CPT, *CP*, and *C* phases, and their effects, in Majorana-particle processes

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In neutrinoless double-β decay, the contributions of two virtual Majorana neutrinos with opposite *CP* parity will interfere destructively. This makes it evident that the amplitudes for reactions involving Majorana particles contain significant new phase factors, reflecting the special discrete-symmetry properties of these particles. To study this phenomenon, we derive and examine the *CPT*, *CP*, and *C* properties of Majorana particles. We then apply these properties, especially to the study of neutrinoless double-β decay, and to the neutral weak and electromagnetic interactions of Majorana particles. We show how the new phase factors in the Feynman amplitudes for Majorana-particle processes arise, and see that their precise form and location within these amplitudes depends on one's choice of formalism.

I. INTRODUCTION

Majorana particles, being their own antiparticles, have special *CPT*, *CP*, and *C* properties, with significant physical consequences. Since these particles occur commonly both in grand unified and supersymmetric theories, one would like to know what these consequences are. In a recent paper,[1] the *CPT* properties of an arbitrary Majorana particle were found and then used to learn about the electromagnetic interactions of such a particle. In addition, it was shown that in neutrinoless double-β decay [$(\beta\beta)_{0\nu}$], the contributions of different virtual Majorana neutrinos of definite mass can oppose each other, even if *CP* is conserved.[2] Thus, $(\beta\beta)_{0\nu}$, whose observation would signal that neutrinos are of Majorana character, may have an invisibly small or vanishing rate even if they *are* of this character. The possibility of opposing contributions from different neutrinos was demonstrated in Ref. 1 without relying on field theory (i.e., without using Feynman's rules), and without requiring any knowledge of the phases of the leptonic mixing matrix U. However, practical calculations of the amplitudes for $(\beta\beta)_{0\nu}$ or for other processes involving Majorana particles would, of course, use Feynman's rules and would demand a knowledge of any special phase factors which may occur in field-theoretic amplitudes when Majorana particles are present. Therefore, in this paper we focus on these new phase factors. We uncover their presence in reaction amplitudes, show how they can appear in different, alternative places in the amplitudes depending on one's choice of formalism, and show how they affect Majorana-particle processes, especially $(\beta\beta)_{0\nu}$.

In Sec. II we present a very plausible argument, based on a Feynman diagram, for the *false* conclusion that different Majorana-neutrino contributions always add in $(\beta\beta)_{0\nu}$, so long as *CP* is conserved. This falacious argument illustrates the traps into which one can fall through neglect of the new phase factors which appear as a result of the special *C*, *CP*, and *CPT* properties of Majorana particles. We proceed to discuss the true situation in $(\beta\beta)_{0\nu}$, as deduced without reliance on field theory or Feynman's rules. In Sec. III we then begin the *proper* field-theoretic treatment of Majorana particles by examining the *C*, *CP*, and *CPT* properties (derived in the Appendix) of Majorana fields and states. Special attention is given to the phase factors which appear, and to the restrictions on their possible values. The physical consequences of these restrictions are illustrated by a simple example. In Sec. IV we find the further constraints on phases which result from *CPT* and *CP* invariance of the interaction of special interest in $(\beta\beta)_{0\nu}$: the charged-current weak interaction with neutrino mixing. Section V discusses two especially convenient field-theoretic formalisms, or "languages" as we shall call them, which deal in alternative ways with the phases encountered in Majorana-neutrino physics when *CP* is conserved. For each of these languages, we see how the physically significant phases are contained in the $(\beta\beta)_{0\nu}$ amplitude given by Feynman's rules. The generalization of the conclusions drawn from this analysis to other problems involving Majorana particles is discussed. In Sec. VI we use the *CPT* properties of Majorana particles to infer the general structure of their neutral weak currents and to gain information on electromagnetic transitions among them. We also discuss consequences of *CP* invariance, and see how electromagnetic transition form factors acquire some of their traits when they are calculated in terms of loop diagrams. Section VII summarizes our results.

II. CANCELLATIONS IN $(\beta\beta)_{0\nu}$

Are neutrinos Majorana particles? The only known practical way to study this question is to search for neutrinoless double-β decay. In this process, a pair of virtual W bosons, generated by two neutrons in a nucleus, produces a pair of outgoing electrons by virtual neutrino exchange (Fig. 1). As Fig. 1 shows, the amplitude for the process is the sum of the contributions from all the neutrino mass eigenstates ν_m which couple to an electron. This coupling is described by the general charged-current weak interaction with neutrino mixing,

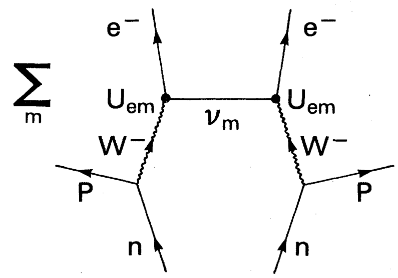

FIG. 1. Neutrinoless double-β decay. The ν_m are the various neutrino mass eigenstates which couple to an electron, each with a factor U_{em}.

$$H(x)=g\sum_{l,m}[W_\mu^- i\bar{l}\gamma_\mu(1+\gamma_5)U_{lm}\nu_m + W_\mu^+ i\bar{\nu}_m U_{lm}^*\gamma_\mu(1+\gamma_5)l] . \tag{2.1}$$

Here l runs over the charged leptons e, μ, and τ, m runs over the neutrino mass eigenstates, g is the (real) overall coupling strength, and U is the lepton mixing matrix. Right-handed currents have been neglected for simplicity.[3]

It is well known that observation of $(\beta\beta)_{0\nu}$ would signal that neutrinos are of Majorana character.[4] However, the absence of this reaction is not necessarily evidence that they are not, because the contributions of the different ν_m in Fig. 1 can interfere destructively and cancel each other out. Now, Doi *et al.* have claimed[5] that destructive interference between these contributions requires CP violation. Were this true, one would not have to worry too much about suppression of $(\beta\beta)_{0\nu}$ through such interference, since CP violation appears to be small in the hadronic sector, and so one can guess that it is also small in the leptonic one. However, Wolfenstein asserts[2] that destructive interference already occurs when CP is conserved but the CP parities of the interfering neutrinos are opposite.[6] Halprin, Petcov, and Rosen[7] agree that destructive interference can occur when CP is conserved, although they say that the sign of the interference depends on that in a "Majorana condition" which appears to express the C, rather than CP, properties of the neutrino field.

A simple argument supports the contention that the interference cannot be destructive if CP is conserved. We shall call it argument A. It is as follows.

(1) The two vertices in Fig. 1 connected by a given ν_m are identical. Therefore, as shown in the figure, the contribution of ν_m is proportional to ${U_{em}}^2$.

(2) If CP is conserved, the mixing matrix U is real. (After all, it is well known that any nonreality in the Kobayashi-Maskawa mixing matrix for quarks is a measure of CP violation.) Then ${U_{em}}^2$ is positive for all m.

(3) Therefore, if CP is conserved, the contributions of all ν_m add constructively.

We hasten to explain that this very plausible argument and its conclusion are incorrect. Indeed, in Ref. 1 it was shown that when CP is conserved, the amplitude for neutrinoless double-β decay, $A[(\beta\beta)_{0\nu}]$, has the form

$$A[(\beta\beta)_{0\nu}]=\sum_{\substack{\text{neutrinos}\\ m}}\tilde{\eta}_z(\nu_m)\,|\,U_{em}\,|^2 M_m\bar{A} . \tag{2.2}$$

Here $\tilde{\eta}_z(\nu_m)$ is the CP parity of ν_m, M_m is its mass, assumed small compared to the momentum transfers typical of $(\beta\beta)_{0\nu}$, and $\bar{A}$ is independent of m. We see that, as Wolfenstein asserts,[2] neutrinos with opposite CP parities do interfere destructively.[8,9] This remains true when the M_m are not small; the factor M_m in Eq. (2.2) is then simply replaced by another kinematical factor which can easily be shown to be positive. (Also, the degree of cancellation between a heavy neutrino [one whose mass is not small compared to typical $(\beta\beta)_{0\nu}$ momentum transfers] and a light one varies from nucleus to nucleus.[7,10])

What, then, is wrong with argument A? Are the two neutrino vertices in Fig. 1 not to be treated identically? Or, is the CP-conserving U matrix not real? Or, are both of these true? In Ref. 1, Eq. (2.2) was obtained in a manner that did not require any knowledge of Feynman's rules for systems involving Majorana particles, or of the phases of the U_{em}. To acquire such knowledge, we now proceed to examine the field-theoretic treatment of Majorana particles, focusing on the new phase factors which appear in amplitudes when these particles are present. It will soon be clear what is wrong with argument A.

III. DISCRETE SYMMETRIES OF MAJORANA FIELDS AND STATES

The symmetry operations which are interesting are those which leave some interactions invariant. Therefore, to define the C, CP, and CPT transformations of Majorana particles, we begin by imposing a requirement: If a Lagrangian involving Dirac fields but no Majorana fields is invariant under some specific discrete symmetry transformations, then it retains its invariances when the Dirac fields are replaced by Majorana ones. That is, invariance under a discrete symmetry transformation shall not depend on whether fermions are of Dirac or Majorana nature.[11] The most obvious way to fulfill this requirement is to define each discrete symmetry operation as having the same effect on a Majorana field as on a Dirac one. However, one must ask whether this definition yields the desired transformation properties of Majorana *states*, and whether it is free of inconsistencies. In the Appendix we show that this is indeed the case *provided* that certain restrictions are obeyed.

To present the effects of C, CP, and CPT, we introduce the general Majorana field

$$\Psi=\sum_{\vec{p},s}\left[\frac{M}{E_{\vec{p}}V}\right]^{1/2}(f_{\vec{p}s}u_{\vec{p}s}e^{ipx}+\lambda f_{\vec{p}s}^\dagger v_{\vec{p}s}e^{-ipx}) . \tag{3.1}$$

Here $f^\dagger$ and f create and destroy the Majorana particle of interest, $u_{\vec{p}s}$ is the usual Dirac spinor, and $v_{\vec{p}s}=\gamma_2 u_{\vec{p}s}^*$. The quantity λ is a phase factor which we shall call the *creation phase factor*. This factor may in general be present in a Majorana field. As we shall see, one can choose it arbitrarily, but, as we shall also see, it is sometimes most convenient *not* to choose $\lambda=1$, so we shall keep this factor explicitly.

The transformation properties and restrictions found in the Appendix are summarized in Table I. In the first column, the symmetry operations C, $CP \equiv z$, and $CPT \equiv \zeta$ are defined to have the same effects on the Majorana field Ψ as they do on a Dirac field. However, conveniently defined phase factors $\eta_{c,z,\zeta}$, which are irrelevant in the Dirac case but not here, are included in the field transformation laws. If one inserts into these laws the plane-wave expansion Eq. (3.1), one can show (see the Appendix) that the corresponding transformation laws for the one-particle state $|\vec{p},s\rangle$ are as given in the second column. Note that each symmetry operation does indeed affect the momentum $\vec{p}$ and spin projection s as it should. Furthermore, under a given symmetry operation, the state picks up a phase factor $\tilde{\eta}$ which is *not* the same as that picked up by the field, but is related to it as shown in the third column. Note that the relations between the "state phases" $\tilde{\eta}$ and the corresponding "field phases" η involve the creation phase factor λ. Finally, one finds (see the Appendix) that our chosen definition of the discrete symmetry operations in the Majorana case is free of inconsistencies only if the phase factors satisfy the restrictions given in the final column. Namely, the C parity of a Majorana-particle state must be real, while its CP parity (and for that matter its ordinary parity) must be imaginary. Furthermore, the CPT "field phase factor" η_ζ must be imaginary.

In spite of appearances, the restrictions on the phase factors do not result from free-field theory alone, but are related to interactions. We have defined each discrete symmetry operation in the Majorana case so that an interaction will be invariant under this operation if it was so in the Dirac case. It is these definitions which have lead to the restrictions.

This connection between the phase-factor restrictions and the interactions can be illustrated by a very simple alternative derivation of the restriction $\tilde{\eta}_z = \pm i$. The decay $Z^0 \rightarrow \nu^D \bar{\nu}^D$, where ν^D is a *Dirac* neutrino, is described by the CP-conserving interaction

$$H = g Z^0_\mu i \bar{\nu}^D \gamma_\mu (1+\gamma_5) \nu^D \ .$$

Let us now demand that this interaction remain CP conserving when the Dirac field ν^D is replaced by a Majorana field ν^M. Then $Z^0 \rightarrow \nu^M \nu^M$ must conserve CP. To see what consequence this has, it suffices to suppose that the outgoing neutrinos are nonrelativistic. Since the final state must obviously be antisymmetric, it must then be a $^3P_1$ state, since this is the only nonrelativistic, antisymmetric state with $J=1$. Now, if

$$z\,|\nu^M(\vec{p},s)\rangle = \tilde{\eta}_z(\nu^M)\,|\nu^M(-\vec{p},s)\rangle \quad (z \equiv CP) \ ,$$

then

$$z\,|\nu^M \nu^M; {}^3P_1\rangle = \tilde{\eta}_z^{\,2}(\nu^M)(-1)^L\,|\nu^M \nu^M; {}^3P_1\rangle \ . \qquad (3.2)$$

Since $z(Z^0) = +1$ and $L=1$, z conservation demands that $\tilde{\eta}_z(\nu^M) = \pm i$.[12]

The constraints $\tilde{\eta}_c = \pm 1$ and $\tilde{\eta}_z = \pm i$ have physical consequences which we illustrate by an example. Consider a decay which may be studied in the future, $\chi_t(2^{++}) \rightarrow \tilde{\gamma}\tilde{\gamma}$, where $\chi_t(2^{++})$ is the $^3P_2$ state of t-quarkonium, and $\tilde{\gamma}$ is the spin-$\frac{1}{2}$ Majorana photino predicted by supersymmetric theories. One expects that this decay is electromagnetic, so that it conserves C and CP. Since $\tilde{\eta}_c(\tilde{\gamma})$ must be real, $C(\tilde{\gamma}\tilde{\gamma}) = \tilde{\eta}_c^{\,2}(\tilde{\gamma}) = +1$, so C is conserved automatically. However, had $\tilde{\eta}_c(\tilde{\gamma})$ been anything but real, C conservation would have been impossible. Furthermore, the requirement that $\tilde{\eta}_c$ be real implies that C-conserving decays of C-*odd* bosons into pairs of identical Majorana fermions are forbidden.[13]

Turning to CP, we suppose for simplicity that the outgoing photinos are nonrelativistic. Now,

$$z\,|\tilde{\gamma}\tilde{\gamma}; {}^{2S+1}L_J\rangle = \tilde{\eta}_z^{\,2}(\tilde{\gamma})(-1)^L\,|\tilde{\gamma}\tilde{\gamma}; {}^{2S+1}L_J\rangle \ , \qquad (3.3)$$

so if CP is conserved, $\tilde{\eta}_z^{\,2}(\tilde{\gamma})(-1)^L = +1$. The constraint $\tilde{\eta}_z(\tilde{\gamma}) = \pm i$ then implies that L is odd. Thus, the outgoing photinos may be in the $^3P_2$ or $^3F_2$ state, but not in the $^1D_2$ state, which is the remaining antisymmetric $J=2$ possibility. Since the angular distributions corresponding to $^1D_2$ and, say, $^3P_2$ differ, the requirement that L be odd rather than even is an observable consequence of the restriction that $\tilde{\eta}_z$ be imaginary rather than real.

As far as we can tell, the CPT phase restriction $\eta_\zeta = \pm$i, which concerns the field phase, not the state phase, has no observable physical consequences. However, as we shall see in Sec. VI, knowledge of this restriction can be of practical use in theoretical calculations.

As shown in Table I, the CPT state phase factor $\tilde{\eta}_\zeta^{\,s}$ depends on s, satisfying the relation

$$\tilde{\eta}_\zeta^{\,-s} = -\tilde{\eta}_\zeta^{\,s} \ . \qquad (3.4)$$

We also see from Table I that, while it must obey this relation, $\tilde{\eta}_\zeta^{\,s}$ is otherwise arbitrary, because the creation phase factor λ is arbitrary. These properties of $\tilde{\eta}_\zeta^{\,s}$ are actually independent of field theory, as was shown in Ref. 1. The demonstration is so simple that for completeness we repeat it here. We consider, in its rest frame, a CPT-self-

TABLE I. Effects of the discrete-symmetry operations on a Majorana field $\Psi(\vec{x},t)$ and on the corresponding Majorana state $|\vec{p},s\rangle$. In the table, Ψ^* is $(\Psi^\dagger)^T$. The quantity λ is the creation phase factor in the field Ψ. The phase factor $\tilde{\eta}_\zeta^{\,s}$ depends on s, as indicated.

Symmetry operation	Effect on $\Psi(\vec{x},t)$	Effect on $\lvert\vec{p},s\rangle$	Relation between phases	Restriction
C	$\eta_c^* \gamma_2 \Psi^*(\vec{x},t)$	$\tilde{\eta}_c \lvert\vec{p},s\rangle$	$\tilde{\eta}_c = \lambda\eta_c$	$\tilde{\eta}_c = \pm 1$
$CP(z)$	$\eta_Z^* \gamma_4 \gamma_2 \Psi^*(-\vec{x},t)$	$\tilde{\eta}_Z \lvert -\vec{p},s\rangle$	$\tilde{\eta}_Z = \lambda\eta_Z$	$\tilde{\eta}_Z = \pm i$
$CPT(\zeta)$	$-\eta_\zeta^* \gamma_5 \Psi^*(-\vec{x},-t)$	$\tilde{\eta}_\zeta^{\,s} \lvert\vec{p},-s\rangle$	$\tilde{\eta}_\zeta^{\,s} = \lambda\eta_\zeta(-1)^{s-1/2}$	$\eta_\zeta = \pm i$

conjugate particle[14] of arbitrary spin J and $J_z=s$. The effect of $CPT\equiv\zeta$ on the state $|J,s\rangle$ of this particle is defined to be

$$\zeta\,|J,s\rangle=\tilde{\eta}^{\,s}_{\zeta}\,|J,-s\rangle\ . \tag{3.5}$$

Let us define an auxiliary operator,

$$b\equiv e^{-i\pi J_y}\zeta\ , \tag{3.6}$$

whose effect on $|J,s\rangle$ is obviously

$$b\,|J,s\rangle=\tilde{\eta}^{\,s}_{b}\,|J,s\rangle\ , \tag{3.7}$$

with $\tilde{\eta}^{\,s}_{b}$ some new phase factor. Like ζ, b is an antiunitary operator. Consequently,

$$\begin{aligned} b^2\,|J,s\rangle&=b(\tilde{\eta}^{\,s}_{b}\,|J,s\rangle)=(\tilde{\eta}^{\,s}_{b})^*b\,|J,s\rangle\\ &=(\tilde{\eta}^{\,s}_{b})^*\tilde{\eta}^{\,s}_{b}\,|J,s\rangle\ . \end{aligned} \tag{3.8}$$

That is, $b^2=1$ when acting on the states $|J,s\rangle$, which from the definition of b implies that

$$1=\zeta^2 e^{-2i\pi J_y}=\zeta^2(-1)^{2J}\ . \tag{3.9}$$

Here, we have used the fact that CPT commutes with rotations, and then the fact that a rotation through 2π yields the original state times $(-1)^{2J}$, independently of conventions. From Eq. (3.9), we learn that for any Majorana (i.e., CPT-self-conjugate) particle of spin J,[15,16]

$$\zeta^2=(-1)^{2J}\ . \tag{3.10}$$

For any such particle, then, either rotation through 360° or $(CPT)^2$ brings one back to the original state, multiplied by precisely the same phase factor, $(-1)^{2J}$.

Now, from Eq. (3.5),

$$\zeta^2\,|J,s\rangle=\zeta(\tilde{\eta}^{\,s}_{\zeta}\,|J,-s\rangle)=(\tilde{\eta}^{\,s}_{\zeta})^*\tilde{\eta}^{\,-s}_{\zeta}\,|J,s\rangle\ . \tag{3.11}$$

Thus, Eq. (3.10) implies that

$$\tilde{\eta}^{\,-s}_{\zeta}=(-1)^{2J}\tilde{\eta}^{\,s}_{\zeta}\ . \tag{3.12}$$

Equation (3.4) is just this relation for $J=\frac{1}{2}$.

Amusingly, for the special case of $J=\frac{1}{2}$, the relation (3.12) can also be obtained in another simple way. For this case there are only two spin states, $|\frac{1}{2},\pm\frac{1}{2}\rangle\equiv|\pm\rangle$. We denote the matrix elements of the raising and lowering operators $J_\pm=J_x\pm iJ_y$ by

$$\langle +|J_+|-\rangle=\langle -|J_-|+\rangle^*\equiv\Gamma\ .$$

Now, the definition (3.5) implies that $\zeta\vec{J}=-\vec{J}\zeta$. Since ζ is antiunitary, it follows that $\zeta J_-=-J_+\zeta$. Applying this equation to the state $|+\rangle$, we have

$$\begin{aligned} \zeta J_-\,|+\rangle&=\zeta\Gamma^*\,|-\rangle=\Gamma\tilde{\eta}^{\,-}_{\zeta}\,|+\rangle\\ &=-J_+\zeta\,|+\rangle=-J_+\tilde{\eta}^{\,+}_{\zeta}\,|-\rangle=-\tilde{\eta}^{\,+}_{\zeta}\Gamma\,|+\rangle\ . \end{aligned} \tag{3.13}$$

Thus, $\tilde{\eta}^{\,-}_{\zeta}=-\tilde{\eta}^{\,+}_{\zeta}$.[17]

Apart from the constraint of Eq. (3.12), the individual phase factors $\tilde{\eta}^{\,s}_{\zeta}$ are arbitrary, because one can always redefine the states $|J,s\rangle$ according to

$$|J,s\rangle\to|J,s\rangle'=e^{i\phi_s}\,|J,s\rangle\ , \tag{3.14}$$

with ϕ_s an arbitrary phase. Under this redefinition

$$\tilde{\eta}^{\,s}_{\zeta}=\langle J,-s\,|\,\zeta\,|\,J,s\rangle\Longrightarrow(\tilde{\eta}^{\,s}_{\zeta})'=e^{-i(\phi_{-s}+\phi_s)}\tilde{\eta}^{\,s}_{\zeta}\ , \tag{3.15}$$

but Eq. (3.12) is still obeyed.

In Table I, the arbitrariness of $\tilde{\eta}^{\,s}_{\zeta}$ is correlated with that of λ. But then, from the previous discussion, the arbitrariness of λ must be connected with the possibility of redefining the states. To exhibit this connection explicitly, let us recall that the field Ψ, Eq. (3.1), annihilates the one-particle states $|\vec{p},s\rangle$ with specific matrix elements:

$$\langle 0\,|\,\Psi(0)\,|\,\vec{p},s\rangle=\left[\frac{M}{E_{\vec{p}}V}\right]^{1/2}u_{\vec{p}s}\ . \tag{3.16}$$

Now, suppose we choose to work with new states $|\vec{p},s\rangle'=e^{i\phi}|\vec{p},s\rangle$, and, correspondingly, a new field $\Psi'=e^{-i\phi}\Psi$, which annihilates the new states with the same matrix elements as usual:

$$\langle 0\,|\,\Psi'(0)\,|\,\vec{p},s\rangle'=\langle 0\,|\,\Psi(0)\,|\,\vec{p},s\rangle\ . \tag{3.17}$$

If we express Ψ' in terms of the annihilation operators $f'_{\vec{p}s}$ for the new states $(\langle 0\,|f'_{\vec{p}s}\,|\,\vec{p}s\rangle'=1)$, it becomes

$$\Psi'=\sum_{\vec{p},s}\left[\frac{M}{E_{\vec{p}}V}\right]^{1/2}[f'_{\vec{p}s}u_{\vec{p}s}e^{ipx}+(e^{-2i\phi}\lambda)f'^{\dagger}_{\vec{p}s}v_{\vec{p}s}e^{-ipx}]\ . \tag{3.18}$$

Note that Ψ' contains a new creation phase factor $\lambda'=e^{-2i\phi}\lambda$. Thus, we can give the creation phase factor any value desired by redefining the states and the corresponding field.

It is amusing to ask how a *composite* Majorana fermion made of non-Majorana particles can have imaginary parity when its constituents have more "normal" parity properties.[18] Let us see how this comes about for a spin-$\frac{1}{2}$ Majorana fermion f which is made of a charged spin-$\frac{1}{2}$ Dirac fermion $e^{\mp}$ and a charged spinless boson $\phi^{\pm}$. Neglecting binding energy, we shall suppose that f and its constituents are all at rest, and shall disregard the spin variables because P does not affect them. Since $|f\rangle$ is a Majorana state, it must have the form

$$|f\rangle=\frac{1}{\sqrt{2}}(\,|e^-\phi^+\rangle+\alpha\,|e^+\phi^-\rangle)\ , \tag{3.19}$$

where α is a phase factor. We are assuming that $C\,|e^{\mp}\rangle=|e^{\pm}\rangle$ and $C\,|\phi^{\pm}\rangle=|\phi^{\mp}\rangle$, so that $C\,|f\rangle=\alpha\,|f\rangle$ if α is real, as Table I requires the C parity of a Majorana state to be. Now, the parity properties of the constituents are

$$P\,|e^-\rangle=\beta\,|e^-\rangle\ , \tag{3.20a}$$

$$P\,|e^+\rangle=-\beta^*\,|e^+\rangle\ , \tag{3.20b}$$

and

$$P\,|\phi^{\pm}\rangle=\gamma\,|\phi^{\pm}\rangle\ , \tag{3.20c}$$

where β may be a complex phase but γ is real (think of $\phi^{\pm}$ as a pion, for example). The relations (3.20a) and (3.20b) follow from the parity transformation law for fermion fields implied by Table I. Together, they imply the well-known result $P(e^-e^+;\ S\ \text{wave})=-1$. From Eqs. (3.20),

$$P\,|f\rangle=\frac{1}{\sqrt{2}}(\beta\gamma\,|e^-\phi^+\rangle-\alpha\beta^*\gamma\,|e^+\phi^-\rangle)\ . \tag{3.21}$$

Thus, if we require $|f\rangle$ to be an eigenstate of parity, we must have $\beta=-\beta^*$. The parity of $|f\rangle$, $\beta\gamma$, is then imaginary. In a similar way, one can also show that the parity of a Majorana composite of three charged spin-$\frac{1}{2}$ fermions is imaginary.

IV. PHASE CONSTRAINTS FROM A SPECIAL INTERACTION

Under *CPT*, a term in the Hamiltonian (2.1) responsible for $(\beta\beta)_{0\nu}$ transforms according to

$$\zeta[W_\mu^- i\bar{l}\gamma_\mu(1+\gamma_5)U_{lm}\nu_m]_x\zeta^{-1} =[\eta_\zeta(W^-)\eta_\zeta(l)\eta_\zeta^*(\nu_m)][W_\mu^+ i\bar{\nu}_m U_{lm}^*\gamma_\mu(1+\gamma_5)l]_{-x}\ . \tag{4.1}$$

Here $\eta_\zeta(\nu_m)$ is the *CPT* field phase factor defined in Table I, and $\eta_\zeta(l)$, $\eta_\zeta(W^-)$ are analogous factors for the other fields. Comparing Eqs. (4.1) and (2.1), we see that *CPT* invariance of $H(x)$ requires that

$$\eta_\zeta(W^-)\eta_\zeta(l)\eta_\zeta^*(\nu_m)=1\ . \tag{4.2}$$

For given l, this relation must hold for all ν_m for which $U_{lm}\neq 0$. Thus, assuming that U is a nontrivial mixing matrix, all the $\eta_\zeta(\nu_m)$ must not only be imaginary, as required by Table I, but equal.

Under *CP*, a term in the interaction (2.1) transforms according to

$$z[W_\mu^- i\bar{l}\gamma_\mu(1+\gamma_5)U_{lm}\nu_m]_{\vec{x},t}z^{-1}=[\eta_z(W^-)\eta_z(l)\eta_z^*(\nu_m)][W_\mu^+ i\bar{\nu}_m U_{lm}\gamma_\mu(1+\gamma_5)l]_{-\vec{x},t}\ . \tag{4.3}$$

Here $\eta_z(\nu_m)$ is the *CP* field phase factor defined in Table I, and $\eta_z(l)$, $\eta_z(W^-)$ are similar factors for the other fields. We see that if H, Eq. (2.1), is to conserve *CP*, we must have

$$U_{lm}^*=U_{lm}[\eta_z(W^-)\eta_z(l)\eta_z^*(\nu_m)]\ . \tag{4.4}$$

Now, if all fermions were Dirac particles, *CP* acting on the state of any one of them would transform it into an antiparticle distinct from the original particle. Any phase encountered in this transformation could be absorbed into the definition of the antiparticle. Hence, the phases η_z in Eq. (4.4) could all be eliminated, so that the *CP*-conserving U matrix would be real. This is the situation familiar from the Kobayashi-Maskawa treatment of the quark sector. However, if the neutrinos are Majorana particles, then *CP* transforms a neutrino state back into itself, with a physically significant phase $\tilde{\eta}_z(\nu_m)$ which represents the intrinsic *CP* parity of the neutrino and cannot be defined away. Thus, the related phase $\eta_z(\nu_m)$ in Eq. (4.4) can no longer simply be eliminated and forgotten. In this situation, one may proceed in one of several ways.

V. LANGUAGES FOR PROCESSES WITH MAJORANA PARTICLES

Given the constraint (4.4) from *CP* conservation, and the relation

$$\tilde{\eta}_z(\nu_m)=\lambda(\nu_m)\eta_z(\nu_m) \tag{5.1}$$

from Table I, there are two alternative formalisms or languages which are particularly convenient for neutrino-physics calculations in the *CP*-conserving case. We describe these in turn.

Language L1. In this language, we exploit the arbitrariness (proved in Sec. III) of the creation phase factors by choosing $\lambda(\nu_m)=1$ for all ν_m. Then the Majorana neutrino fields have conventional plane-wave expansions with no extra phases. However, the *CP*-conserving U matrix is not real. To see this, note from Eq. (5.1) that $\eta_z(\nu_m)=\tilde{\eta}_z(\nu_m)$. Thus, unless all neutrinos $|\nu_m\rangle$ happen to have the same *CP* parity, Eq. (4.4) implies that when *CP* is conserved the phase of U_{lm} varies with m. One can, of course, still choose $\eta_z(l)$ so that, say, U_{l1} is real. Then, since $\tilde{\eta}_z(\nu_m)=\pm i$ for all ν_m, each U_{lm} will be either real or purely imaginary. With this choice of $\eta_z(l)$, Eq. (4.4) implies that

$$U_{lm}{}^2=|U_{lm}|^2\frac{\tilde{\eta}_z(\nu_m)}{\tilde{\eta}_z(\nu_1)}\ . \tag{5.2}$$

Language L2. Here we set out to make the *CP*-conserving U matrix real. This is accomplished by choosing $\lambda(\nu_m)=\tilde{\eta}_z(\nu_m)/i$. Then $\eta_z(\nu_m)=+i$ for all ν_m, so that according to Eq. (4.4) the phase of U_{lm} does not vary with m. We may then choose $\eta_z(l)$ and $\eta_z(W^-)$ so that the U matrix is real. Of course, nontrivial creation phase factors $\lambda(\nu_m)=\pm 1$ are now present in the Majorana neutrino fields, and these factors will appear in reaction amplitudes.

In Sec. III it was shown that changing a creation phase factor is equivalent to introducing a new one-particle state and a corresponding new field which keeps the one-particle annihilation amplitude at its usual value, as in Eq. (3.17). In particular, from Eq. (3.18) and the discussion following it, going from language L1 $[\lambda(\nu_m)=1]$ to language L2 $[\lambda(\nu_m)=\tilde{\eta}_z(\nu_m)/i]$ is equivalent to introducing new neutrino states

$$|\nu_m\rangle_{\mathrm{L2}}=(-i)^{\delta_{\tilde{\eta}_z(\nu_m),-i}}\,|\nu_m\rangle_{\mathrm{L1}} \tag{5.3}$$

and new fields

$$(\nu_m)_{\mathrm{L2}}=i^{\delta_{\tilde{\eta}_z(\nu_m),-i}}(\nu_m)_{\mathrm{L1}}\ , \tag{5.4}$$

where δ is the Kronecker δ. Obviously, the ν_m mass and kinetic-energy terms in the Lagrangian, being of the form $\bar{\nu}_m\nu_m$, are invariant under the transformation (5.4). So, similarly, is the diagonal neutrino neutral-current interaction of the standard model. However, in L2 the charged-current interaction (2.1) obviously has a new U matrix related to the U of L1 by

$$(U_{lm})_{L2}(\nu_m)_{L2}=(U_{lm})_{L1}(\nu_m)_{L1} . \tag{5.5}$$

From Eq. (5.2) for U_{L1}, U_{L2} is real, as previously stated.

Consider, now, the contribution of ν_m to $(\beta\beta)_{0\nu}$, depicted in Fig. 1. From Eq. (2.1), a Feynman-rule calculation of this contribution, carried out in any language, involves the product of currents

$$[\bar{e}\gamma_\mu(1+\gamma_5)U_{em}\nu_m][\bar{e}\gamma_\nu(1+\gamma_5)U_{em}\nu_m] . \tag{5.6}$$

Now, the conventional neutrino propagator is a contraction of the field ν_m with $\bar{\nu}_m$. To make possible such a contraction here, we rewrite the current for one of the vertices in terms of the fields ν_m^c and e^c, where for any fermion field Ψ, $\Psi^c\equiv\gamma_2\Psi^*$. It is a mathematical identity that

$$\bar{e}\gamma_\nu(1+\gamma_5)U_{em}\nu_m=\overline{\nu_m^c}\gamma_\nu(-1+\gamma_5)U_{em}e^c . \tag{5.7}$$

Furthermore, from Eq. (3.1) it follows trivially that when Ψ is a Majorana field,[19]

$$\gamma_2\Psi^*=\lambda^*\Psi . \tag{5.8}$$

Thus, $\overline{\nu_m^c}=\lambda(\nu_m)\bar{\nu}_m$, and we my now contract ν_m with $\bar{\nu}_m$, whereupon (5.6) becomes

$$\lambda(\nu_m)U_{em}{}^2[\bar{e}\gamma_\mu(1+\gamma_5)\nu_m^{\cdot}][\bar{\nu}_m^{\cdot}\gamma_\nu(-1+\gamma_5)e^c] . \tag{5.9}$$

This expression will obviously lead to a $(\beta\beta)_{0\nu}$ amplitude of the form

$$A[(\beta\beta)_{0\nu}]\propto\sum_m\lambda(\nu_m)U_{em}{}^2M_m , \tag{5.10}$$

where the mass M_m comes from the ν_m propagator.[20,21] In language L1, $\lambda(\nu_m)=1$, but (assuming now that CP is conserved) $U_{em}{}^2\propto\tilde{\eta}_z(\nu_m)\,|\,U_{em}\,|^2$ [see Eq. (5.2)], so the result (5.10) agrees with Eq. (2.2). In this language, it is the assumption that the CP-conserving U matrix is real that is the false part of argument A. In language L2, $U_{em}{}^2=|\,U_{em}\,|^2$, but $\lambda(\nu_m)\propto\tilde{\eta}_z(\nu_m)$, so again Eq. (5.10) agrees with Eq. (2.2). In this language, it is the assumption that the two vertices connected by ν_m are identical that is the false step in argument A. Yes, they *are* identical, but they are not *treated* equally. The current at one of them must be rewritten in terms of $\bar{\nu}_m$ and e^c, whereupon $\lambda(\nu_m)$ appears as an extra factor in the amplitude.[22,22a]

We see that in the field-theoretic amplitudes for processes involving Majorana particles, new phases related to these particles appear. The precise location of these phases within the amplitudes depends on the choice of formalism.[23] However, the phases are present *somewhere* in the amplitudes, and can have important physical consequences.

While we have uncovered these phases in $(\beta\beta)_{0\nu}$, a process produced by the C- and P-violating weak interactions, it should be clear that they are a general feature and will also occur where the interactions conserve all of the discrete symmetries. For example, our analysis of $(\beta\beta)_{0\nu}$ would be practically unaffected if we were to suppose that the weak interactions are, say, pure axial vector, and hence C and P conserving. Furthermore, imagine that we do suppose this and also that, for instance, the neutrinos ν_1 and ν_2 couple to a C-odd spinless boson ϕ through the interaction

$$H=g\phi\bar{\nu}_1\nu_2+\text{H.c.} \tag{5.11}$$

Then one can show that if the creation phase factors in the ν_m fields are adjusted to make the C- and P-conserving U matrix real, the C-conserving scalar coupling g cannot be real, and vice versa. Calculations ignoring the complexity of these quantities will give erroneous results.

VI. CURRENTS OF MAJORANA PARTICLES

A great deal can easily be learned about the electromagnetic and neutral weak interactions of Majorana particles just by exploiting their CPT and CP properties. For example, suppose we want to know how a $\tilde{\gamma}$ couples to a Z^0. There is no direct coupling, but an effective one is induced through loop effects. This effective coupling may be described by giving $\langle\tilde{\gamma}\,|\,N_\mu(0)\,|\,\tilde{\gamma}\rangle$, where $N_\mu(x)$ is the complete neutral weak current to which Z^0 couples. Now, the Z^0 is known[24] to couple, in particular, to the electron neutral current $i\bar{e}\gamma_\mu(a+b\gamma_5)e$. The latter is explicitly CPT odd. Thus, if the neutral weak interaction

$$H=Z_\mu^0N_\mu \tag{6.1}$$

is to conserve CPT, the entire neutral current N_μ must be CPT-odd. From the antiunitarity of CPT and Table I, it then follows that

$$\langle\tilde{\gamma}(\vec{p}_f,s_f)\,|\,N_\mu(0)\,|\,\tilde{\gamma}(\vec{p}_i,s_i)\rangle=-[\tilde{\eta}_\zeta^{s_i}(\tilde{\gamma})]^*\tilde{\eta}_\zeta^{s_f}(\tilde{\gamma})\langle\tilde{\gamma}(\vec{p}_i,-s_i)\,|\,N_\mu(0)\,|\,\tilde{\gamma}(\vec{p}_f,-s_f)\rangle . \tag{6.2}$$

Lorentz invariance implies that the left-hand side of this constraint can be written in the form

$$\langle\tilde{\gamma}(\vec{p}_f,s_f)\,|\,N_\mu(0)\,|\,\tilde{\gamma}(\vec{p}_i,s_i)\rangle=i\bar{u}_{\vec{p}_f,s_f}[\gamma_\mu(V+A\gamma_5)+\sigma_{\mu\nu}q_\nu(M+Ei\gamma_5)+q_\mu(S+Pi\gamma_5)]u_{\vec{p}_i,s_i} . \tag{6.3}$$

Here $q=p_f-p_i$, and V, A, M, E, S, and P are form factors depending on q^2. Similarly, the right-hand side (RHS) of Eq. (6.2) can, after some algebra, be written as

$$\text{RHS}=\{[\tilde{\eta}_\zeta^{s_i}(\tilde{\gamma})]^*\tilde{\eta}_\zeta^{s_f}(\tilde{\gamma})(-1)^{s_i-s_f}\}i\bar{u}_{\vec{p}_f,s_f}[\gamma_\mu(-V+A\gamma_5)+\sigma_{\mu\nu}q_\nu(-M-Ei\gamma_5)+q_\mu(S+Pi\gamma_5)]u_{\vec{p}_i,s_i} . \tag{6.4}$$

From Eq. (3.12), the combination of phases in curly brackets is $+1$. Thus, comparing Eqs. (6.3) and (6.4), we see that $V=M=E=0$. Hence, the most general expression for the matrix element of the neutral weak current of a photino, or of any other spin-$\frac{1}{2}$ Majorana particle, is

$$\langle \tilde{\gamma}(\vec{p}_f,s_f) \,|\, N_\mu(0) \,|\, \tilde{\gamma}(\vec{p}_i,s_i) \rangle = i\bar{u}_{\vec{p}_f,s_f}[A\gamma_u\gamma_5 + q_\mu(S+Pi\gamma_5)]u_{\vec{p}_i,s_i} \,. \tag{6.5}$$

Our derivation of this result illustrates the use of the *CPT* properties of Majorana states, and especially of Eq. (3.12), to constrain the matrix elements of operators with known *CPT* characteristics.[25] To find $\langle \tilde{\gamma} \,|\, J_\mu^{\rm EM}(0) \,|\, \tilde{\gamma} \rangle$, where $J_\mu^{\rm EM}(0)$, the electromagnetic current, is also a *CPT*-odd operator, we need only impose on Eq. (6.5) the requirement that the current be conserved. This yields[1,26]

$$\langle \tilde{\gamma}(\vec{p}_f,s_f) \,|\, J_\mu^{\rm EM}(0) \,|\, \tilde{\gamma}(\vec{p}_i,s_i) \rangle = i\bar{u}_{\vec{p}_f,s_f}[G(q^2)(q^2\gamma_\mu - \not{q}q_\mu)\gamma_5]u_{\vec{p}_i,s_i} \,. \tag{6.6}$$

We see that a spin-$\frac{1}{2}$ Majorana particle has only one electromagnetic form factor, G.[27–29]

For the electromagnetic matrix element connecting two *different* spin-$\frac{1}{2}$ Majorana fermions, f_1 and f_2, we find from Lorentz invariance and current conservation the form

$$\langle f_2(\vec{p}_f,s_f) \,|\, J_\mu^{\rm EM}(0) \,|\, f_1(\vec{p}_i,s_i) \rangle = i\bar{u}_2(\vec{p}_f,s_f)[(q^2\gamma_\mu - \not{q}q_\mu)(F_{21}+G_{21}\gamma_5) + \sigma_{\mu\nu}q_\nu(M_{21}+E_{21}i\gamma_5)]u_1(\vec{p}_i,s_i) \,. \tag{6.7}$$

Here F_{21}, G_{21}, M_{21}, and E_{21} are transition form factors. As discussed in Ref. 1, the *CPT* constraint on $\langle f_2 \,|\, J_\mu^{\rm EM} \,|\, f_1 \rangle$ analogous to Eq. (6.2), when combined with the Hermiticity of $J_\mu^{\rm EM}$, implies that

$$F_{21} = -\xi F_{21}^*, \quad G_{21} = \xi G_{21}^* \,,$$
$$M_{21} = -\xi M_{21}^*, \quad E_{21} = -\xi E_{21}^* \,. \tag{6.8}$$

In these relations,

$$\xi \equiv \tilde{\eta}_\xi^*(f_1)\tilde{\eta}_\xi(f_2) \,, \tag{6.9}$$

where $\tilde{\eta}_\xi(f_i) \equiv \tilde{\eta}_\xi^{s=+1/2}(f_i)$.[30]

As explained in Ref. 1, the most important feature of Eqs. (6.8) is their implication that the phases of the form factors are independent of q^2. If these overall phases are divided out, the form factors become real functions suitable for representation by conventional dispersion relations. The actual overall phase of each form factor has no absolute significance, because we can always decide to work with new initial states

$$|\, f_1(\vec{p}_i,s_i) \rangle' = e^{i\phi} \,|\, f_1(\vec{p}_i,s_i) \rangle \,.$$

If the transition form factors are still defined by a relation of the form (6.7), they will then clearly change by a factor $e^{i\phi}$. Nevertheless, the phase of a form factor *relative* to that of other amplitudes contributing to the same process obviously *is* significant. For example, in $\nu_1 + d \to \nu_2 + d$, where d is the down quark, one-photon exchange, which depends on form factors of the kind in Eq. (6.7), interferes with two-W exchange. Thus, it is useful to be able to check that, given one's arbitrary conventions, the phase of a form factor resulting from an actual calculation is correct. This one can do with aid of Eqs. (6.8). From Table I, we see that once the creation phase factors are chosen, ξ is determined apart from a sign, so that Eqs. (6.8) specify the phase of each form factor, except for a fourfold ambiguity. More usefully, if $f_{1,2}$ in Eq. (6.7) are neutrinos $\nu_{1,2}$ which mix under the weak interactions, then as shown in Sec. IV all $\eta_\xi(\nu_m)$ are equal, so that

$$\xi = \frac{\lambda(\nu_2)}{\lambda(\nu_1)} \,. \tag{6.10}$$

Hence, once the creation phase factors are chosen, the arbitrary phases of the form factors are fixed, apart from a sign, and may be read off from Eqs. (6.8).

When *CP* is conserved, $F_{21} = M_{21} = 0$ ($G_{21} = E_{21} = 0$) if f_1 and f_2 have the same (opposite) *CP* parity.[31] For radiative decay involving a real photon, $f_1 \to f_2 + \gamma$, only the M and E terms contribute [see Eq. (6.7)]. Thus, when *CP* is conserved, the radiation will be pure $E1$ (pure $M1$) when f_1 and f_2 have the same (opposite) *CP* parity.[32,31] This selection rule is easily understood by going to the cross channel $\gamma \to f_1 + f_2$, where one quickly discovers that it is the $\sigma_{\mu\nu}q_\nu\gamma_5$ ($\sigma_{\mu\nu}q_\nu$) coupling which conserves *CP* when f_1 and f_2 have the same (opposite) *CP* parity.[33]

It is instructive to see, in a language-independent way, how the phases of neutrino transition form factors behave, and how some of these form factors vanish in the *CP*-conserving case, when the form factors are calculated in terms of the one-loop diagrams of Fig. 2. Of the two diagrams, only diagram S would be present for Dirac neutrinos. This diagram by itself gives

$$\langle \nu_2 \,|\, J_\mu^{\rm EM} \,|\, \nu_1 \rangle = U_{l2}^* U_{l1} \bar{u}_2 \Gamma_\mu(\gamma_5) u_1 \,, \tag{6.11}$$

with some specific $\Gamma_\mu(\gamma_5)$ of the form shown in Eq. (6.7), involving γ_5. Diagrams S and S' together then yield

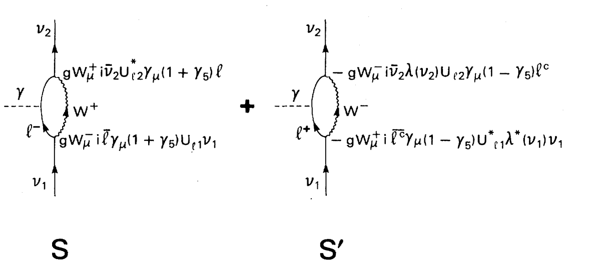

FIG. 2. Gauge-boson loop diagrams for the transition $\nu_1 \to \nu_2 + \gamma$. The term in the Hamiltonian (2.1) which is active at each vertex is written next to it. In the diagram S', the vertex terms are rewritten in terms of the fields l^c and ν_m^c using Eqs. (5.7) and (5.8). (It is understood that these diagrams are accompanied by similar ones where the photon attaches to the W line.)

$$\langle \nu_2 | J_\mu^{\rm EM} | \nu_1 \rangle = U_{l2}^* U_{l1} \bar{u}_2 \left[\Gamma_\mu(\gamma_5) - \frac{U_{l1}^*/U_{l1}}{U_{l2}^*/U_{l2}} \frac{\lambda^*(\nu_1)}{\lambda^*(\nu_2)} \Gamma_\mu(-\gamma_5) \right] u_1 , \tag{6.12}$$

where the minus sign in front of the S' contribution is due to the photon vertex. From Eqs. (4.4) and (5.1), we see that in the CP-conserving case, the quantity in brackets in Eq. (6.12) is just

$$\left[\Gamma_\mu(\gamma_5) - \frac{\tilde{\eta}_z(\nu_1)}{\tilde{\eta}_z(\nu_2)} \Gamma_\mu(-\gamma_5) \right] . \tag{6.13}$$

Thus, if the neutrinos have the same (opposite) CP parity, the form factors F_{21}, M_{21} (G_{21}, E_{21}) vanish, as required.[34] Whether or not CP is conserved, imagine now that we change the language by replacing the creation phase factors $\lambda(\nu_m)$ by new ones,

$$\lambda'(\nu_m) = \lambda(\nu_m) e^{-2i\phi_m} . \tag{6.14}$$

To preserve the physical content of the theory, we must simultaneously change the U matrix so that the net effect is a rephasing of the ν_m states, plus a corresponding rephasing of the ν_m fields so that Eq. (3.17) is obeyed. From the discussions surrounding Eqs. (3.18) and (5.5), the new states are $|\nu_m\rangle' = e^{i\phi_m} |\nu_m\rangle$, and the new U matrix is given by

$$U'_{lm} = U_{lm} e^{i\phi_m} . \tag{6.15}$$

Hence,

$$U'^*_{l2} U'_{l1} = U^*_{l2} U_{l1} e^{i(\phi_1 - \phi_2)} ,$$

so that according to Eq. (6.12) the new form factors are phase rotated from the old ones by the amount that they should be in view of the new states. Equation (6.8) correctly reflects this change in the form factors, since according to Eq. (6.10) the new ξ is $\xi' = \xi e^{2i(\phi_1 - \phi_2)}$.[35]

VII. SUMMARY

The amplitudes for reactions involving Majorana particles contain new phase factors, characteristic of these particles, with important physical consequences. The precise location of these phase factors within the amplitudes varies with one's formalism or language. Going from one language to another involves a change in the creation phase factors which occur in the Majorana fields, and amounts to a redefinition of the states and fields of the Majorana particles, with corresponding changes in the coupling constants so as to keep the physics unchanged.

The new phase factors in the reaction amplitudes are a reflection of the discrete symmetry properties of a Majorana particle, and in particular of the fact that such a particle can have a physically significant CP quantum number, and sometimes a C quantum number as well. The CP parity must be imaginary, and the C parity real, and we saw examples of the consequences of these rules. We also showed that for a Majorana particle of arbitrary spin J, $(CPT)^2 = (-1)^{2J}$, which implies that $\tilde{\eta}\,_{CPT}^{-s} = (-1)^{2J} \tilde{\eta}\,_{CPT}^{s}$. This constraint was used to find the general forms for the neutral-weak-current and electromagnetic interactions of spin-$\frac{1}{2}$ Majorana particles.

If neutrinos are Majorana particles, then, even if CP is conserved, either the U matrix is not real, or else there are creation phase factors in the Feynman amplitudes for neutrino reactions, or both, depending on one's language. In particular, when CP is conserved, two neutrinos of opposite (like) CP parity will contribute with opposite (like) sign to the amplitude for $(\beta\beta)_{0\nu}$. Now, the radiative decay of the heavier of these two neutrinos to the lighter one will yield pure $M1$ ($E1$) radiation when the neutrinos are of opposite (like) CP parity. Thus, when CP is conserved, there is a model-independent correlation between the character of the radiation in this decay, and the character of the interference between the contributions of these same two neutrinos to $(\beta\beta)_{0\nu}$.

It is, of course, important to ask whether neutrinos not only *can* interfere destructively in $(\beta\beta)_{0\nu}$, but actually *do so* in popular, reasonable models. Unfortunately, it is very easy to construct gauge models in which there is significant (or even total[36]) cancellation. If all M_m are small compared to typical $(\beta\beta)_{0\nu}$ momentum transfers, one can accomplish this simply by making the electron-electron element of the neutrino mass matrix small or zero.[2] What is worse, there are popular models which were constructed without $(\beta\beta)_{0\nu}$ in mind, and in which severe cancellation occurs. The severity can be expressed in terms of the effective mass

$$M_{\rm eff} = \sum_m \frac{\tilde{\eta}_z(\nu_m)}{i} |U_{em}|^2 M_m , \tag{7.1}$$

to which the $(\beta\beta)_{0\nu}$ amplitude is proportional. Chang and Pal have found that in some realizations of SO(10) grand unified theory, $M_{\rm eff}$ is an order of magnitude below the lightest of the actual masses M_m of the neutrinos.[37] Neutrinoless double-β decay remains an extremely interesting experiment, representing as it does the only currently known way to tell whether neutrinos are Majorana or Dirac particles. However, we must bear in mind that an experimental upper limit on the $(\beta\beta)_{0\nu}$ amplitude does not really imply an upper limit on the mass of any Majorana neutrino.

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APPENDIX: DISCRETE-SYMMETRY TRANSFORMATIONS OF MAJORANA PARTICLES

Here we show that if C, CP, and CPT are defined to affect Majorana fields as shown in Table I, then they will affect the corresponding states as indicated in that table. We also prove that this definition of the discrete-symmetry operations requires that the phase factors be restricted as stated in Table I.

C

From Table I

$$C\Psi C^{-1} \equiv \eta_c^* \gamma_2 \Psi^* \; . \tag{A1}$$

Using Eq. (5.8), this becomes

$$C\Psi C^{-1} = (\eta_c \lambda)^* \Psi \; . \tag{A2}$$

From the plane-wave expansion of ψ, Eq. (3.1), this implies that

$$C f_{\vec{p}s} C^{-1} = (\eta_c \lambda)^* f_{\vec{p}s} \tag{A3a}$$

and

$$C f^\dagger_{\vec{p}s} C^{-1} = (\eta_c \lambda)^* f^\dagger_{\vec{p}s} \; . \tag{A3b}$$

If we Hermitian conjugate Eq. (A3a), it becomes

$$C f^\dagger_{\vec{p}s} C^{-1} = (\eta_c \lambda) f^\dagger_{\vec{p}s} \; . \tag{A3a$'$}$$

Applying this relation to the vacuum, which we define to be C even, we find that

$$C \mid \vec{p},s \rangle = (\eta_c \lambda) \mid \vec{p},s \rangle \; . \tag{A4}$$

Thus, the one-particle state transforms, as desired, into itself under C, except for a phase factor $\tilde{\eta}_c$ given by

$$\tilde{\eta}_c = \eta_c \lambda \; . \tag{A5}$$

Furthermore, the consistency of Eqs. (A3a′) and (A3b) requires that this $\tilde{\eta}_c$, the C parity of the Majorana state, be real.

CP

From Table I, we define CP ($\equiv z$) by

$$z\Psi(\vec{x},t)z^{-1} \equiv \eta_z^* \gamma_4 \gamma_2 \Psi^*(-\vec{x},t) \; . \tag{A6}$$

Using Eq. (5.8), this reads

$$z\Psi(\vec{x},t)z^{-1} = (\eta_z \lambda)^* \gamma_4 \Psi(-\vec{x},t) \; . \tag{A7}$$

Inserting the plane-wave expansion of Ψ into this equation, and using the relations $\gamma_4 u_{\vec{p}s} = u_{-\vec{p}s}$, $\gamma_4 v_{\vec{p}s} = -v_{-\vec{p}s}$, we obtain

$$z\Psi(\vec{x},t)z^{-1} = (\eta_z \lambda)^* \sum_{\vec{p},s} \left[\frac{M}{E_{\vec{p}} V} \right]^{1/2} (f_{-\vec{p},s} u_{\vec{p}s} e^{ipx} - \lambda f^\dagger_{-\vec{p},s} v_{\vec{p}s} e^{-ipx}) \; . \tag{A7$'$}$$

Comparing this with Eq. (3.1), we conclude that

$$z f_{\vec{p},s} z^{-1} = (\eta_z \lambda)^* f_{-\vec{p},s} \tag{A8a}$$

and

$$z f^\dagger_{\vec{p},s} z^{-1} = -(\eta_z \lambda)^* f^\dagger_{-\vec{p},s} \; . \tag{A8b}$$

Upon Hermitian conjugation, Eq. (A8a) becomes

$$z f^\dagger_{\vec{p},s} z^{-1} = (\eta_z \lambda) f^\dagger_{-\vec{p},s} \; . \tag{A8a$'$}$$

Applying this result to the vacuum, which we define to be CP even, we have

$$z \mid \vec{p},s \rangle = (\eta_z \lambda) \mid -\vec{p},s \rangle \; . \tag{A9}$$

This reversal of $\vec{p}$ but not s is indeed the desired effect of CP on the one-particle state. Note that the phase factor $\tilde{\eta}_z$ picked up by the state under CP (that is, the CP parity of the state) is

$$\tilde{\eta}_z = \eta_z \lambda \; . \tag{A10}$$

Most importantly, note that the consistency of Eqs. (A8a′) and (A8b) demands that $\tilde{\eta}_z$ be *imaginary*.

As we see, the restriction on $\tilde{\eta}_z$ (and similarly on $\tilde{\eta}_c$) arises from the fact that the Majorana field Ψ both annihilates and creates the same particle, so that Eqs. (A8a) and (A8b) both constrain the same operator. That the restricted $\tilde{\eta}_z$ is required, in particular, to be imaginary is due to the minus sign in Eq. (A7′), which, in turn, is due to that in the relations $\gamma_4 u_{\vec{p}s} = u_{-\vec{p}s}$, $\gamma_4 v_{\vec{p}s} = -v_{-\vec{p}s}$.[38] The latter minus sign is the same one which causes the s-wave states of positronium and quarkonium to have *negative* parity.

CPT

Here, due to the antiunitarity of CPT ($\equiv \zeta$), we encounter some special features. Applying ζ to the plane-wave expansion of Ψ yields

$$\zeta\Psi(x)\zeta^{-1} = \sum_{\vec{p},s} \left[\frac{M}{E_{\vec{p}} V} \right]^{1/2} [\zeta f_{\vec{p}s} \zeta^{-1} (u_{\vec{p}s} e^{ipx})^* + \lambda^* \zeta f^\dagger_{\vec{p}s} \zeta^{-1} (v_{\vec{p}s} e^{-ipx})^*] \; . \tag{A11}$$

From Table I, this is to be equated to $-\eta_\zeta^* \gamma_5 \Psi^*(-x)$. Since

$$\gamma_5 v_{\vec{p}s} = (-1)^{s-1/2} u_{\vec{p},-s}$$

and

$$\gamma_5 u_{\vec{p}s} = -(-1)^{s-1/2} v_{\vec{p},-s} \; ,$$

we have

$$-\eta_\zeta^* \gamma_5 \Psi^*(-x) = \eta_\zeta^* \sum_{\vec{p},s} \left[\frac{M}{E_{\vec{p}} V} \right]^{1/2} (-1)^{s-1/2} \times [\lambda^* f_{\vec{p},-s} (u_{\vec{p}s} e^{ipx})^* - f^\dagger_{\vec{p},-s} (v_{\vec{p}s} e^{-ipx})^*] . \tag{A12}$$

Comparing Eqs. (A11) and (A12), we see that

$$\zeta f_{\vec{p},s} \zeta^{-1} = (\eta_\zeta \lambda)^* (-1)^{s-1/2} f_{\vec{p},-s} \tag{A13a}$$

and

$$\zeta f^\dagger_{\vec{p},s} \zeta^{-1} = -\eta_\zeta^* \lambda (-1)^{s-1/2} f^\dagger_{\vec{p},-s} . \tag{A13b}$$

Since $\zeta^{-1} \neq \zeta^\dagger$, it is not so obvious how to Hermitian conjugate Eq. (A13a). Therefore, rather than try to compare Eqs. (A13a) and (A13b) directly, we see whether the consistency between them restricts the *CPT* phase factors by constructing a relation which depends on both of them. We recall that

$$\langle b \mid Q \mid a \rangle = \langle \zeta a \mid \zeta Q^\dagger \zeta^{-1} \mid \zeta b \rangle$$

for any operator Q, so that if we define $\zeta \mid 0 \rangle = \mid 0 \rangle$,

$$\begin{aligned} 1 &= \langle 0 \mid f_{\vec{p}s} f^\dagger_{\vec{p}s} \mid 0 \rangle \\ &= \langle 0 \mid \zeta f_{\vec{p}s} \zeta^{-1} \zeta f^\dagger_{\vec{p}s} \zeta^{-1} \mid 0 \rangle . \end{aligned} \tag{A14}$$

Using Eqs. (A13a) and (A13b), we then have

$$1 = -\eta_\zeta^{*2} , \tag{A15}$$

so that η_ζ must be imaginary. In view of this result, the application of Eq. (A13b) to the vacuum gives

$$\zeta \mid \vec{p}, s \rangle = \eta_\zeta \lambda (-1)^{s-1/2} \mid \vec{p}, -s \rangle . \tag{A16}$$

That is, the action of *CPT* on a one-particle state reverses s but not $\vec{p}$, as desired, and introduces the s-dependent phase factor[39]

$$\tilde{\eta}^s_\zeta = \eta_\zeta \lambda (-1)^{s-1/2} . \tag{A17}$$

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[1]B. Kayser and A. S. Goldhaber, Phys. Rev. D **28**, 2341 (1983).

[2]L. Wolfenstein, Phys. Lett. **107B**, 77 (1981).

[3]Our analysis is in terms of the neutrinos of definite mass. For recent, interesting discussions of the relationship beween the properties of these neutrinos and the structure of the neutrino mass matrix in the charged-lepton basis, see S. P. Rosen, Los Alamos National Laboratory Report No. LA-UR-83-3546 (unpublished); R. Mohapatra, in *Proceedings of the Third LAMPF II Workshop, Los Alamos, 1983,* edited by J. Allred, T. Bhatia, K. Ruminer, and B. Talley (Los Alamos National Laboratory Report No. LA-9933-C, 1983), p. 510; C. Leung and S. Petcov, Phys. Lett. **125B**, 461 (1983).

[4]S. P. Rosen, in *Neutrino 81,* Proceedings of the International Conference on Neutrino Physics and Astrophysics, Maui, Hawaii, 1981, edited by R. J. Cence, E. Ma, and A. Roberts (University of Hawaii, High Energy Physics Group, Honolulu, 1981), p. 76.

[5]M. Doi *et al.*, Phys. Lett. **102B**, 323 (1981).

[6]From the observations in J. Schechter and J. Valle, Phys. Rev. D **25**, 283(E) (1982), it follows that they agree.

[7]A. Halprin, S. Petcov, and S. P. Rosen, Phys. Lett. **125B**, 335 (1983).

[8]When *CP* is *violated,* the matrix elements U_{em} can have any phases at all, and the contributions of different neutrinos, being proportional to $U_{em}{}^2$, can interfere with *arbitrary* relative phases. The *CP*-violating phases which may occur in the U matrix in the Majorana case have been analyzed by S. Bilenky, J. Hošek, and S. Petcov [Phys. Lett. **94B**, 495 (1980)], by J. Schechter and J. Valle [Phys. Rev. D **22**, 2227 (1980)], by Dori *et al.* (Ref. 5), and by J. Bernabeu and P. Pascual [Nucl. Phys. **B228**, 21 (1983)]. The latter authors also discuss the *CP*-conserving case.

[9]The modifications induced in Eq. (2.2) by *CP* violation are discussed for the two-generation case by C. Kim and H. Nishiura [Johns Hopkins University Report No. JHU-HET 8309 (unpublished)].

[10]J. Vergados, Phys. Rev. D **28**, 2887 (1983).

[11]The substitution of Majorana fields for Dirac ones in a theory of Dirac particles *does* yield a sensible theory of Majorana particles.

[12]We thank Lincoln Wolfenstein for an interesting conversation in which he independently put forward the essence of this argument. We would also like to thank him, Darwin Chang, Ling-Fong Li, and Palash Pal for a very enlightening general discussion of how phase-factor restrictions are related to interactions.

[13]This is an easy way to see that the *CP*-conserving decays of the $^3S_1$ ($C=-1$) states of quarkonium into $\tilde{\gamma}\tilde{\gamma}$ or $\tilde{g}\tilde{g}$ ($\tilde{g}$=Majorana gluino) must violate parity. Such decays of t-quarkonium are discussed as a possible way to detect photinos or gluinos by J. Ellis and S. Rudaz [Phys. Lett. **128B**, 248 (1983)].

[14]Since the weak interactions are maximally C violating, a Majorana neutrino is not an eigenstate of C, but of *CPT*, as discussed by B. Kayser [Phys. Rev. D **26**, 1662 (1982)], and in Ref. 1. Obviously, Majorana particles not involved in the weak interactions are also self-conjugate under *CPT*. Therefore, to be completely general, we define an arbitrary Majorana particle to be one which is self-conjugate under *CPT*.

[15]The result $\zeta^2 = (-1)^{2J}$ already appears in P. Carruthers, Phys. Lett. **26B**, 158 (1968); and *Spin and Isospin in Particle Physics* (Gordon and Breach, New York, 1971); and in E. Wigner, in *Group Theoretical Concepts and Methods in Elementary Particle Physics,* edited by F. Gürsey (Gordon and Breach, New York, 1964), p. 37. See also G. Feinberg and S. Weinberg, Nuovo Cimento **14**, 571 (1959). The present derivation of this result demonstrates, however, that it does not depend on field-theoretic assumptions, or on any conventions, such as those pertaining to Dirac γ matrices, Dirac spinors, or representations of the angular momentum operators. This derivation also shows how surprisingly simply the result can be obtained.

[16]The properties of ζ that we have used to obtain Eq. (3.10) for a Majorana particle also hold for T applied to *any* particle in its rest frame. Thus, the result (3.10) is related to the fact, discussed by J. J. Sakurai, in *Invariance Principles and Elementary Particles* (Princeton University Press, Princeton, NJ, 1964), that $T^2 = (-1)^{2J}$.

[17]We thank Alfred S. Goldhaber for the conversation in which this argument was constructed.

[18]We thank R. Mohapatra for asking us this question.

[19]Note from Table I that $\eta_c^* \Psi^c$, and not $\Psi^c \equiv \gamma_2 \Psi^*$, is the charge-conjugate field $C\Psi C^{-1}$. In the Majorana case, it is dangerous to call Ψ^c the "charge-conjugate field" because, as Eq. (5.8) indicates, the phase factor relating Ψ^c to Ψ is the arbitrary creation phase factor, rather than the C parity of the Majorana particle. Indeed, in language L2, for example, $\nu_m^c = [\tilde{\eta}_{CP}(\nu_m)/i]\nu_m$. Furthermore, the field Ψ^c is a useful concept even when, as here, C is not conserved. When it is conserved, $\eta_c^* \Psi^c = \tilde{\eta}_c \Psi$ bears the proper relation of a charge-conjugate field to Ψ.

[20]It is trivial to check that the propagator $\nu_m \bar{\nu}_m$ contains no unconventional phases, even though a creation phase factor is in general present in the ν_m field.

[21]If one follows the approach used in Ref. 1 to obtain Eq. (2.2), but invokes only CPT invariance, rather than CP invariance, one finds that

$$A[(\beta\beta)_{0\nu}] \propto \sum_m \tilde{\eta}_\xi^{s=+1/2}(\nu_m) U_{em}{}^2 M_m \ .$$

From Table I and the equality of the $\eta_\xi(\nu_m)$,

$$\tilde{\eta}_\xi^{s=+1/2}(\nu_m) \propto \lambda(\nu_m) \ .$$

Thus, one confirms Eq. (5.10).

[22]In Ref. 2, Wolfenstein was implicitly using the language we are calling L2.

[22(a)]*Note added in proof.* After submission of this work for publication, we received a Joint Institute for Nuclear Research (Dubna) report by S. Bilenky, N. Nedelcheva, and S. Petcov (unpublished), which, in a spirit very similar to that of the present paper, stresses the arbitrariness of the factors $\lambda(\nu_m)$ appearing in Eq. (5.8), but shows that $A[(\beta\beta)_{0\nu}]$ has the form of Eq. (2.2) independently of these factors. Bilenky *et al.* favor a formalism which we might call language L3 in which the elements of the CP-conserving U matrix have phases of $\pm(\pi/4)$.

[23]After this work was essentially completed, we received an interesting paper by A. Barroso and J. Maalampi [Phys. Lett. **132B**, 355 (1983)] which discusses the possibility of locating significant phases in alternative places in the *CP-violating,* two-generation case.

[24]C. Prescott *et al.*, Phys. Lett. **77B**, 347 (1978).

[25]That $\langle \tilde{\gamma} \,|\, N_\mu \,|\, \tilde{\gamma} \rangle$ must involve only three form factors can be seen very easily by considering the cross-channel process $Z^0(q^2) \rightarrow \tilde{\gamma}\tilde{\gamma}$, where the Z^0 is allowed to be off shell and so can have either $J=1$ or $J=0$. Supposing for simplicity that the photinos are nonrelativistic, there are then three allowed antisymmetric $\tilde{\gamma}\tilde{\gamma}$ final states: $^3P_1$, $^3P_0$, and $^1S_0$.

[26]The use of CPT constraints to determine the electromagnetic matrix element of a Majorana fermion was first attempted by J. Nieves [Phys. Rev. D **26**, 3152 (1982)], and by B. McKellar [Los Alamos National Laboratory Report No. LA-UR-82-1197 (unpublished)]. However, their calculations needed revision to take into account the nontrivial CPT phase factors $\tilde{\eta}_\xi^s$.

[27]Equation (6.6) agrees with the result obtained by other means by Kayser (Ref. 14), by Nieves (Ref. 26), and by R. Shrock [Nucl. Phys. **B206**, 359 (1982)]. See also J. Schechter and J. Valle, Phys. Rev. D **24**, 1883 (1981).

[28]Note that for a Majorana particle f of well-defined C and P, the electromagnetic coupling $\gamma \rightarrow ff$ is C and P violating, since $C(ff) = \tilde{\eta}_c^2(f) = 1$ and $P(ff;{}^3P_1) = -\tilde{\eta}_P^2(f) = 1$.

[29]In a Comment on Ref. 1 [Phys. Rev. D **29**, 1542 (1984)], A. Khare and J. Oliensis have just used these same CPT techniques to constrain the *gravitational* interaction of a spin-$\frac{1}{2}$ Majorana particle.

[30]For the special case $\xi = 1$, the relations (6.8) were obtained earlier from CPT by McKellar (Ref. 26), and through another method by Nieves (Ref. 26). These earlier treatments did not uncover the fact that ξ, which in general is not unity, is present in these relations.

[31]Nieves (Ref. 26) and McKellar (Ref. 26).

[32]P. Pal and L. Wolfenstein, Phys. Rev. D **25**, 766 (1982).

[33]See Kayser (Ref. 14) and McKellar (Ref. 26).

[34]For language L2, essentially this analysis is given in Ref. 32.

[35]For $\xi = 1$, the requirements of Eq. (6.8) are verified for the form factors resulting from loop diagrams by Nieves (Ref. 26).

[36]A. Zee, Phys. Lett. **93B**, 389 (1980).

[37]D. Chang and P. Pal, Phys. Rev. D **26**, 3113 (1982).

[38]We thank P. Pal for showing explicitly that these relations are representation-independent.

[39]Carruthers (in *Spin and Isospin in Particle Physics,* Ref. 15) has carried out the reverse analysis in which the discrete-symmetry transformations are defined in terms of their effects on states, and the implied effects on fields are then found and required to be simple. The conclusions of his analysis and the one given here are mostly in agreement. See, in addition, S. Weinberg, Phys. Rev. **133**, B1318 (1964). Also, after this work was essentially completed, we received a paper by M. Doi *et al.* [Prog. Theor. Phys. **70**, 1331 (1983)], which treats the case $\lambda = 1$ and follows a third approach; namely, the discrete-symmetry transformations are required to leave the free Lagrangian, expressed in terms of two-component Majorana fields, invariant, and the consequences of this requirement are found.