertices referring to UCP lines that stay below the cutting plane are the complex conjugates of those which stay above the cutting plane.

In Fig. 3 we have listed the various ways of cutting the correction diagrams and their associated weights. Each weight carries a factor of r_0^2 or r_0^4 with it depending on the order of the diagram. These factors are included in the vertex factors given below in the Feynman rules. These rules are essentially the same as those given in Ref. 6.

(i) Draw all topologically distinct digraphs (graphs with arrows indicating the direction of momentum flow).

(ii) $\int d^Dq\, dE_q$ around each loop.

(iii) At each vertex put $-ir_0/(2\pi)^{(D+1)/2}$.

(iv) For each Reggeon of momentum $\vec{k}$ and energy E use the propagator

$$G_0^{(1,1)}(E,\vec{k})=i/(E-\alpha_0'\vec{k}^2-\Delta+i\epsilon)\ ,$$

where for a CP $\Delta=b_0(z-1)$, and for a UCP $\Delta=0$.

(v) Conserve E and $\vec{q}$ at each vertex.

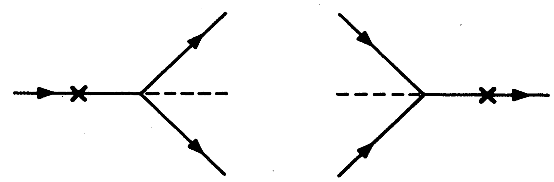

FIG. 2. The allowed ways the cutting plane, shown here as a dashed line, may pass between two uncut incoming or outgoing Pomeron lines.

(vi) For each two-Reggeon loop with both momenta flowing from left to right, multiply by $\frac{1}{2}$.

(vii) Because of the $i\epsilon$ prescription in item (iv), telling us that only the retarded propagator enters this theory, Reggeon loops in which all the momenta circulate in the same direction are zero.

(viii) Multiply by the appropriate weight as obtained from the cutting rules.

Figure 3 shows the momentum conventions for the three diagrams we wish to calculate. Note that a—e, 1—p, q, and r are zero or one according to whether a line is uncut or cut, respectively. Applying the Feynman rules we have the following integrals:

$$\Gamma_A^{(1,1)}(E,\vec{k}^2=0)=f_A i^5\left[\frac{-ir_0}{(2\pi)^{(D+1)/2}}\right]^4\int d^Dk_1d^Dk_2dE_1dE_2[E_1-\alpha_0'\vec{k}_1^2-\Delta_a+i\epsilon]^{-2}$$
$$\times[E_1-E_2-\alpha_0'(\vec{k}_1-\vec{k}_2)^2-\Delta_b+i\epsilon]^{-1}$$
$$\times(E_2-\alpha_0'\vec{k}_2^2-\Delta_c+i\epsilon)^{-1}[E-E_1-\alpha_0'\vec{k}_1^2-\Delta_d+i\epsilon]^{-1}\ , \tag{38}$$

$$\Gamma_B^{(1,1)}(E,\vec{k}^2=0)=f_B i^5\left[\frac{-ir_0}{(2\pi)^{(D+1)/2}}\right]^4\int d^Dk_1d^Dk_2dE_1dE_2(E_1-\alpha_0'\vec{k}_1^2-\Delta_l+i\epsilon)^{-1}$$
$$\times[E_1+E_2-\alpha_0'(\vec{k}_1+\vec{k}_2)^2-\Delta_m+i\epsilon]^{-1}(E_2-\alpha_0'\vec{k}_2^2-\Delta_n+i\epsilon)^{-1}$$
$$\times[E-E_1-E_2-\alpha_0'(\vec{k}_1+\vec{k}_2)^2-\Delta_o+i\epsilon]^{-1}[E-E_1-\alpha_0'\vec{k}_1^2-\Delta_p+i\epsilon]^{-1}\ , \tag{39}$$

$$\Gamma_C^{(1,1)}(E,\vec{k}^2=0)=f_C i^2\left[\frac{-ir_0}{(2\pi)^{(D+1)/2}}\right]^2\int d^Dk_1dE_1(E_1-\alpha_0'\vec{k}_1^2-\Delta_q+i\epsilon)^{-1}[E-E_1-\alpha_0'\vec{k}_1^2-\Delta_r+i\epsilon]^{-1}\ , \tag{40}$$

where we have used the fact that $a=e$ for every A diagram. The f_A, f_B, and f_C are the weight factors given in Fig. 3. These integrals are renormalized and evaluated in the Appendix.

Suppose now that we have all these correction integrals evaluated and have summed over all the cut diagrams A, B, and C with their appropriate weights. Call this sum $\Pi(E,\vec{k}^2=0)$, where

$$\Pi=\sum_A\Gamma_A+\sum_B\Gamma_B+\sum_C\Gamma_C\ . \tag{41}$$

We then have the following Dyson series for G

$$2G=2G_0+2G_0\Pi 2G_0+2G_0\Pi 2G_0\Pi 2G_0+\cdots$$
$$=2G_0\frac{1}{1-2G_0\Pi}\ ,$$

or (42)

$$G(E,0)=(1/G_0-2\Pi)^{-1}\ .$$

Define $\Gamma(E,0)=G^{-1}(E,0)$, so that

$$\Gamma = -i[E - b_0(z-1)] - 2\Pi \,. \tag{43}$$

Now the renormalization constant Z is defined by

$$Z^{-1} = -\frac{\partial}{\partial E} i\Gamma^{(1,1)} \bigg|_{\substack{z=1 \\ k^2=0 \\ E=-E_N}} , \tag{44}$$

and the renormalized inverse propagator Γ_R by

$$\Gamma_R = Z\Gamma \,. \tag{45}$$

Now we showed in Sec. II that we need only find $\Gamma_R^{(1,1)}(-E_N,0,z)$ to obtain, through the scaling relation, $\Gamma_R^{(1,1)}(E,0,z)$. This is what is done in the Appendix. The result is a power series in x, where

$$x = -\frac{b}{E_N}(z-1)\left[-\frac{E}{E_N}\right]^{-(1-\gamma)} . \tag{46}$$

This is merely the scaling form of b, given in Eq. (36), multiplied by $-(z-1)/E_N$. The results from the Appendix are summarized by

$$-i\Gamma_R^{(1,1)}(E,0,z) = E_N\left[-\frac{E}{E_N}\right]^{1-\gamma}\left[1 - x - \sum_{n=0}^{\infty} x^n \frac{g^2}{(8\pi)^2}[1+\tfrac{1}{2}\epsilon(\delta+3\ln 2)](a_n + \tfrac{1}{2}\epsilon e_n) + \frac{g^4}{(8\pi)^4} f_n\right] . \tag{47}$$

The factor $[1+\frac{1}{2}\epsilon(\delta+3\ln 2)]$ arises from the expansion of $\frac{1}{2}\epsilon\Gamma(\frac{1}{2}\epsilon)(8\pi)^{\epsilon/2}$ in powers of ϵ. We have let $\delta = \ln\pi - \gamma_{\rm EM}$, where $\gamma_{\rm EM}$ is the Euler-Mascheroni constant. If one expanded g^2 to order ϵ^2 one would find the factor δ is canceled out. This point is discussed in Ref. 3.

The coefficients a_n, e_n, and f_n are given up to $n=8$ in Table I. We emphasize that the letters a,e,f do not imply relative contributions from diagrams A, B, or C. Indeed the a's and e's come from diagram C alone while the f's arise from diagrams A and B as well as cross terms from the multiplicative renormalization $\Gamma_R = Z\Gamma$. We emphasize at this point that we have an expansion in both g^2 and ϵ, the only connection between the two being that we know g^2 is of order ϵ; thus, we see that while the g^2 term has order ϵ^0 and ϵ^1 coefficients, the g^4 term has only ϵ^0 coefficients. *A priori* one might expect $1/\epsilon$ coefficients in the g^4 term, but these poles are exactly what are canceled by renormalization. Thus, we see that only the first-order diagram can ever make a contribution to order ϵ within the framework of the ϵ expansion simply because g^2 is of order ϵ.

From here one may proceed in several ways. One way is to use the order-ϵ^2 expansion for g_1 as given in Eq. (37), where g_1 is the zero of the Gell-Mann—Low function, and expand $-i\Gamma_R^{(1,1)}$ to order ϵ^2. From there one computes the multiplicity moments in terms of an ϵ expansion. We will outline this method first before returning to an alternate prescription.

The result of plugging Eq. (37) into Eq. (47) is

$$-i\Gamma_R^{(1,1)} = E_N\left[-\frac{E}{E_N}\right]^{1-\gamma}\left[1 - x - \sum_{n=0}^{\infty} x^n(a_n\epsilon/6 + b_n\epsilon^2/36)\right] , \tag{48}$$

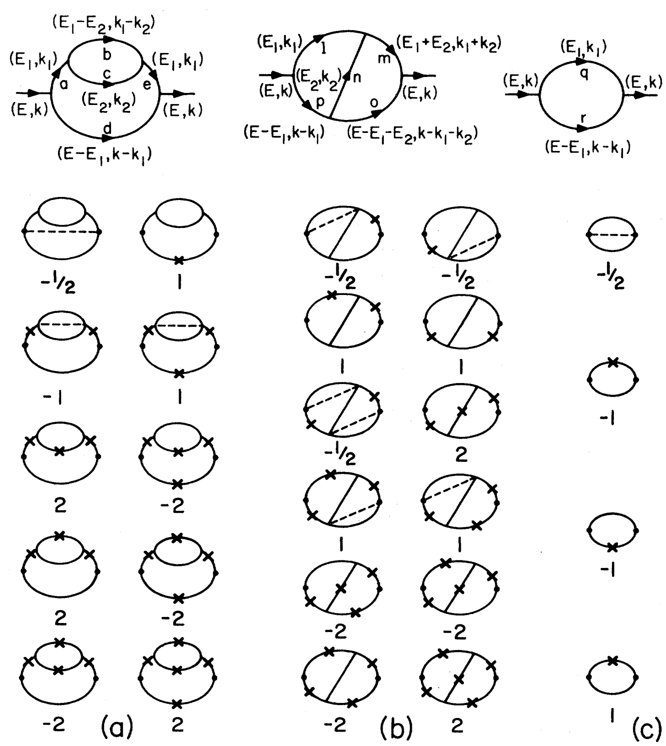

FIG. 3. The three large diagrams at the top show the momentum conventions used in evaluating the Feynman integrals. a—e, l—p, q, and r, are one or zero according to whether the corresponding line is cut or uncut. The smaller figures show the various ways of cutting the diagrams. The numbers underneath each figure are the weights as obtained by the cutting rules.

TABLE I. Coefficients of x^n. See the Appendix for their calculation.

n	a_n	e_n	f_n
0	0.5	0.5	-1.8640
1	0	0	0
2	1.0	0	3.3611
3	1.0	1.0	4.6673
4	1.1667	1.75	10.8869
5	1.5	2.75	24.3594
6	2.0667	4.3056	52.2078
7	3.0	6.85	109.756
8	4.5357	11.1125	229.548

where

$$b_n = f_n + 3e_n + 3a_n[3\ln 2 - (28\ln 2 + 106\ln 3 + 23)/144] . \quad (49)$$

One now inverts $\Gamma_R^{(1,1)}$ to find G_R. Again, keeping terms only to ϵ^2 one obtains, letting $\lambda = \epsilon/6$ for brevity,

$$iG_R^{(1,1)} = \frac{1}{E_N}\left[-\frac{E}{E_N}\right]^{-(1-\gamma)} \sum_{n=0}^{\infty} x^n(1 + c_n\lambda + d_n\lambda^2) , \quad (50)$$

where

$$c_n = \sum_{k=0}^{n} a_k(n-k+1) , \quad (51)$$

$$d_n = \sum_{k=0}^{n} \left[b_k(n-k+1) + \tfrac{1}{2}(n-k+1)(n-k+2) \sum_{m=0}^{k} a_m a_{k-m} \right] . \quad (52)$$

The values are tabulated up to $n=8$ in Table II.

Recalling the definition of x in Eq. (46) gives us a power series in $z-1$, which, it may be in need of recalling, is what we are after. [See Eq. (3).]

TABLE II. Coefficients of x^n. The a_n and b_n are the order-λ and order-λ^2 ($\lambda = \epsilon/6$) coefficients for the expansion of Γ_R. The c_n and d_n are the order-λ and order-λ^2 coefficients for the expansion of $G_R = (\Gamma_R)^{-1}$.

n	a_n	b_n	c_n	d_n
0	0.5	1.1004	0.5	1.3504
1	0	0	1.0	2.9507
2	1.0	6.2898	2.5	12.0909
3	1.0	10.5960	5.0	34.0771
4	1.1667	19.5537	8.6667	80.0337
5	1.5	37.0025	13.8333	170.9095
6	2.0667	71.1771	21.0667	346.279
7	3.0	139.092	31.3	682.390
8	4.5357	276.169	46.069	1329.35

$$iG_R^{(1,1)}(E,0,z) = \frac{1}{E_N} \sum_{0}^{\infty} (-b)^n(-E_N)^{(n+1)(1-\gamma)}(z-1)^n \times (E)^{-(n+1)(1-\gamma)}[1 + c_n\lambda + d_n\lambda^2] . \quad (53)$$

We now inverse Mellin transform from E space to s space, where s is the invariant energy squared, with the aid of the following identity:

$$\int_0^{\infty} dE\, s^{-E} E^{-(1+w)} = \frac{(\ln s)^w}{\Gamma(1+w)} . \quad (54)$$

Thus,

$$iG_R(s,0,z) = \frac{1}{E_N} \sum_{0}^{\infty} (z-1)^n(1 + c_n\lambda + d_n\lambda^2)(-b)^n(-E_N)^{(n+1)(1-\gamma)}(\ln s)^{-\gamma + n(1-\gamma)}/\Gamma((n+1)(1-\gamma)) . \quad (55)$$

Now,

$$\sigma_T(s) = iG_R(s,0,1) = \frac{1}{E_N}(1 + c_0\lambda + d_0\lambda^2)(-E_N)^{1-\gamma}(\ln s)^{-\gamma/\Gamma(1-\gamma)} , \quad (56)$$

and

$$n_p(s)\sigma_T(s) = \frac{\partial^p}{\partial z^p} iG(s,0,z)\bigg|_{z=1} ,$$

so we read off

$$n_p(s) = p!(-b)^p(-E_N)^{p(1-\gamma)}(\ln s)^{p(1-\gamma)} \frac{\Gamma(1-\gamma)}{\Gamma((p+1)(1-\gamma))} \frac{1 + c_p\lambda + d_p\lambda^2}{1 + c_0\lambda + d_0\lambda^2} . \quad (57)$$

Thus,

$$C_p = \frac{n_p(s)}{[n_1(s)]^p} = \frac{p!\Gamma^p(2(1-\gamma))(1 + c_0\lambda + d_0\lambda^2)^{p-1}(1 + c_p\lambda + d_p\lambda^2)}{\Gamma^{p-1}(1-\gamma)\Gamma((p+1)(1-\gamma))(1 + c_1\lambda + d_1\lambda^2)^p} . \quad (58)$$

One way of proceeding is to expand the numerator and denominator separately to order λ^2 and to leave it in the form of a Padé approximant. Another way is to expand the denominator and again truncate to order λ^2. To do these expansions one needs the following:

$$\Gamma[(n+1)(1-\gamma)] \simeq \Gamma(n+1)[1 + (n+1)(-\gamma)\psi_{n+1} + \tfrac{1}{2}(n+1)^2(-\gamma)^2(\psi'_{n+1} + \psi_{n+1}^2)] , \quad (59)$$

where

$$\psi_{n+1}=\frac{d}{dx}\ln\Gamma(x)\bigg|_{x=n+1}=\psi_1+\sum_{k=1}^{n}\frac{1}{k}\ , \tag{60}$$

$$\psi'_{n+1}=\frac{d}{dx}\psi(x)\bigg|_{x=n+1}=\psi'_1-\sum_{k=1}^{n}\frac{1}{k^2}\ , \tag{61}$$

and $\psi_1=-0.577\,216$ and $\psi'_1=\pi^2/6$.

For the Padé method one obtains

$$C_p=N/D\ , \tag{62}$$

where

$$\begin{aligned}D=&1+\eta\lambda[pc_1+p\psi_1+\tfrac{1}{2}(p+1)S_p^{(1)}]\\&+\tfrac{1}{2}\lambda^2\{2(p-1)\psi_1[pc_1+\tfrac{1}{2}(p+1)(\psi_1+S_p^{(1)})]+\tfrac{1}{4}(p-1)(\psi'_1+2C'\psi_1)+\tfrac{1}{4}(p-1)^2\psi_1{}^2\\&+\tfrac{1}{4}(p+1)^2[\psi'_1-S_p^{(2)}+(\psi_1+S_p^{(1)})^2]+\tfrac{1}{2}(p+1)C'(\psi_1+S_p^{(1)})\\&+p(p+1)c_1(\psi_1+S_p^{(1)})+2pd_1+p(p-1)c_1{}^2\}\ ,\end{aligned} \tag{63}$$

$$\begin{aligned}N=&1+\lambda[C_p+(p-1)c_0+p(1+\psi_1)]\\&+\lambda^2(\tfrac{1}{2}p\{\psi'_1-1+p(\psi_1+1)^2+(\psi_1+1)[C'+2c_0(p-1)+2c_p]\}\\&+d_p+(p-1)(d_0+c_0c_p)+\tfrac{1}{2}(p-1)(p-2)c_0{}^2)\ ,\end{aligned} \tag{64}$$

and

$$C'=\tfrac{161}{12}\ln\tfrac{4}{3}+\tfrac{37}{12}\ , \tag{65}$$

$$S_p^{(1)}=\sum_{k=1}^{p}\frac{1}{k}\ , \tag{66}$$

$$S_p^{(2)}=\sum_{k=1}^{p}\frac{1}{k^2}\ . \tag{67}$$

One should recognize the value of C' from Eq. (34) which, written in terms of λ and C', is

$$-\gamma=\lambda/2+C'\lambda^2/4\ . \tag{68}$$

The results for the multiplicity moments are tabulated in Table III under the column headed "Padé."

One could now expand the denominator into the numerator. Consider the following expression:

$$\begin{aligned}C_p&=\frac{1+t_1\lambda+t_2\lambda^2}{1+b_1\lambda+b_2\lambda^2}\\&=1+\lambda(t_1-b_1)+\lambda^2(t_2-b_2+b_1{}^2-b_1t_1)+O(\lambda^3)\ ,\end{aligned} \tag{69}$$

where the definitions of $t_{1,2},b_{1,2}$ can be read off from Eqs. (63) and (64). The results for the multiplicity moments are tabulated in Table III under the column headed "ϵ expanded."

We now return to Eq. (47) to discuss another way of proceeding. One takes the point of view that the ϵ expansion is a device for ordering our calculations and regularizing integrals. Now that we have obtained an expression for $\Gamma_R^{(1,1)}$ consistently expanded to order ϵ^2 we may set $\epsilon=2$. Our point here is that once the value of g is deter-

TABLE III. Multiplicity moments to order $p=8$. The first-order results are due to diagram C alone. The second-order results are due to diagrams A, B, and C. The experimental results are from Alpgard *et al.*, Ref. 7. The β_p are the experimentally measured moments $\beta_p=\langle n^p\rangle/\langle n\rangle^p$. The C_p's are corrected to correspond to our definition $C_p=\langle n(n-1)\cdots(n-p+1)\rangle/\langle n\rangle$ by assuming some model for the average multiplicity $\langle n\rangle$.

	First order		Second order				Experimental		
n	Padé	Expanded	ϵ Padé	ϵ expanded	g Padé	g expanded	β_p	π model	3π model
2	1.123	1.250	1.286	1.772	1.160	1.184	1.28	1.26	1.21
3	1.294	1.778	1.697	3.675	1.428	1.424	2.01	1.91	1.75
4	1.501	2.653	2.252	7.590	1.794	1.544	3.61	3.33	2.85
5	1.751	3.994	3.017	15.303	2.281	1.321			
6	2.062	5.997	4.110	30.353	2.935	0.437			
7	2.465	8.976	5.730	59.786	3.829	−1.646			
8	3.006	13.446	8.220	117.835	5.083	−5.911			

mined, we want to expand in powers of g, our coupling constant, and not in terms of ϵ.

If we take this point of view, then we must take the expression from Ref. 3 for $\beta(g)$ in terms of g and ϵ and solve for a numerical value of g [actually $g^2/(8\pi)^2$]. Let g_1 be a fixed point of $\beta(g)$. Equation (73) of Ref. 9 tells us

$$\beta(g_1)=0=g_1\left[-\frac{\epsilon}{4}+\frac{g_1^2}{(8\pi)^2}[\tfrac{3}{2}+\epsilon(\tfrac{15}{16}+\tfrac{5}{2}\ln 2+\tfrac{3}{4}\delta)] -\frac{g_1^4}{(8\pi)^4}(\tfrac{157}{32}+\tfrac{53}{16}\ln\tfrac{4}{3})\right]. \quad (70)$$

Set $\epsilon=2$, and assuming $g_1\neq 0$, one obtains

$$\left[\frac{g_1^2}{(8\pi)^2}\right]_{\pm}=\begin{cases}0.9247\\0.0923\,.\end{cases} \quad (71)$$

Since we want an infrared fixed point we must pick

$$\frac{g_1^2}{(8\pi)^2}=0.0923\,. \quad (72)$$

We must now express γ in terms of g. From Ref. 3 [Eq. (76)],

$$-\gamma=\tfrac{1}{2}(1+3\ln 2+\delta)\frac{g^2}{(8\pi)^2} +(\tfrac{5}{2}\ln 2-\tfrac{9}{4}\ln 3+\tfrac{5}{8})\frac{g^4}{(8\pi)^4} \quad (73)$$

$$=h_1\frac{g^2}{(8\pi)^2}-h_2\frac{g^4}{(8\pi)^4}\,, \quad (74)$$

where in (73) we set $\epsilon=2$ and (74) defines h_1 and h_2.

We now define α_1 as

$$\alpha_1=\frac{g_1^2}{(8\pi)^2}=0.0923\,. \quad (75)$$

Return to Eq. (47), expand in ϵ, and set $\epsilon=2$ to obtain

$$-i\Gamma_R^{(1,1)}=E_N\left[-\frac{E}{E_N}\right]^{1-\gamma}\times\left[1-x-\sum_{n=0}^{\infty}x^n(\alpha_1 a_n'+\alpha_1^2 b_n')\right], \quad (76)$$

where

$$a_n'=a_n+e_n+a_n(\delta+3\ln 2)\,, \quad (77)$$

$$b_n'=f_n\,. \quad (78)$$

Invert and expand in α_1 exactly as before in terms of λ to obtain

$$iE_N G_R^{(1,1)}(E,0,z)=\left[-\frac{E}{E_N}\right]^{-(1-\gamma)}\sum_{n=0}^{\infty}(1+\alpha_1 c_n'+\alpha_1^2 d_n')(z-1)^n\left[-\frac{b}{E_N}\right]^n\left[-\frac{E}{E_N}\right]^{-n(1-\gamma)}, \quad (79)$$

where

$$c_n'=\sum_{k=0}^{n}a_k'(n-k+1)\,, \quad (80)$$

$$d_n'=\sum_{k=0}^{n}\left[b_k'(n-k+1)+\tfrac{1}{2}(n-k+1)(n-k+2)\sum_{m=0}^{k}a_m'a_{k-m}'\right]. \quad (81)$$

These are tabulated up to $n=8$ in Table IV. Proceeding exactly as before, one can obtain the following for C_p, in a Padé approximant,

$$C_p=\frac{N}{D}\,, \quad (82)$$

where

$$N=1+\alpha_1[c_p+(p-1)c_0+2ph_1(\psi_1+1)] +\alpha_1^2\{d_p+(p-1)d_0+\tfrac{1}{2}(p-1)(p-2)c_0^2+(p-1)c_pc_0+2ph_1[c_p+(p-1)c_0](\psi_1+1) +2p^2h_1^2(\psi_1+1)^2+2p[h_1^2(\psi_1'-1)-h_2(\psi_1+1)]\}\,, \quad (83)$$

$$D=1+\alpha_1\{pc_1+h_1[(p+1)S_p^{(1)}+2p\psi_1]\} +\alpha_1^2\{pd_1+\tfrac{1}{2}(p-1)c_1^2+pc_1h_1[(p+1)S_p^{(1)}+2p\psi_1]+\tfrac{1}{2}(p+1)^2h_1^2(S_p^{(1)2}-S_p^{(2)}) -h_2(p+1)S_p^{(1)}-2ph_2\psi_1+2h_1^2p^2\psi_1^2+2h_1^2p(p+1)\psi_1S_p^{(1)}+\tfrac{1}{2}h_1^2\psi_1'p(p+3)\}\,. \quad (84)$$

TABLE IV. Coefficients of x^n. The a_n' and b_n' are the order-α_1 and order-α_1^2 [$\alpha_1 = g_1^2/(8\pi)^2 = 0.0923$] coefficients for the expansion of Γ_R. The c_n' and d_n' are the order-α_1 and order-α_1^2 coefficients for the expansion of $G_R = (\Gamma_R)^{-1}$.

n	a_n'	b_n'	c_n'	d_n'
0	2.3235	−1.864	2.3235	3.5345
1	0	0	4.6470	12.4676
2	3.6470	3.3611	10.6174	47.1076
3	4.6470	4.6673	21.2348	130.355
4	6.0048	10.8869	37.8569	309.633
5	8.2204	24.3594	62.6996	670.509
6	11.843	52.2078	99.3847	1361.26
7	17.791	109.756	153.861	2637.86
8	27.654	229.548	235.991	4947.47

The values for C_p are listed in Table III under the column heading g Padé.

If one now uses Eq. (69), replacing λ by α_1, one can obtain an expression for C_p with D expanded into N up to order α_1^2. The results for the multiplicity moments C_p are listed in Table III under column heading g expanded.

One may verify that in both the ϵ-expanded and g-expanded analytic expressions, the dependence of C_p on $\psi_1 = -\gamma_{\rm EM} = -0.577\,216$ completely cancels out. [The factor ψ_l comes from expanding Γ functions. These occur in the combination

$$F = \Gamma^p(2(1-\gamma))/\Gamma^{p-1}(1-\gamma)\Gamma((p+1)(1-\gamma)) .$$

Using the identity

$$\psi(1+Z) = \psi_1 + \sum_{n=2}^{\infty} (-1)^n \zeta(n) Z^{n-1} = \psi_1 + r(Z) ,$$

where $\psi(z) = d/dz \ln\Gamma(z)$, one may show that F is independent of ψ_1.]

For completeness we wish to give the analytic form for C_p to first order. This was the case considered in Refs. 1 and 2. Let

$$a_p = \sum_{k=2}^{p} (p-k+1) \frac{2^k - 2}{k(k-1)} , \qquad (85)$$

and let $S_p^{(1)}$ be as in Eq. (66). Then to first order

$$C_p = \frac{1 + \lambda[a_p + p(\psi_1 + 2)]}{1 + \lambda[(p+1/2) S_p^{(1)} + p(\psi_1 + 1)]} \qquad (86)$$

for the Padé method and

$$C_p = 1 + \lambda \left[a_p + p - \frac{p+1}{2} S_p^{(1)} \right] \qquad (87)$$

for the ϵ expansion. As before $\lambda = \epsilon/6$.

The multiplicity moments calculated by both methods are given in Table III under major column heading "first order" and minor column headings "Padé" and "ϵ expanded". Notice that to first order there is no distinction between an ϵ expansion and a $g^2/(8\pi)^2$ expansion. Thus, both methods to second order are to be compared to the same values in first order.

IV. DISCUSSION

We have calculated the multiplicity moments to second order by two methods, in each case giving the final expression in the form of a Padé approximant as well as a formula for expanding the denominator into a power series to obtain what we have termed the "expanded" form. The full summary of our labors is given in Table III. The experimental values given under the column heading β_p are taken from Alpgard *et al.*[11] As these numbers are measurements of $\langle n^p \rangle / \langle n \rangle^p$, they are not suitable for direct comparison to our

$$C_p = \langle n(n-1) \cdots (n-p+1) \rangle / \langle n \rangle^p$$

except when $\langle n \rangle \to \infty$, i.e., $s \to \infty$. However, by choosing some suitable model for the basis emission unit along the multiperipheral chain for the bare Pomeron, we may "correct" these numbers for direct comparison.

For illustrative purposes, two emission ansatz are considered: single-pion emission and the nominal 3π-cluster emissions. A comparison between the theory and the corrected experimental data at $\sqrt{s} = 540$ GeV, taking into account the emission ansatz, is shown under the columns headed π and 3π models.

One can see the best comparison is in the first-order expanded terms. Indeed, it was this close agreement which prompted our investigation of whether this would hold to higher order in perturbation theory.

The most discouraging fact is not that we get poor agreement to experiment, but that even with the large number of prescriptions we have given for calculating C_p, not one of them seems to give sensible results in terms of perturbation theory. For each method the second-order contributions are as large as or larger than the first-order contribution. This leads us to doubt whether the ϵ expansion even converges. Perhaps it is in the nature of an asymptotic series in which the first term gives reasonably good results but further terms in the series lead to poorer and poorer agreement. In the context of this optimistic interpretation, the straightforward expansion is somehow preferred over the Padé method.

Our conclusions here, together with our comments in the Introduction approximately summarize the present status of the phenomenological applications of the critical Pomeron solution of the Reggeon field theory to hadron collisions. In closing we mention that currently there is yet another approach to hadron phenomenology which appears to be promising. It is based on the so-called modified eikonal model, which in the context of Reggeon field theory corresponds to the strong-coupling solution of the theory. We refer the interested reader to Ref. 12 and also to Ref. 10 together with the references quoted therein for details.

ACKNOWLEDGMENTS

One of us (C.B.C.) would like to thank P. Suranyi and A. White for useful conversations. Thanks also go to Professor George Sudarshan for careful reading of the manuscript. The work done is supported in part by the Department of Energy under Contract Number DE-AS05-76ER03992, and is also supported in part by a National Science Foundation Graduate Fellowship.

APPENDIX

In this appendix we will calculate the integrals corresponding to the second- and fourth-order correction diagrams. From these it will be possible to compute the renormalization function Z and so obtain the renormalized inverse propagator $\Gamma_R^{(1,1)}$.

For diagram C the Feynman rules give us

$$\Gamma_C^{(1,1)}(E,k^2,z)=f_C i^2\left[\frac{-ir_0}{(2\pi)^{(D+1)/2}}\right]^2 \tfrac{1}{2}\int d^Dk_1 dE_1(E_1-\alpha_0' k_1{}^2-\Delta_q+i\epsilon)^{-1}[E-E_1-\alpha_0'(\vec{k}-\vec{k}_1)^2-\Delta_r+i\epsilon]^{-1} . \tag{A1}$$

Use Cauchy's theorem to evaluate the integral over E_1 and the formulas of 't Hooft and Veltman[13] to evaluate the integral over $\vec{k}_1$. One obtains, at $k^2=0$,

$$\Gamma_C^{(1,1)}(E,0,z)=\frac{if_C r_0{}^2}{2(8\pi\alpha_0')^{D/2}}\Gamma\left[1-\frac{D}{2}\right](-E+\Delta_q+\Delta_r)^{D/2-1} . \tag{A2}$$

Recall the definition of g_0 from Eq. (22) and let $\epsilon=4-D$. Then Eq. (A2) becomes

$$\Gamma_C^{(1,1)}(E,0,z)=if_C E_N\frac{g_0{}^2}{2(8\pi)^2}(8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}-1\right]\left[\frac{-E+\Delta_q+\Delta_r}{E_N}\right]^{1-\epsilon/2} . \tag{A3}$$

Turning to diagram A, we have from the Feynman rules,

$$\begin{aligned}\Gamma_A^{(1,1)}(E,0,z)=f_A i^5\left[\frac{-ir_0}{(2\pi)^{(D+1)/2}}\right]^4 \tfrac{1}{2}\int d^Dk_1 d^Dk_2 dE_1 dE_2&(E_1-\alpha_0'\vec{k}_1{}^2-\Delta_a+i\epsilon)^{-2}\\ &\times[E_1-E_2-\alpha_0'(\vec{k}_1-\vec{k}_2)^2-\Delta_b+i\epsilon]^{-1}\\ &\times(E_2-\alpha_0'\vec{k}_2{}^2-\Delta_c+i\epsilon)^{-1}(E-E_1-\alpha_0'\vec{k}_1{}^2-\Delta_d+i\epsilon)^{-1} ,\end{aligned} \tag{A4}$$

where we have used the fact that $a=e$ for each possible cutting of diagram A. Use Cauchy's theorem twice and introduce a Feynman parameter x to collect denominators to obtain

$$\begin{aligned}\Gamma_A^{(1,1)}(E,0,z)=\frac{2if_A r_0{}^4}{2(2\pi)^{2D}(2\alpha_0')^3}\int_0^1 &dx\,(1-x)d^Dk_1 d^Dk_2\\ &\times\left[\vec{k}_1{}^2+x\vec{k}_2{}^2-x\vec{k}_1\cdot\vec{k}_2+\frac{1}{2\alpha_0'}[-E+\Delta_a+\Delta_d+x(-\Delta_a+\Delta_b+\Delta_c)]\right]^{-3} .\end{aligned} \tag{A5}$$

We make use of the following integral, given in Ref. 3:

$$\int d^Dk_1 d^Dk_2(a\vec{k}_1{}^2+\vec{b}k_2{}^2+c\vec{k}_1\cdot\vec{k}_2+d+e\vec{k}\cdot\vec{k}_1+f\vec{k}\cdot\vec{k}_2)^{-\sigma}=(2\pi)^D\tilde{d}^{\,D-\sigma}\Gamma(\sigma-D)(4ab-c^2)^{-D/2}/\Gamma(\sigma) , \tag{A6}$$

where

$$\tilde{d}=d-\frac{\vec{k}^2}{4ab-c^2}(be^2+af^2-cef) . \tag{A7}$$

The result of using this, inserting the dimensionless constant g_0, and letting $\epsilon=4-D$ is

$$\Gamma_A^{(1,1)}(E,0,z)=if_A\frac{g_0{}^4}{2(4\pi)^4}(4\pi E_N)^{\epsilon}\Gamma(\epsilon-1)\int_0^1 dx(1-x)(4x-x^2)^{\epsilon/2-2}[-E+\Delta_a+\Delta_d+x(-\Delta_a+\Delta_b+\Delta_c)]^{1-\epsilon} . \tag{A8}$$

Leaving Γ_A in its present form we turn to diagram B. We have

$$\begin{aligned}\Gamma_B^{(1,1)}(E,0,z)=f_B i^5\left[\frac{-ir_0}{(2\pi)^{(D+1)/2}}\right]^4\int d^Dk_1 d^Dk_2 dE_1 dE_2&(E_1-\alpha_0'\vec{k}_1{}^2-\Delta_l+i\epsilon)^{-1}\\ &\times[E_1+E_2-\alpha_0'(\vec{k}_1+\vec{k}_2)^2-\Delta_m+i\epsilon]^{-1}(E_2-\alpha_0'\vec{k}_2{}^2-\Delta_n+i\epsilon)^{-1}\\ &\times[E-E_1-E_2-\alpha_0'(\vec{k}_1+\vec{k}_2)^2-\Delta_o+i\epsilon]^{-1}(E-E_1-\alpha_0'\vec{k}_1{}^2-\Delta_p+i\epsilon)^{-1} .\end{aligned} \tag{A9}$$

Using Cauchy's theorem twice we have

$$\Gamma_B^{(1,1)}(E,0,z)=\frac{-if_B r_0^{\,4}}{(2\pi)^{2D}}\int d^Dk_1 d^Dk_2[E-2\alpha_0'(\vec{k}_1+\vec{k}_2)^2-\Delta_m-\Delta_o]^{-1}$$
$$\times[E-2\alpha_0'(\vec{k}_1^{\,2}+\vec{k}_2^{\,2}+\vec{k}_1\cdot\vec{k}_2)-\Delta_n-\Delta_o-\Delta_l]^{-1}(E-2\alpha_0'\vec{k}_1^{\,2}-\Delta_p-\Delta_l)^{-1}\,. \tag{A10}$$

Collect denominators with Feynman parameters x and y, use the identity of Eq. (A6), and let $\epsilon=4-D$ to obtain

$$\Gamma_B^{(1,1)}(E,0,z)=if_B\frac{g_0^{\,4}}{(4\pi)^4}(4\pi E_N)^\epsilon\Gamma(\epsilon-1)\int_0^1 dx\int_0^{1-x}dy[3-2(x+y)-(x-y)^2]^{\epsilon/2-2}[c+x(a-c)+y(b-c)]^{1-\epsilon}\,, \tag{A11}$$

where

$$a=-E+\Delta_l+\Delta_p,\quad b=-E+\Delta_m+\Delta_o,\quad c=-E+\Delta_l+\Delta_n+\Delta_o\,. \tag{A12}$$

Change variables by letting $u=x+y$ and $v=x-y$ and defining

$$\alpha=\frac{1}{2E_N}(a+b-2c)=\frac{1}{2E_N}(-\Delta_l+\Delta_m-\Delta_o+\Delta_p-2\Delta_n)\,,$$
$$\beta=\frac{1}{2E_N}(a-b)=\frac{1}{2E_N}(\Delta_l-\Delta_m-\Delta_o+\Delta_p)\,. \tag{A13}$$

One obtains

$$\Gamma_B^{(1,1)}(E,0,z)=if_B\frac{g_0^{\,4}}{2(4\pi)^4}(4\pi E_N)^\epsilon\Gamma(\epsilon-1)$$
$$\times\left[\int_0^1 dv\int_v^1 du+\int_{-1}^0 dv\int_{-v}^0 du\right](3-2u-v^2)^{(\epsilon-2)/2}[c+E_N(\alpha u+\beta v)]^{1-\epsilon}\,. \tag{A14}$$

As shown in the text, summing the Dyson series leads to the following expression for the unrenormalized inverse propagator:

$$i\Gamma^{(1,1)}=E-b_0(z-1)-2i\left[\sum_A\Gamma_A+\sum_B\Gamma_B+\sum_C\Gamma_C\right]. \tag{A15}$$

Applying Eq. (17), we have for Z^{-1}

$$Z^{-1}=1+\frac{g_0^{\,2}}{(8\pi)^2}(8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}\right]\sum_C f_C$$
$$+\frac{g_0^{\,4}}{(4\pi)^4}(4\pi)^\epsilon\Gamma(\epsilon)\left[\sum_A f_A\int_0^1 dx(1-x)(4x-x^2)^{\epsilon/2-2}\right.$$
$$\left.+\sum_B f_B\left[\int_0^1 dv\int_v^1 du+\int_{-1}^0 dv\int_{-v}^1 du\right](3-2u-v^2)^{\epsilon/2-2}\right]. \tag{A16}$$

Thus,

$$Z=1-\frac{g_0^{\,2}}{(8\pi)^2}(8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}\right]\sum_C f_C+\frac{g_0^{\,4}}{(8\pi)^4}\left[\Gamma\left[\frac{\epsilon}{2}\right]\sum_C f_C\right]^2$$
$$-\frac{g_0^{\,4}}{(4\pi)^4}(4\pi)^\epsilon\Gamma(\epsilon)\left[\sum_A f_A\int_0^1 dx(1-x)(4x-x^2)^{\epsilon/2-2}\right.$$
$$\left.+\sum_B f_B\left[\int_0^1 dv\int_v^1 du+\int_{-1}^0 dv\int_{-v}^1 du\right](3-2u-v^2)^{\epsilon/2-2}\right]. \tag{A17}$$

We pause in our discussion to note a consequence of the cutting rules on the value of b. The renormalized parameter b was defined in Eq. (21) as

$$\frac{b}{Z}=-\frac{\partial}{\partial z}i\Gamma^{(1,1)}(E,k^2,z)\Bigg|_{\substack{k^2=0\\E=-E_N\\z=1}} . \tag{A18}$$

From Eq. (A15), and Eqs. (A3), (A8), and (A14), we see that the right-hand side of Eq. (A18) is equal to b_0 plus terms linear in the Δ's. It is a general consequence of the cutting rules that the sum of the weights of diagrams with a given line being cut is zero. This may be explicitly verified for the diagrams under consideration by glancing at Fig. 3. Thus, Eq. (A18) assures us that

$$\frac{b}{Z}=b_0 . \tag{A19}$$

We now write down the renormalized inverse propagator $\Gamma_R^{(1,1)}=Z\Gamma^{(1,1)}$ as

$$\begin{aligned}
-i\Gamma_R^{(1,1)}(-E_N,0,z)
&=E_N\Bigg\{1+\frac{b}{E_N}(z-1)+\frac{g_0^2}{(8\pi)^2}(8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}\right]\sum_C f_C\left[\frac{(1+\xi Z^{-1})^{1-\epsilon/2}}{1-\epsilon/2}-1\right]\\
&\quad-\frac{g_0^4}{(8\pi)^4}(8\pi)^{\epsilon}\left[\Gamma\left[\frac{\epsilon}{2}\right]\right]^2\left[\sum_C f_C\right]\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\epsilon/2}-1\right]\\
&\quad+\frac{g_0^4}{(4\pi)^4}(4\pi)^{\epsilon}\Gamma(\epsilon)\Bigg[\sum_A f_A\int_0^1 dx(1-x)(4x-x^2)^{\epsilon/2-2}\\
&\qquad\times\left[\frac{\{1+(b/E_N)(z-1)[a+d+x(-a+b+c)]\}^{1-\epsilon}}{1-\epsilon}-1\right]\\
&\qquad+\sum_B f_B\left[\int_0^1 dv\int_v^1 du+\int_{-1}^0 dv\int_{-v}^1 du\right](3-2u-v^2)^{\epsilon/2-2}\\
&\qquad\times\left[\frac{\{1+(b/E_N)(z-1)[l+n+o+\frac{1}{2}u(p+m-l-o-2n)+\frac{1}{2}v(-p+m+l-o)]\}^{1-\epsilon}}{1-\epsilon}-1\right]\Bigg]\Bigg\},
\end{aligned} \tag{A20}$$

where $\xi=(b/E_N)(z-1)(q+r)$. Define

$$I(C,D)=\int_0^1 dx(1-x)(4x-x^2)^{\epsilon/2-2}\left[\frac{(C+Dx)^{1-\epsilon}}{1-\epsilon}-1\right], \tag{A21}$$

$$J(G,\alpha,\beta)=\int_0^1 dv\int_v^1 du(3-2u-v^2)^{\epsilon/2-2}\left[\frac{(G+\alpha u+\beta v)^{1-\epsilon}}{1-\epsilon}-1\right], \tag{A22}$$

where

$$C=1+\frac{b}{E_N}(z-1)(a+d) , \tag{A23}$$

$$D=\frac{b}{E_N}(z-1)(-a+b+c) , \tag{A24}$$

$$G=1+\frac{b}{E_N}(z-1)(l+n+o) , \tag{A25}$$

$$\alpha=\frac{b}{2E_N}(z-1)(p+m-l-o-2n) , \tag{A26}$$

$$\beta=\frac{b}{2E_N}(z-1)(p-m+l-o) . \tag{A27}$$

Further, let

$$K=\frac{{g_0}^2}{(8\pi)^2}(8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}\right]. \tag{A28}$$

Then we have

$$-i\Gamma_R^{(1,1)}(-E_N,0,z)=E_N\left[1+\frac{b}{E_N}(z-1)+K\sum_C f_C\left[\frac{(1+\xi Z^{-1})^{1-\epsilon/2}}{1-\epsilon}-1\right]-K^2\left[\sum_C f_C\right]\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\epsilon/2}-1\right]\right.$$
$$\left.+\frac{{g_0}^4}{(4\pi)^4}(4\pi)^\epsilon\Gamma(\epsilon)\left[\sum_A f_A I(C,D)+\sum_B f_B[J(G,\alpha,\beta)+J(G,\alpha,-\beta)]\right]\right]. \tag{A29}$$

We now wish to remove the Z^{-1} in the order-K (order-${g_0}^2$) term. To this end we note that to order ${g_0}^2$, Z^{-1} is

$$Z^{-1}=1+K\sum_C f_C\,. \tag{A30}$$

Putting this into the order-K term and expanding to order K^2 gives us

$$K\sum_C f_C\left[\frac{(1+\xi Z^{-1})^{1-\epsilon/2}}{1-\epsilon/2}-1\right]=K\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\epsilon/2}-1\right]+K^2\left[\sum_C f_C\right]\sum_C f_C\xi(1+\xi)^{-\epsilon/2}\,. \tag{A31}$$

We recall from Ref. 3 that to order g^4

$$\frac{{g_0}^2}{(8\pi)^2}=\frac{g^2}{(8\pi)^2}\left[1+\frac{g^2}{(8\pi)^2}\left[\frac{6}{\epsilon}+(\tfrac{15}{4}+5\ln 2+3\delta)\right]\right], \tag{A32}$$

where

$$\delta=\ln\pi-\gamma_{\rm EM}\,. \tag{A33}$$

Thus, $\Gamma_R^{(1,1)}$ can be written completely in terms of renormalized quantities as

$$-i\Gamma_R^{(1,1)}(-E_N,0,z)=E_N\left\{1+\frac{b}{E_N}(z-1)+\frac{g^2}{(8\pi)^2}(8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}\right]\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\epsilon/2}-1\right]\right.$$
$$+\frac{g^4}{(8\pi)^4}\left[\frac{6}{\epsilon}+(\tfrac{15}{4}+5\ln 2+3\delta)\right](8\pi)^{\epsilon/2}\Gamma\left[\frac{\epsilon}{2}\right]\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\epsilon/2}-1\right]$$
$$-\frac{g^4}{(8\pi)^4}(8\pi)^\epsilon\left[\Gamma\left[\frac{\epsilon}{2}\right]\right]^2\left[\sum_C f_C\right]\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\epsilon/2}-1-\xi(1+\xi)^{-\epsilon/2}\right]$$
$$\left.+\frac{g^4}{(4\pi)^4}(4\pi)^\epsilon\Gamma(\epsilon)\left[\sum_A f_A I(C,D)+\sum_B f_B[J(G,\alpha,\beta)+J(G,\alpha,-\beta)]\right]\right\}. \tag{A34}$$

It remains to calculate $I(C,D)$ and $J(G,\alpha,\beta)$. We will treat $I(C,D)$, defined in Eq. (A21), first. One extracts the singularities in ϵ by using the following identities:

$$\int_0^1 dx\,x^{\epsilon/2-1}f(x)=\frac{2}{\epsilon}f(0)-\int_0^1 dx\,\ln(x)f'(x)\,, \tag{A35}$$

$$\int_0^1 dx\,x^{\epsilon/2-2}f(x)=\left[\frac{2}{\epsilon}+1\right]f'(0)-f(1)-\int_0^1 dx\,\ln(x)f''(x)\,, \tag{A36}$$

where for $I(C,D)$

$$f(x)=(4-x)^{\epsilon/2-2}\left[\frac{(C+Dx)^{1-\epsilon}}{1-\epsilon}-1\right]. \tag{A37}$$

At some steps we will make use of the fact noted above, namely, that the sum over weights of terms linear in the Δ's is zero. This allows us to ignore certain terms. In particular,

$$\frac{(C+Dx)^{1-\epsilon}}{1-\epsilon}-1=\frac{C+DX}{1-\epsilon}\left[1-\epsilon\ln(C+Dx)+\frac{\epsilon^2}{2}\ln^2(C+Dx)+\cdots\right]-1$$
$$=C+Dx-1+O(\epsilon).$$

We may set $C+Dx$ equal to 1 since $C=1+$ terms linear in Δ's and thus will disappear later in the sum over weights. We now apply Eqs. (A35) and (A36) to Eq. (A21) for $I(C,D)$.

After considerable algebra one finds

$$I(C,D)=\tfrac{1}{16}((C-2D)\ln C-C+\tfrac{1}{2}\epsilon\{\ln C[(C-2D)(1+2\ln 2)+\tfrac{1}{2}C]$$
$$-(C-2D)\ln^2 C+\tfrac{32}{9}(C+D)\ln(C+D)+C(\ln 3-4\ln 2-5)$$
$$-D(6\ln\tfrac{3}{4}+\tfrac{16}{3})+32\tilde{I}(C,D)\}), \tag{A38}$$

where

$$\tilde{I}(C,D)=\int_0^1 dx\,\ln(x)\{D^2(4-x)^{-2}(C+Dx)^{-1}+\ln(C+Dx)[D(4-x)^{-2}-2(C+5D)(4-x)^{-3}+6(C+4D)(4-x)^{-4}]\}. \tag{A39}$$

In obtaining $I(C,D)$ use has been made of the following integrals:

$$\int_0^1 dx\frac{\ln(x)}{(4-x)^2}=\tfrac{1}{4}\ln\tfrac{3}{4}, \tag{A40}$$

$$\int_0^1 dx\frac{\ln(x)}{(4-x)^3}=\tfrac{1}{32}(\ln\tfrac{3}{4}-\tfrac{1}{3}), \tag{A41}$$

$$\int_0^1 dx\frac{\ln(x)}{(4-x)^4}=\tfrac{1}{192}(\ln\tfrac{3}{4}-\tfrac{1}{3}-\tfrac{7}{18}). \tag{A42}$$

Using the above and the following formulas,

$$\int_0^1 dx\frac{\ln(x)}{(4-x)^2}\ln(C+Dx)=\tfrac{1}{4}\ln 3\ln(C+D)-\tfrac{1}{4}\ln 4\ln C+\frac{D}{4}\int_0^1 dx\frac{\ln x-\ln(4-x)}{C+Dx}-D\int_0^1 dx\frac{\ln(x)}{(C+Dx)(4-x)}, \tag{A43}$$

$$\int_0^1 dx\frac{\ln(x)}{(4-x)^3}\ln(C+Dx)=\left[\frac{\ln 3}{32}-\frac{1}{24}\right]\ln(C+D)-\left[\frac{\ln 4-1}{32}\right]\ln C$$
$$+\frac{D}{32}\int_0^1 dx\frac{\ln x-\ln(4-x)}{C+Dx}+\frac{D}{8}\int_0^1 dx\frac{1}{(C+Dx)(4-x)}-\frac{D}{2}\int_0^1 dx\frac{\ln(x)}{(C+Dx)(4-x)^2}, \tag{A44}$$

$$\int_0^1 dx\frac{\ln(x)}{(4-x)^4}\ln(C+Dx)=\left[\frac{\ln 3}{192}-\frac{5}{192}\right]\ln(C+D)-\left[\frac{\ln 4}{192}-\frac{3}{384}\right]\ln C$$
$$+\frac{D}{192}\int_0^1 dx\frac{\ln x-\ln(4-x)}{C+Dx}-\frac{D}{3}\int_0^1 dx\frac{\ln x}{(C+Dx)(4-x)^3}$$
$$+\frac{D}{48}\int_0^1 dx\frac{1}{(C+Dx)(4-x)}+\frac{D}{24}\int_0^1 dx\frac{1}{(C+Dx)(4-x)^2}, \tag{A45}$$

$$\frac{1}{(C+Dx)(4-x)}=\frac{1}{C+4D}\left[\frac{D}{C+Dx}+\frac{1}{4-x}\right], \tag{A46}$$

$$\frac{1}{(C+Dx)(4-x)^2}=\frac{1}{C+4D}\left[\frac{D^2}{C+4D}\frac{1}{C+Dx}+\frac{D}{C+4D}\frac{1}{4-x}+\frac{1}{(4-x)^2}\right], \tag{A47}$$

$$\frac{1}{(C+Dx)(4-x)^3}=\frac{1}{C+4D}\left[\frac{D^3}{(C+4D)^2}\frac{1}{C+Dx}+\frac{D^2}{(C+4D)^2}\frac{1}{4-x}+\frac{D}{C+4D}\frac{1}{(4-x)^2}+\frac{1}{(4-x)^3}\right]. \tag{A48}$$

One may, after some algebra, obtain the following for $\tilde{I}(C,D)$:

$$\tilde{I}(C,D)=\tfrac{1}{32}\ln C[(C-2D)2\ln 2-\tfrac{1}{2}C]-\tfrac{1}{32}\ln(C+D)[(C-2D)\ln 3-\tfrac{4}{3}(C+D)]$$
$$+\tfrac{1}{32}D(10\ln\tfrac{3}{4}+\tfrac{4}{3})-\tfrac{1}{32}D(C-2D)\int_0^1 dx\frac{\ln x-\ln(4-x)}{C+Dx}. \tag{A49}$$

Plugging into Eq. (A40) and changing the dummy variable in the integral from x to y, we obtain

$$I(C,D)=\tfrac{1}{16}[(C-2D)\ln C-C]+\frac{\epsilon}{32}\Bigg[\ln C(C-2D)(1+4\ln 2)-(C-2D)\ln^2 C$$
$$-\ln(C+D)[(C-2D)\ln 3-4(C+D)]-C(5+4\ln 2-\ln 3)$$
$$-4D(1-\ln\tfrac{3}{4})-D(C-2D)\int_0^1 dx\frac{\ln(y)-\ln(4-y)}{C+Dy}\Bigg]. \tag{A50}$$

For $J(G,\alpha,\beta)$ we take Eq. (A22) and integrate by parts once to obtain

$$J(G,\alpha,\beta)=\frac{1}{2-\epsilon}\int_0^1 dv\Bigg[(1-v^2)^{\epsilon/2-1}\left[\frac{(G+\alpha+\beta v)^{1-\epsilon}}{1-\epsilon}-1\right]$$
$$-(3-2v-v^2)^{\epsilon/2-1}\left[\frac{[G+(\alpha+\beta)v]^{1-\epsilon}}{1-\epsilon}-1\right]-\int_v^1 du(3-2u-v^2)^{\epsilon/2-1}(G+\alpha u+\beta v)^{-\epsilon}\Bigg]. \tag{A51}$$

In the last term expand

$$(G+\alpha u+\beta v)^{-\epsilon}=1-\epsilon\ln(G+\alpha u+\beta v).$$

Note that we have an α in front and $\sum_B f_B\alpha=0$, hence, we may ignore the term involving 1. Use the identity

$$\int_0^1 dv(1-v)^{\epsilon/2-1}f(v)=\frac{2}{\epsilon}f(1)+\int_0^1 dv\ln(1-v)f'(v) \tag{A52}$$

with

$$f(v)=(1+v)^{\epsilon/2-1}\left[\frac{(G+\alpha+\beta v)^{1-\epsilon}}{1-\epsilon}-1\right]-(3+v)^{\epsilon/2-1}\left[\frac{[G+(\alpha+\beta)v]^{1-\epsilon}}{1-\epsilon}-1\right] \tag{A53}$$

to obtain, after a moderate amount of algebra,

$$J(G,\alpha,\beta)=\tfrac{1}{4}\Bigg[\{(G+\alpha+\beta)[1-\ln(G+\alpha+\beta)]\}+\frac{\epsilon}{2}\Bigg[(G+\alpha+\beta)[3-3\ln(G+\alpha+\beta)+\ln^2(G+\alpha+\beta)]$$
$$+\ln 3G(1-\ln G)+4\alpha\int_0^1 dv\int_v^1 du\frac{\ln(G+\alpha u+\beta v)}{3-2u-v^2}$$
$$+\int_0^1 dv\{2\beta\ln(G+\alpha+\beta v)[\ln(1+v)-\ln(1-v)]$$
$$-(\alpha+\beta)\ln[G+(\alpha+\beta)v][\ln(3+v)-\ln(1-v)]\}\Bigg]\Bigg]. \tag{A54}$$

We are now ready to expand in powers of $z-1$. To this end we define

$$x=\frac{b}{E_N}(z-1). \tag{A55}$$

Since we are interested in $z \simeq 1$ we treat x as a small parameter. In terms of x we have

$$C = 1 - x(a+d) , \tag{A56}$$

$$D = -x(-a+b+c) . \tag{A57}$$

Now consider the sum over weights of $I(C,D)$. The following sums hold:

$$\sum_A f_A (C-2D)\ln C = 2(1-2x)\ln(1-x) - (1-4x)\ln(1-2x) , \tag{A58}$$

$$\sum_A f_A (C+d)\ln(C+D) = 6(1-x)\ln(1-x) - 6(1-2x)\ln(1-2x) + 2(1-3x)\ln(1-3x) , \tag{A59}$$

$$\sum_A f_A (C-2D)\ln(C+D) = (6-9x)\ln(1-x) - 6(1-x)\ln(1-2x) + 2\ln(1-3x) , \tag{A60}$$

$$\sum_A f_A (C-2D)\ln^2 C = 2(1-2x)\ln^2(1-x) - (1-3x)\ln^2(1-2x) . \tag{A61}$$

Let

$$A_{\text{int}}(x) = \sum_A f_A D(C-2D) \int_0^1 dy \frac{\ln y - \ln(4-y)}{C+Dy} . \tag{A62}$$

By expanding

$$1/(C+Dy) = 1/\{1 - x[a+d+y(a+b+c)]\}$$

in a geometric series, integrating, and summing over weights one can obtain the values listed in Table V for $A_{\text{int}}(x)$.

By using Eqs. (A58) and (A61) one can obtain

$$\begin{aligned}16 \sum_A f_A I(C,D) = {} & -\tfrac{1}{2} + 2(1-2x)\ln(1-x) - (1-4x)\ln(1-2x) \\ & + \frac{\epsilon}{2}\{ -A_{\text{int}}(x) - \tfrac{1}{2}(5 + 4\ln 2 - \ln 3) - 2(1-2x)\ln^2(1-x) + (1-4x)\ln^2(1-2x) \\ & + \ln(1-x)[(1-x)(28 + 16\ln^2 - 9\ln 3) - (2 + 8\ln 2 - 3\ln 3)] \\ & - \ln(1-2x)[(1-2x)(26 + 8\ln 2 - 3\ln 3) - (1 + 4\ln 2 + 3\ln 3)] + \ln(1-3x)[(1-3x)8 - 2\ln 3]\} .\end{aligned} \tag{A63}$$

Now let

$$B_{D\,\text{int}}(x) = \sum_B f_B 4\alpha \int_0^1 dv \int_v^1 du \frac{\ln(G+\alpha u+\beta v)}{3-2u-v^2} + (\beta \to -\beta) , \tag{A64}$$

$$\begin{aligned}B_{S\,\text{int}}(x) = \sum_B f_B \int_0^1 dv \{ & 2\beta \ln(G+\alpha+\beta v)[\ln(1+v) - \ln(1-v)] \\ & - (\alpha+\beta)\ln[G+(\alpha+\beta)v][\ln(3+v) - \ln(1-v)]\} + (\beta \to -\beta) .\end{aligned} \tag{A65}$$

TABLE V. Results up to $n=8$ of evaluating the integrals left over in the evaluation of $I(C,D)$ and $J(G,\alpha,\beta)$. B_{int} is the sum of $B_{S\,\text{int}}$ and $B_{D\,\text{int}}$.

n	A_{int}	$B_{D\,\text{int}}$	$B_{S\,\text{int}}$	B_{int}
0	0.0	0.0	0.0	0.0
1	0.0	0.0	0.0	0.0
2	0.0	−1.4522	1.4987	0.0465
3	16.048	−5.291	8.353	3.062
4	52.555	−12.481	21.901	9.420
5	144.42	−27.225	51.792	24.567
6	379.80	−58.431	120.01	61.583
7	989.11	−125.79	278.87	153.08
8	2581.9	−273.66	655.03	381.36

Let $B_{\rm int}(x)$ be the sum of $B_{D\,\rm int}$ and $B_{S\,\rm int}$. All three are listed up to $n=8$ in Table V. The following sums hold:

$$\sum_B f_B(G+\alpha\pm\beta)\ln(G+\alpha\pm\beta)=2(1-x)\ln(1-x)-(1-2x)\ln(1-2x)\,, \tag{A66}$$

$$\sum_B f_B G\ln G=6(1-x)\ln(1-x)-6(1-2x)\ln(1-2x)+2(1-3x)\ln(1-3x)\,. \tag{A67}$$

One then can obtain

$$\begin{aligned}4\sum_B f_B[I(G,\alpha,\beta)+I(G,\alpha,-\beta)]=&1+2(1-2x)\ln(1-2x)-4(1-x)\ln(1-x)\\&+\frac{\epsilon}{2}[3+\ln3-B_{\rm int}(x)+4(1-x)\ln^2(1-x)-2(1-2x)\ln^2(1-2x)\\&-12(1+\ln3)(1-x)\ln(1-x)+6(1+2\ln3)(1-2x)\ln(1-2x)\\&-4\ln3(1-3x)\ln(1-3x)]\,.\end{aligned} \tag{A68}$$

Putting it all together one has

$$\begin{aligned}16\left[\sum_A f_A I(C,D)+\sum_B f_B[J(G,\alpha,\beta)+J(G,\alpha,-\beta)]\right]\\=&\tfrac{7}{2}-(14-12x)\ln(1-x)+(7-12x)\ln(1-2x)\\&+\frac{\epsilon}{2}\{-(4B_{\rm int}+A_{\rm int})+\tfrac{17}{2}-2\ln2+\tfrac{9}{2}\ln3+(14-12x)\ln^2(1-x)-(7-12x)\ln^2(1-2x)\\&-\ln(1-x)[(20-16\ln2+57\ln3)(1-x)+(2+8\ln2-3\ln3)]\\&+\ln(1-2x)[(-2-8\ln2+51\ln3)(1-2x)+(1+4\ln2+3\ln3)]\\&-\ln(1-3x)[(16\ln3-8)(1-3x)+2\ln3]\}\,.\end{aligned} \tag{A69}$$

If one looks back to Eq. (A34), one will see we have two more terms to do to get all the g^4 terms. Recalling Eq. (A20) where ξ is defined, one may obtain

$$\sum_C f_C\left[\frac{(1+\xi)^{1-\epsilon/2}}{1-\frac{\epsilon}{2}}-1-\xi(1+\xi)^{-\epsilon/2}\right]$$

$$=-\frac{\epsilon}{2}\left[\tfrac{1}{2}+\ln(1-2x)-2\ln(1-x)+\frac{\epsilon}{2}[\tfrac{1}{2}+(1-2x)\ln(1-2x)-2(1-x)\ln(1-x)-\tfrac{1}{2}\ln^2(1-2x)+\ln^2(1-x)]\right]. \tag{A70}$$

The remaining sum over the C term is done in a similar fashion. Finally one can put the above together with Eq. (A69) to obtain the g^4 terms of Eq. (A34) to be

$$\begin{aligned}\frac{g^4}{(8\pi)^4}\{&\tfrac{9}{8}-\tfrac{5}{2}\ln2+\tfrac{9}{4}\ln3+\tfrac{1}{2}(7-6x)\ln^2(1-x)-\tfrac{1}{4}(7-12x)\ln^2(1-2x)\\&-\ln(1-x)[(1-x)(-\tfrac{9}{2}-12\ln2+\tfrac{57}{2}\ln3)+(1+2\ln2-\tfrac{3}{2}\ln3)]\\&+\ln(1-2x)[(1-2x)(-\tfrac{33}{4}-6\ln2+\tfrac{51}{2}\ln3)+(\tfrac{1}{2}+\ln2+\tfrac{3}{2}\ln3)]\\&-\ln(1-3x)[(1-3x)(8\ln3-4)+\ln3]-\tfrac{1}{2}(4B_{\rm int}+A_{\rm int})\}\,.\end{aligned} \tag{A71}$$

The g^2 term of Eq. (A34) are easily obtained to be

$$\frac{g^2}{(8\pi)^2}[1+\epsilon\tfrac{1}{2}(\delta+3\ln2)]\{-\tfrac{1}{2}+2(1-x)\ln(1-x)-(1-2x)\ln(1-2x)$$
$$+\epsilon\tfrac{1}{2}[-\tfrac{1}{2}+2(1-x)\ln(1-x)-(1-2x)\ln(1-2x)$$
$$-(1-x)\ln^2(1-x)+\tfrac{1}{2}(1-2x)\ln^2(1-2x)]\}\ . \tag{A72}$$

By using the expansion

$$\ln(1-ax)=-\sum_{n=1}^{\infty}\frac{(ax)^n}{n}\ , \tag{A73}$$

one can easily compute the curly brackets, { }, terms of Eqs. (A71) and (A72). Writing the two as

$$-\frac{g^2}{(8\pi)^2}\sum_{n=0}^{\infty}x^n\left[1+\frac{\epsilon}{2}(\delta+3\ln2)\right]\left[a_n+\frac{\epsilon}{2}e_n\right]\ , \tag{A74}$$

$$-\frac{g^4}{(8\pi)^4}\sum_{n=0}^{\infty}x^n f_n\ , \tag{A75}$$

the values of a_n, e_n, and f_n are tabulated in Table I in Sec. III.

[1]For a review on the early development of the phenomenological applications of Regge theory, see for example, L. Van Hove, in *Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967); also C. B. Chiu, Annu. Rev. Nucl. Sci. **22**, 255 (1972), and references quoted therein.

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