

The above theorem certainly holds. We note there that this theorem is applicable to all the limit-integral exchanges done in this paper. The integrands always consist of an integrable function along with either the denominator D above or a term like $(1/u\epsilon) \ln D$ which is also bounded.⁶

Getting back to our commutator, we have found

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int d^3y [T^{*k10}(y_4 = \epsilon) - T^{*k10}(y_4 = -\epsilon)] \\ = \frac{2ic}{\pi^4} \frac{z_4 z_\alpha}{z^8} \epsilon^{k01\alpha} \\ = \frac{2c}{\pi^4} \frac{z^0 z_\alpha}{(z^2)^4} \epsilon^{k01\alpha}. \quad (\text{A10}) \end{aligned}$$

⁶ Uniform convergence does not hold for some of the integrals, necessitating the weaker dominated-convergence theorem.

This means that the equal-time commutator has a $\delta^3(y)$ with the above coefficient. In particular,

$$\begin{aligned} (1/i)\delta(x^0 - y^0) \langle T\{[A^k(x), V^l(y)]V^0(z)\} \rangle \\ = -\frac{2ic}{\pi^4} \frac{(z-x)_\alpha (z-x)^0}{(z-x)^8} \delta^4(x-y), \quad (\text{A11}) \end{aligned}$$

where we have reinserted x . This is seen to be a particular term of

$$\begin{aligned} \frac{ic}{\pi^4} \epsilon^{0\nu\rho\alpha} [(z-x)^2 g_\alpha^\rho - 2(z-x)_\alpha (z-x)^\rho] \\ \times \frac{1}{(z-x)^8} \delta^4(x-y), \quad (\text{A12}) \end{aligned}$$

as quoted in the text. We will not bother to illustrate the veracity of the above expression for other choices of $\mu\nu\rho$.

Group-Theoretical Construction of Dual Amplitudes

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(Received 5 October 1970)

We present general group-theoretical requirements for the construction of factorizable, dual N -point functions. Dual amplitudes for the scattering of arbitrary numbers of spinning particles are built as an example of this approach.

ALTHOUGH the $SU(1,1)$ invariance of the Veneziano N -point function has long been recognized,¹ it has not yet been systematically exploited in the construction of new dual amplitudes. In the following, we would like to generalize the group-theoretic structure of the multi-Veneziano function, discussed earlier by the authors,² to include external particles of different types. As an example of our techniques, we give a simple, factorizable dual amplitude for the absorption of arbitrary numbers of high-spin particles.

For the purpose of this paper, we regard duality as a purely group-theoretical concept, implying $SU(1,1)$ invariance but not necessarily any particular asymptotic behavior (e.g., Regge). Basic to our description are the three generators of $SU(1,1)$, namely, L_0 , L_+ , and L_- , which satisfy the algebra

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = -L_0. \quad (1)$$

We present the following minimal set of group-theoretical conditions for the construction of dual amplitudes.

¹Z. Koba and H. B. Nielsen, Nucl. Phys. B12, 517 (1969).

²L. Clavelli and P. Ramond, Phys. Rev. D 2, 973 (1970).

(i) Associate with the absorption of a particle with momentum k_μ , spin j , j_3 , internal quantum numbers $\{\lambda\}$, a vertex operator $V(k_\mu, j, j_3, \{\lambda\}; z)$, where $z = e^{-i\tau}$ is a complex variable on the unit circle.

(ii) Require that $V(k_\mu, j, j_3, \{\lambda\}; z)$ transforms under $SU(1,1)$ as a spin J_S representation, that is to say,

$$[L_0, V] = -z \frac{d}{dz} V, \quad (2a)$$

$$[L_\pm, V] = \frac{-1}{\sqrt{2}} z^{\pm 1} \left(z \frac{d}{dz} \mp J_S \right) V, \quad (2b)$$

where we take J_S to be in general a function of the Casimir operators of the Lorentz and internal symmetry groups

$$J_S = J_S(m^2, j, c^\lambda). \quad (3)$$

(iii) Under the Lorentz and internal symmetry groups, V is required to transform in the same way as the field of the absorbed particle. This ensures the correct selection rules at each vertex.

(iv) Any number of external particles 1, 2, . . . , l , . . . can interact in a dual manner only if they have the same $SU(1,1)$ spin, i.e.,

$$J_S(m_1^2, j_{(1)}, c_{(1)}^\lambda) = J_S(m_l^2, j_{(l)}, c_{(l)}^\lambda) \quad \text{for all } l. \quad (4)$$

This implies relations among the quantum numbers of different particles.

(v) Then the factorizable dual amplitude for the scattering of particles 1, 2, . . . , l , . . . , N , in that order, is given by

$$A_N = \frac{1}{C} \left\langle 0 \left| \oint \prod_{l=1}^N \left(\frac{dz_l}{z_l} |z_l - z_{l+1}|^{-1-J_S \theta(\arg z_{l+1} - \arg z_l)} \right. \right. \right. \\ \left. \left. \left. \times V(k_{l\mu}, j_{(l)}, j_{3(l)}, \{\lambda\}_{(l)}; z_l) \right) \right| 0 \right\rangle, \quad (5)$$

where C is the integrated Haar measure, and the contour is taken around the unit circle. This type of expression was shown to be $SU(1,1)$ invariant in Ref. 2. If V is a matrix in the internal group space, the vacuum expectation value must be defined to include the trace of the product of the vertices.

We now proceed to illustrate these rules by considering several examples of dual amplitudes. The best known example of a vertex satisfying criteria (i)-(iv) is the one for scalar isoscalar particles:

$$V(k_\mu, 0, 0; z) = e^{-k^2/2\epsilon} e^{ik \cdot F^\dagger(z)} e^{ik \cdot F(z)}, \quad (6)$$

where $F(z)$ satisfies^{2,3}

$$[L_0, F_\rho(z)] = -z \frac{d}{dz} F(z), \quad (7a)$$

$$[L_{\pm}, F_\rho(z)] = -\frac{z^{\pm 1}}{\sqrt{2}} \left(\frac{d}{dz} \pm \frac{\epsilon}{2} \right) F(z). \quad (7b)$$

This vertex obeys Eqs. (2), with $J_S = -\frac{1}{2}k^2$. Its insertion in Eq. (5) yields the usual multi-Veneziano amplitude, provided [criterion (iv)] that all the particles have the same mass. We will let the factor $e^{-k^2/2\epsilon}$ be understood from here on.

A simple extension of this vertex for isovector scalar particles

$$V(k_\mu, 0, L_k; z) = \sigma_k V(k_\mu, 0, 0; z) \quad (8)$$

leads to the Chan-Paton amplitude.⁴

We now apply our criteria to the construction of dual vertices for particles of higher spin. Our basic building

³ S. Fubini and G. Veneziano, Nuovo Cimento **67**, 29 (1970).

⁴ J. E. Paton and Chan Hong-Mo, Nucl. Phys. **B10**, 516 (1969).

blocks will be the Nambu field⁵

$$Q_\mu(z) = F_\mu(z) + F_\mu^\dagger(z) \quad (9a)$$

$$= \sum_{m=0}^{\infty} \left(\frac{(m-1+\epsilon)!}{m!} \right)^{1/2} \\ \times (a_\mu^{(m)} z^{m+\epsilon/2} + a_\mu^{(m)\dagger} z^{-m-\epsilon/2}) \quad (9b)$$

and its conjugate momentum

$$P_\mu(z) = i[L_0, Q_\mu(z)] \quad (10a)$$

$$= -i \sum_{m=0}^{\infty} (m + \frac{1}{2}\epsilon) \left(\frac{(m-1+\epsilon)!}{m!} \right)^{1/2} \\ \times (a_\mu^{(m)} z^{m+\epsilon/2} - a_\mu^{(m)\dagger} z^{-m-\epsilon/2}), \quad (10b)$$

where ϵ has to be taken to zero at the end of all calculations. It follows from Eqs. (10) and (7) that

$$[L_0, P_\mu(z)] = -z \frac{d}{dz} P_\mu(z), \quad (11a)$$

$$[L_{\pm}, P_\mu(z)] = -\frac{z^{\pm 1}}{\sqrt{2}} \left[\frac{d}{dz} \pm \left(1 + \frac{\epsilon}{2} \right) \right] P_\mu(z) \\ + \frac{i\epsilon}{2\sqrt{2}} z^{\pm 1} Q_\mu(z). \quad (11b)$$

As a first attempt at constructing a vertex for the absorption of a vector meson, we might consider $e^{ik \cdot F^\dagger(z)} e^{ik \cdot F(z)} P_\mu(z)$. This indeed transforms irreducibly under the group, but its lack of normal ordering leads to a divergent dual amplitude. From this stems the general necessity to consider only normal-ordered vertices, as is common in field theory. On the other hand, the normal-ordered vertex $e^{ik \cdot F^\dagger(z)} P_\mu(z) e^{ik \cdot F(z)}$ does not transform irreducibly:

$$[L_+, e^{ik \cdot F^\dagger(z)} P_\mu(z) e^{ik \cdot F(z)}] \\ = -\frac{z}{\sqrt{2}} \left(\frac{d}{dz} + 1 + \frac{k^2}{2} \right) e^{ik \cdot F^\dagger(z)} P_\mu(z) e^{ik \cdot F(z)} \\ + \frac{k_\mu z}{2\sqrt{2}} e^{ik \cdot F^\dagger(z)} e^{ik \cdot F(z)}. \quad (12)$$

The undesirable extra term can be eliminated only if one takes the transverse part of $P_\mu(z)$:

$$\hat{P}_\mu(z) = (g_{\mu\rho} - k_\mu k_\rho / k^2) P_\rho(z). \quad (13)$$

Then, a well-behaved normal-ordered vertex for vector mesons is

$$V_\mu(k; z) = e^{ik \cdot F^\dagger(z)} \hat{P}_\mu(z) e^{ik \cdot F(z)} \quad (14)$$

⁵ Y. Nambu, University of Chicago Report No. EFI 6964, 1969 (unpublished).

and satisfies

$$[L_0, V_\mu(k, z)] = -z \frac{d}{dz} V_\mu(k, z), \quad (15a)$$

$$[L_\pm, V_\mu(k, z)] = -\frac{z^{\pm 1}}{\sqrt{2}} \left[z \frac{d}{dz} \pm \left(1 + \frac{k^2}{2} \right) \right] V_\mu(k, z). \quad (15b)$$

Thus V_μ transforms as a $J_S = -1 - \frac{1}{2}k^2$ representation of $SU(1,1)$. According to our rules, for any fixed value of k^2 , we can construct a dual vector-meson N -point amplitude. However, if we wish to have a dual amplitude containing both scalar and vector mesons, criterion (iv) forces the mass relation

$$1 + \frac{1}{2}k_V^2 = \frac{1}{2}k_S^2, \quad (16)$$

which fixes the vector meson as the first Regge recurrence of the ground-state scalar.

As an example we will calculate the amplitude $S+V \rightarrow S+S$. Putting $\frac{1}{2}k_S^2 = \alpha_0$, we have from Eq. (5)

$$A_\mu = \frac{1}{C} \oint \prod_{i=1}^4 \left(\frac{dz_i}{z_i} |z_i - z_{i+1}|^{\alpha_0 - 1} \theta(\arg z_{i+1} - \arg z_i) \right) \times \langle 0 | V(k_1, z_1) V_\mu(k_2, z_2) V(k_3, z_3) V(k_4, z_4) | 0 \rangle \quad (17a)$$

$$= \left(g_{\mu\rho} - \frac{k_{2\mu} k_{2\rho}}{k_2^2} \right) \frac{1}{C} \oint \prod_{i=1}^4 \left(\frac{dz_i}{z_i} |z_i - z_{i+1}|^{\alpha_0 - 1} \right) \times \theta(\arg z_{i+1} - \arg z_i) \prod_{i < j} |z_i - z_j|^{k_i \cdot k_j} \times \left(\sum_{l \neq 2} \frac{1}{2} k_{l\rho} \frac{z_l + z_2}{z_l - z_2} \right) \quad (17b)$$

$$= \left(g_{\mu\rho} - \frac{k_{2\mu} k_{2\rho}}{k_2^2} \right) \int_0^1 dx x^{-1-\alpha(s)} (1-x)^{-1-\alpha(t)} \times [k_{1\rho}(1-x) - k_{3\rho}x] \quad (17c)$$

$$= \int_0^1 dx x^{-1-\alpha(s)} (1-x)^{-1-\alpha(t)} \times [k_{1\mu}(1-x) - k_{3\mu}x + k_{2\mu}(\frac{1}{2}-x)]. \quad (17d)$$

This amplitude coincides with the one proposed by Bouchiat, Gervais, and Sourlas⁶ to represent a vector

⁶ C. Bouchiat, J. L. Gervais, N. Sourlas, Nuovo Cimento Letters **3**, 767 (1970); see also R. Bower and J. Weis, Phys. Rev. **188**, 2486 (1969); **188**, 2495 (1969); Phys. Rev. D **3**, 451 (1971).

current interacting with three scalars. From the group-theoretical point of view, however, it is clear that the amplitude more appropriately describes a vector meson since the mass must remain fixed for duality. Indeed in the present $SU(1,1)$ coupling scheme, a dual theory with currents must await the construction of a current vertex whose $SU(1,1)$ transformation properties are independent of its squared four-momentum.

Needless to say, it is trivial to calculate a dual amplitude involving any number of scalars and vectors using Eq. (5) and maintaining the mass relation (16).

We would now like to investigate dual vertices for higher-spin particles. If we write symbolically

$$\hat{P}_\mu^{(n)}(a) \equiv \prod_{i=1}^n \hat{P}_{\mu_i}(z_i), \quad (18)$$

then

$$V_\mu^{(n)}(z) = e^{ik \cdot F^\dagger(z)} : \hat{P}_\mu^{(n)}(z) : e^{ik \cdot F(z)} \quad (19)$$

transforms reducibly under the Lorentz group, containing spins up to $j_{\max} = n$. Furthermore, under $SU(1,1)$

$$[L_\pm, V_\mu^{(n)}(z)] = -\frac{z^{\pm 1}}{\sqrt{2}} \left[z \frac{d}{dz} \pm \left(n + \frac{k^2}{2} \right) \right] V_\mu^{(n)}(z), \quad (20)$$

i.e., $V_\mu^{(n)}$ transforms as a $J_S = -n - \frac{1}{2}k^2$ representation. In order for the particles represented by the various $V_\mu^{(n)}$ to interact in a dual manner, we must have

$$n + \frac{1}{2}k_n^2 = \alpha_0 \quad (n=0, 1, \dots). \quad (21)$$

This mass relation identifies $V_\mu^{(n)}$ as representing particles at the $(n+1)$ th mass level. By operating on $V_\mu^{(n)}$ with the various spin-projection operators, we can separate distinct vertices for particles of spin $n, n-2, n-4, \dots$, etc. The odd daughters, however, cannot be constructed in this scheme since $V_\mu^{(n)}$ is transverse in all its indices. Furthermore, we have no way to distinguish degenerate states at the same spin.

Although we have investigated here only a very limited class of dual vertices, it seems clear that the group-theoretical approach should prove useful in the consideration of internal symmetries, off-shell currents, and the positive-intercept problem. For example, the identities found by Virasoro⁷ require only that $J_S = -1$, thereby perhaps avoiding the necessity of introducing an unphysical pole.

⁷ M. A. Virasoro, Phys. Rev. D **1**, 2933 (1970).