phases would be given by expressions analogous to Eq. (40).[12]

[12] After this work had been completed we learned that the two-ladder diagrams discussed in Sec. II have been studied independently by B. Hasslacher and D. K. Sinclair, and by I. Muzinich. Their results are in agreement with ours.

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Conformal Symmetry and Three-Point Functions*

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The general three-point functions invariant under conformal transformations are constructed for both axial-vector and vector currents. These are used to study the anomalies present in the Ward-Takahashi identities. The axial-vector anomalies are considered in detail and Schwinger terms are calculated using equal-time limits.

I. INTRODUCTION

IT has been generally recognized in recent months that the conformal group provides a useful tool for the investigation of certain problems in elementary-particle physics. The symmetries involved are obeyed by systems which include no massive particles and a dimensionless coupling constant; there is thus nothing to set the scale for measurements of length. Maxwell's equations, as an example, are invariant under conformal transformations since the photon mass is zero. The relevance of these considerations to high-energy physics is suggested by the hypothesis that when the energies involved in an experiment are much greater than the masses of the particles, it should not matter too much if we set those masses equal to zero. The behavior of a dimensionless form factor $f(p^2/m^2)$ as $p^2 \to \infty$ is the same as for $m^2 \to 0$. One might also claim that a canonical theory's fundamental behavior, which is probed at high energies, is governed by the commutation relations; these should be independent of the mass. The conformal group suggests itself as a useful technique for investigating any phenomenon which appears to be mass independent.

Once the masses in a theory are set formally to zero, calculational simplifications are great. In particular, demanding that a given function transform properly under an inversion of coordinates can in certain instances determine its algebraic structure, leaving only one or more arbitrary constants unspecified. In this paper, we use the simplifications imposed by conformal invariance to study three-point functions.

In Sec. II we briefly show how the algebraic structure of the covariant time-ordered products (T^*) of two and three currents are determined using inversion symmetry. Our currents will be assumed to be defined via fermion fields, although this is only relevant for comparison purposes. It will be seen that the results are very close to perturbation theory for quantum electrodynamics with zero fermion mass. We will also introduce the relevance of this general approach to the problem of the ambiguities present in triangle diagrams.

In Sec. III we use our expression for the general axial-vector three-point function to study the anomalies present in the Ward-Takahashi identities. We will compare our results for these anomalies with those obtained by Gerstein and Jackiw,[1] who investigated anomalous triangle diagrams in momentum space. We will then compute the Schwinger terms of the axial-vector three-point function by calculating the anomalous Ward identities in both covariant and noncovariant ways (using Johnson-Low-Bjorken-type limits).

II. INVERSION-SYMMETRIC T^* PRODUCTS

A. Introduction

The general conformal group under which Maxwell's equations are invariant is a 15-parameter group which includes the Poincaré transformations (ten parameters), scale transformations, and the special conformal transformations. These latter form a four-parameter group which transforms coordinates as

$$x^\mu \to \tilde{x}^\mu = \frac{x^\mu + a^\mu x^2}{1+2a\cdot x+a^2x^2}. \quad (1)$$

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[1] I. Gerstein and R. Jackiw, Phys. Rev. 181, 1955 (1969).

Rather than continually deal with this four-parameter transformation throughout the paper, we will instead deal with the much simpler discrete symmetry, the coordinate inversion

$$x^\mu \to \tilde{x}^\mu = x^\mu/x^2 . \qquad (2)$$

In order to show the equivalence between the two approaches, we consider the conformal transformation for large a and expand to order a^{-2}:

$$\tilde{x}^\mu = \frac{a^\mu}{a^2} + \frac{1}{a^2}\left(\frac{x^\mu}{x^2} - \frac{2a\cdot x a^\mu}{a^2x^2}\right) + O\left(\frac{1}{a^3}\right).$$

Therefore,

$$x^\mu \to \tilde{x}^\mu \approx \frac{a^\mu}{a^2} + \frac{1}{a^2}\left(g^{\mu\nu} - 2\frac{a^\mu a^\nu}{a^2}\right)\frac{x_\nu}{x^2} . \qquad (3)$$

We can immediately recognize a^μ/a^2 as a translation, $1/a^2$ as a scale transformation, and $g^{\mu\nu}-2a^\mu a^\nu/a^2$ as a Lorentz transformation[2] (we recognize it as an orthogonal transformation in four dimensions). Since the 15-parameter group includes all of these as symmetries, the remaining transformation must also be one: $x^\mu \to x^\mu/x^2$.

We note that the simplified technique which we are presenting here would not necessarily hold if we were to study arbitrary representations of the conformal group, since in general, inversion is not contained in a general $O(4,2)$ representation. However, we will only be dealing with vector and axial-vector operators which are a part of the O(4,2) representations of the type which do contain the inversion operator.

The essence of setting up an inversion-symmetric covariant time-ordered product is very simple: If we know how an individual current must transform, then we know how a product of currents must transform. In particular, if we consider a vector current, we know its dimensions to be L^{-3}. We write

$$V^\mu(x) = M^{\mu\nu}(x)V_\nu(1/x) , \qquad (4)$$

where $1/x$ is symbolic of x_α/x^2. The transformation matrix M may be calculated trivially by considering as a special example of a vector current the operator $(1/x^2)\partial/\partial x^\mu$, which has the appropriate dimensions. It is easily seen that

$$M^{\mu\nu}(x) = (1/x^6)(g^{\mu\nu} - 2x^\mu x^\nu/x^2) . \qquad (5)$$

B. Two-Point Functions

If we wish to calculate the covariant two-point function

$$T^{\mu\nu}(x,y) = \langle T^*\{V^\mu(x)V^\nu(y)\}\rangle , \qquad (6)$$

we impose the transformation equation

$$T^{\mu\nu}(x,y) = M^{\mu\alpha}(x)M^{\nu\beta}(y)T_{\alpha\beta}(1/x,1/y) . \qquad (7)$$

It is seen to be important not to make use of translation invariance too soon; translation and inversion do not commute. Setting one coordinate equal to zero would make the inversion not well defined. Only after the algebraic structure is determined can translation invariance be invoked to "remove" a variable.

The straightforward approach to determining the form of $T^{\mu\nu}(x,y)$ is to write down all possible Lorentz-covariant tensors with arbitrary constants, insert them in the transformation equation, and find the conditions imposed on the constants. This is very simple for the two-point function, as there are only two tensors—$g_{\mu\nu}$ and $(x-y)_\mu(x-y)_\nu$. But in more complicated instances, it is advantageous to write the functions as derivatives of invariant (under inversion) functions. The matrix transformation is then explicitly accounted for and all that is left is to count dimensions. Thus, we might write

$$T_{\mu\nu}(x-y) = f((x-y)^2)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}g((x-y)^2) . \qquad (8)$$

Noting that under inversion $(x-y)^2 \to (x-y)^2/x^2y^2$, it is easily seen that the only acceptable form for the two-point function with dimension L^{-6} is

$$T_{\mu\nu}(x-y) = \frac{C}{(x-y)^4}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\ln(x-y)^2 , \qquad (9)$$

where C is an arbitrary constant. We note that this is the most general two-point function since one cannot form an invariant from two coordinates.

C. Vector Three-Point Function

We consider the vector three-point functions

$$\langle T^*\{V^\mu(x)V^\nu(y)V^\rho(z)\}\rangle , \quad \langle T^*\{V^\mu(x)A^\nu(y)A^\rho(z)\}\rangle$$

and other permutations whose over-all character is vector. We start by observing that we already have a two-point function. Thus, we first construct a one-index function of three variables which transforms properly by itself. Using the same method as for the two-point function, we find

$$R_\rho(z;x,y) = \frac{(x-y)^2}{(x-z)^2(z-y)^2}\frac{\partial}{\partial z^\rho}\ln\frac{(z-x)^2}{(z-y)^2} , \qquad (10)$$

along with two permutations—$R_\mu(x;y,z)$ and $R_\nu(y;z,x)$.

Again, we note that there are no invariant functions of three points which can multiply our one- and two-point functions. It can be easily seen that it requires at least four coordinates to produce a true invariant

[2] It may be noted that although the transformation as written is a reflection $(x\to -x)$ implying that the coordinate inversion is an improper conformal transformation, the entire formalism could be redone, with the inversion being defined by $x^\mu \to -x^\mu/x^2$. This would remove any apparent restriction of the results to theories invariant under parity transformations.

under inversion; for example,

$$(w-x)^2(y-z)^2/(w-y)^2(x-z)^2 .$$

It is this fact which allows us to study three-point functions in a relatively simple way.

Thus, we can combine our one- and two-index functions in four independent ways to obtain our inversion-symmetric vector three-point function $S^{\mu\nu\rho}$.

$$S^{\mu\nu\rho}(x,y,z)=c_1S_1{}^{\mu\nu\rho}+c_2S_2{}^{\mu\nu\rho}+c_3S_3{}^{\mu\nu\rho}+c_4S_4{}^{\mu\nu\rho} , \tag{11}$$

$$S_1{}^{\mu\nu\rho}=\frac{1}{(x-y)^2(y-z)^2(z-x)^2}\frac{\partial}{\partial x^\mu}\ln\frac{(x-y)^2}{(x-z)^2} \times\frac{\partial}{\partial y^\nu}\ln\frac{(y-z)^2}{(y-x)^2}\frac{\partial}{\partial z^\rho}\ln\frac{(z-x)^2}{(z-y)^2} , \tag{12}$$

$$S_2{}^{\mu\nu\rho}=\frac{1}{(x-y)^2(y-z)^2(z-x)^2} \times\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\ln(x-y)^2\frac{\partial}{\partial z^\rho}\ln\frac{(z-x)^2}{(z-y)^2} . \tag{13}$$

We obtain S_3 and S_4 by a cyclical permutation of (x,y,z) and (μ,ν,ρ). We may reduce the number of independent constants and at the same time make our three-point function more physical by imposing Ward identities on any conserved currents. Since in a massless theory we naively expect all currents to be conserved (under PCAC, for example, $\partial_\mu A^\mu \sim m$), we require

$$\frac{1}{i}\frac{\partial}{\partial x^\mu}S^{\mu\nu\rho}(x,y,z)=T^{\nu\rho}(y-z)[\delta^4(x-y)-\delta^4(x-z)] , \tag{14}$$

along with two permutations. $T^{\nu\rho}$ is of course the two-point function previously discussed. The differentiations are all straightforward if we remember that

$$\frac{\partial}{\partial x^\mu}\left(\frac{1}{x^2}\frac{\partial}{\partial x_\mu}\ln x^2\right)=-\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\frac{1}{x^2}=4\pi^2 i\delta^4(x) . \tag{15}$$

We do have to define certain behavior at the δ-function singularities, so we adopt the symmetric limiting convention

$$\delta^4(x)x^\alpha x^\beta/x^2 \to \tfrac{1}{4}g^{\alpha\beta}\delta^4(x) . \tag{16}$$

This, of course, is equivalent to the usual momentum-space technique,

$$\int d^4k_E k^\alpha k^\beta f(k^2) \to \tfrac{1}{4}g^{\alpha\beta}\int d^4k_E k^2 f(k^2) .$$

With this in mind, we find that S_1 satisfies each Ward identity, but only a certain combination of S_2, S_3, and S_4 can be used:

$$S^{\mu\nu\rho}=c_1S_1{}^{\mu\nu\rho}+c_2(S_2{}^{\mu\nu\rho}-S_3{}^{\mu\nu\rho}-S_4{}^{\mu\nu\rho}) . \tag{17}$$

Of course, there is a linear relation between them and the coefficient of the two-point function. We will dwell on this no further.

D. Axial-Vector Three-Point Function

In finding the general form of the inversion-symmetric covariant axial-vector three-point functions $\langle T^*\{A^\mu(x)A^\nu(y)A^\rho(z)\}\rangle$ or $\langle T^*\{A^\mu(x)V^\nu(y)V^\rho(z)\}\rangle$ plus permutations, we cannot use the same simple techniques as in the vector case. The indices μ, ν, and ρ now appear in the four-dimensional antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. The transformation matrices will mix up different types of terms, e.g.,

$$\epsilon^{\mu\nu\rho\sigma} \to \left(g^{\mu\alpha}-2\frac{x^\mu x^\alpha}{x^2}\right)\epsilon_\alpha{}^{\nu\rho\sigma}=\epsilon^{\mu\nu\rho\sigma}-2\frac{x^\mu x^\alpha}{x^2}\epsilon_\alpha{}^{\nu\rho\sigma} .$$

It is thus advantageous to write the various tensors in a more or less "unique" way in order to correctly equate terms in the transformation equation. We will outline the general method used as it illustrates a convenient but not widely known identity.[3]

We start by noting that there are only two types of terms allowed by the antisymmetry of the ϵ function and the presence of only two independent coordinate differences. We thus write

$$T^{*\mu\nu\rho}(x,y,z)=\epsilon^{\mu\nu\rho}{}_\alpha f_{(1)}{}^\alpha+(x-y)_\alpha(y-z)_\beta \times(\epsilon^{\mu\nu\alpha\beta}f_{(2)}{}^\rho+\epsilon^{\mu\rho\alpha\beta}f_{(3)}{}^\nu+\epsilon^{\nu\rho\alpha\beta}f_{(4)}{}^\mu) , \tag{18}$$

where

$$f_{(i)}{}^\sigma=f_{i1}x^\sigma+f_{i2}y^\sigma+f_{i3}z^\sigma ,$$

$$\sum_{j=1}^{3} f_{ij}=0 , \tag{19}$$

$$f_{ij}=f_{ij}((x-y)^2,(y-z)^2,(z-x)^2) .$$

The transformation equation reads

$$T^{*\mu\nu\rho}(x,y,z)=-M^{\mu\alpha}(x)M^{\nu\beta}(y)M^{\rho\gamma}(z) \times T^*_{\alpha\beta\gamma}(1/x,1/y,1/z) , \tag{20}$$

where the minus sign is due to the axial-vector nature of T^*. We now must reduce both sides of the equation to unique terms which may be equated term by term.

In order to do this, we introduce the identity

$$\epsilon^{\mu\nu\rho\sigma}=(x_\alpha x_\beta/x^2)(\epsilon^{\alpha\nu\rho\sigma}g^{\mu\beta}+\epsilon^{\mu\alpha\rho\sigma}g^{\nu\beta} +\epsilon^{\mu\nu\alpha\sigma}g^{\rho\beta}+\epsilon^{\mu\nu\rho\alpha}g^{\sigma\beta}) , \tag{21}$$

where x_α is any four-vector. We immediately see that this allows us to define a "unique" single contraction $\epsilon^{\mu\nu\rho\sigma}x_\sigma$. For we have

$$\epsilon^{\mu\nu\rho\sigma}y_\sigma=\epsilon^{\mu\nu\rho}{}_\sigma x^\sigma x\cdot y/x^2+\text{double contractions} ,$$

[3] In keeping with our more intuitive approach to conformal group techniques, we do not use the usual projection of the group in 6-space to find invariants. See, e.g., D. G. Boulware *et al.*, Phys. Rev. D 2, 293 (1970).

and similarly for $\epsilon^{\mu\nu\rho\sigma}z_\sigma$. We thus find that

$$\epsilon^{\mu\nu\rho}{}_\alpha f_{(1)}{}^\alpha = \epsilon^{\mu\nu\rho\alpha}x_\alpha\left(f_{11}+\frac{x\cdot y}{x^2}f_{12}+\frac{x\cdot z}{x^2}f_{13}\right)$$
$$+\text{double contractions}. \quad (22)$$

We may go further and contract two indices on the identity equation; this allows us to express $\epsilon^{\mu\nu\alpha\beta}y_\alpha z_\beta$ in terms of $\epsilon^{\mu\nu\alpha\beta}x_\alpha y_\beta$, $\epsilon^{\mu\nu\alpha\beta}x_\alpha z_\beta$, and terms like $\epsilon^{\mu\alpha\beta\gamma}x_\alpha y_\beta z_\gamma$. We can in this way reduce both sides of the equation to our unique tensors:

$$\epsilon^{\mu\nu\rho\alpha}x_\alpha\,,\quad \epsilon^{\mu\nu\alpha\beta}x_\alpha y_\beta\,,\quad \epsilon^{\mu\nu\alpha\beta}x_\alpha z_\beta\,,\quad \epsilon^{\mu\alpha\beta\gamma}x_\alpha y_\beta z_\gamma$$

plus permutations of $\mu\nu\rho$, but *not* xyz. The calculation is very messy, as should be apparent by now; we shall omit the mechanics. The answer, however, is far from obscure—the general covariant inversion-symmetric axial-vector three-point function has the same structure as the lowest-order diagram, the triangle graph with vertices connected by fermion propagators:

$$T^{*\mu\nu\rho}(x,y,z)=c\,\mathrm{Tr}\{\gamma^5\gamma^\mu S(x-y)\gamma^\nu S(y-z)\times\gamma^\rho S(z-x)\}\,, \quad (23)$$

where

$$c=\text{const},$$
$$S(x-y)=(1/2\pi^2)[\gamma\cdot(x-y)/(x-y)^4]\,.$$

E. Three-Point-Function Ambiguities

With the current interest in anomalies of Ward identities, we will see how we can adapt our general approach to the subject. We expect these terms to be singularities on the light cone, corresponding to polynomials in momentum space. We will thus see if we can form functions involving δ functions and their derivatives which satisfy out transformation equations. We note that under inversion the δ function transforms as

$$\delta^4(x-y)\to x^4y^4\delta^4(x-y)\,.$$

The details of finding the correct combinations of terms are not hard. We use as a guide the known ambiguities and the necessary dimensions. In particular, the ambiguities are linear in momentum. The only way to construct such terms in configuration space is via expressions like $\partial_\mu\delta^4(x-y)\delta^4(y-z)$. The axial-vector ambiguity is particularly simple—any term of the form

$$\epsilon^{\mu\nu\rho\alpha}(\partial/\partial x^\alpha)\delta^4(x-y)\delta^4(y-z)$$

will transform properly. This, or course, corresponds to momentum-space ambiguities like

$$\epsilon^{\mu\nu\rho\alpha}p_\alpha\,,\quad \epsilon^{\mu\nu\rho\alpha}q_\alpha\,. \quad (24)$$

The vector ambiguity is slightly more involved. We find

$$\left(g_{\mu\rho}\frac{\partial}{\partial x^\nu}-g_{\mu\nu}\frac{\partial}{\partial x^\rho}\right)\delta^4(x-y)\delta^4(y-z)$$

plus permutations of xyz and $\mu\nu\rho$. In momentum space, the three-vector ambiguities are

$$p_\nu g_{\mu\rho}-p_\rho g_{\mu\nu}\,,$$
$$q_\mu g_{\nu\rho}-q_\rho g_{\mu\nu}\,, \quad (25)$$
$$(p+q)_\mu g_{\nu\rho}-(p+q)_\nu g_{\mu\rho}\,.$$

We now compare these ambiguities to those found by Gerstein and Jackiw[4] considering only the structure, not the constants. We see that their axial-vector triangle ambiguity is perfectly consistent with ours. However, it is impossible to construct their vector ambiguity from any combination of our terms:

$$g_{\mu\nu}(p-q)_\rho+g_{\mu\rho}(p-q)_\nu+g_{\nu\rho}(p-q)_\mu\,. \quad (26)$$

This means that the ambiguity which they calculated from a conformally symmetric expression, namely, the lowest-order triangle diagram, is not itself conformally invariant. It would appear that the divergence present in the vector triangle breaks the conformal symmetry. The presence of the γ^5 in the axial-vector triangle makes that graph only superficially divergent, serving to preserve the conformal symmetry.

III. ANOMALOUS AXIAL-VECTOR THREE-POINT FUNCTION

A. Introduction

Starting with our expression for the conformally invariant covariant axial-vector three-point function, we will investigate the anomalies present in the Ward identities, finding the anomalous Schwinger terms. Working in configuration space, we will isolate the anomalous part of the three-point function. This term will then be used in two ways. First, we will calculate the "covariant" Ward identity by a straightforward differentiation. Then we will form equal-time commutators by a limiting procedure. It is this last technique which will lead to a noncovariant result. By subtracting the two expressions, we will in the end find the noncovariant Schwinger terms.

We define $T^{*\mu\nu\rho}(x,y,z)$ to be our covariant conformally invariant axial-vector three-point function, either $\langle T^*\{A^\mu(x)V^\nu(y)V^\rho(z)\}\rangle$ or $\langle T^*\{A^\mu(x)A^\nu(y)A^\rho(z)\}\rangle$. We let $T^{\mu\nu\rho}(x,y,z)$ be the corresponding time-ordered product; thus

$$T^{*\mu\nu\rho}=T^{\mu\nu\rho}+C^{\mu\nu\rho}\,, \quad (27)$$

where $C^{\mu\nu\rho}$ is the noncovariant "seagull" term which by definition makes T into the covariant T^*. The Schwinger term, which comes from noncanonical behavior of the equal-time commutators, will thus arise when we take a derivative to form a Ward identity. In particular, for

[4] See Ref. 1. Note that there is a misprint in the AAA ambiguity. To preserve crossing symmetry, $b+a$ must replace $b-a$.

example, the Schwinger term $S^{\nu\rho}$ is given by

$$S^{\nu\rho}=\frac{\partial}{\partial x^{\mu}}C^{\mu\nu\rho}. \qquad (28)$$

We will also define corresponding anomalous parts to each of the above, whose exact meaning will become clear later in the section:

$$\tilde{T}^{*\mu\nu\rho}=\tilde{T}^{\mu\nu\rho}+\tilde{C}^{\mu\nu\rho}. \qquad (29)$$

We recall that the most general $T^{*\mu\nu\rho}$ has the same structure as the lowest-order triangle graph:

$$T^{*\mu\nu\rho}=c\,\mathrm{Tr}\{\gamma^5\gamma^\mu S(x-y)\gamma^\nu S(y-z)\gamma^\rho S(z-x)\}. \qquad (30)$$

This represents equally well $\langle T^*\{A^\mu(x)V^\nu(y)V^\rho(z)\}\rangle$ and $\langle T^*\{A^\mu(x)A^\nu(y)A^\rho(z)\}\rangle$ since the two extra γ^5's anticommute themselves away. We remark here that because of this symmetry, each current gets treated the same, e.g., $A^\mu(x)$ is not singled out in $\langle T^*\{A^\mu(x)V^\nu(y)\times V^\rho(z)\}\rangle$ since $\langle T^*\{V^\mu(x)A^\nu(y)V^\rho(z)\}\rangle$ has the same structure. Thus, there is complete crossing symmetry. (This will be used later in discussing our results.)

B. Naive Ward Identity

The naive (nonanomalous) Ward identity may be written down by inspection. Since

$$\frac{1}{i}\gamma^\mu\frac{\partial}{\partial x^\mu}S(x-y)=\delta^4(x-y), \qquad (31)$$

we see that, e.g.,

$$\frac{1}{i}\frac{\partial}{\partial y^\nu}T^{*\mu\nu\rho}=c\,\mathrm{Tr}\{\gamma^5\gamma^\mu S(x-z)\gamma^\rho S(z-x)\}\times[\delta^4(y-z)-\delta^4(y-x)]=0. \qquad (32)$$

($\mathrm{Tr}\{\gamma^5\gamma\cdot a\gamma\cdot b\gamma\cdot c\gamma\cdot d\}=4i\epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma$. If any two vectors are the same, the trace equals zero.) This, of course, mirrors the fact that $\langle T^*\{A^\mu(x)V^\nu(y)\}\rangle=0$, i.e., that it is impossible to construct a covariant antisymmetric two-point function. We may also show that there exist no nonanomalous Schwinger terms. We do this by forming equal-time commutators via the standard limiting procedure, ignoring, of course, the anomalies:

$$[j^\mu(\bar{x},0),j^\nu(0,0)]=(\lim_{\epsilon\to 0+}-\lim_{\epsilon\to 0-})T\{j^\mu(\bar{x},\epsilon)j^\nu(0,0)\}. \qquad (33)$$

Here T refers to the usual noncovariant time-ordered product. But since the ambiguous seagull terms are local in time, we may form the equal-time limits from the covariant T^* product.

The method of finding the equal-time commutators is as follows. We take $T^{*\mu\nu\rho}(x,y,z)$ for a given choice of $\mu\nu\rho$ and go to a Euclidean metric: $x^0\to ix^4$, etc. Let us say we are looking for the equal-time commutator in x and y:

$$\langle T^*\{[A^\mu(x),V^\nu(y)]_{\mathrm{e.t.}}V^\rho(z)\}\rangle.$$

We set y equal to zero for simplicity and form the expression

$$T^{*\mu\nu\rho}(\bar{x},x^4=\epsilon;0,z)-T^{*\mu\nu\rho}(\bar{x},x^4=-\epsilon;0,z). \qquad (34)$$

Realizing that we are looking for a term proportional to $\delta^3(x)$ or $\partial_k\delta^3(x)$, we will integrate over d^3x or x_kd^3x to extract the appropriate coefficient. After integrating, we finally set ϵ equal to zero. If we find the result diverging in this limit, it characteristically means that there is a derivative of a δ function. The details of a typical calculation are given in the Appendix. We here quote the result:

$$\delta(x^0-y^0)\langle T^*\{[A^\mu(x),V^\nu(y)]V^\rho(z)\}\rangle =-(c/\pi^4)\epsilon^{0\mu\nu\alpha}[(z-x)^2g_\alpha{}^\rho-2(z-x)^\rho(z-x)_\alpha]\times\delta^4(x-y)/(z-x)^8. \qquad (35)$$

It may be recognized that the $(z-x)$ expression is just the conformally invariant two-point function.

Interchanging $\nu\leftrightarrow\rho$, $y\leftrightarrow z$, we get a similar expression for the $[A^\mu,V^\rho]$ commutator. By adding the two expressions and setting μ equal to zero, we get a representation of the Ward identity as calculated from the T product.

$$\frac{1}{i}\frac{\partial}{\partial x^\mu}\langle T\{A^\mu(x)V^\nu(y)V^\rho(z)\}\rangle =(1/i)\delta(x^0-y^0)\langle T\{[A^0(x),V^\nu(y)]V^\rho(z)\}\rangle +(1/i)\delta(x^0-z^0)\langle T\{[V^\rho(z),A^0(x)]V^\nu(y)\}\rangle. \qquad (36)$$

But of course we see that when $\mu=0$, each commutator term is identically zero. We have thus shown that

$$\frac{\partial}{\partial x^\mu}T^{*\mu\nu\rho}=\frac{\partial}{\partial x^\mu}T^{\mu\nu\rho}=0, \qquad (37)$$

with similar equations for the other two Ward identities. There are, then, no Schwinger terms in the nonanomalous part of the T product.

C. Anomalous Ward Identities

We are now ready to show that the Schwinger terms come from the anomalous parts of the T^* product. We first must calculate these anomalous parts. The anomalies in Ward identities occur when, in the formation of an equal-time commutator, the third current gets "pinched" between the two in the commutator. If one thinks of the limiting procedure involved, say,

$$\lim_{\epsilon\to 0}\langle T\{j^\mu(\bar{x},y^0+\epsilon)j^\nu(y)j^\rho(z)\}\rangle$$

for the $[j^\mu,j^\nu]$ commutator, then the anomaly will occur when $y^0<z^0<y^0+\epsilon$. We will evaluate the effects of this pinching by integrating over d^4z. This has the effect of extracting those parts of $T^{\mu\nu\rho}$ which occur when z lies between x and y. A slightly more detailed analysis

of this point can be found in Jackiw and Johnson,[5] which also discusses more generally the type of anomalies associated with the axial-vector current. On a more naive level, since we know that anomalies are associated with the more singular behavior of the T product, we may consider the integration over one of the coordinates as isolating a four-dimensional δ function (or derivatives) in that variable. If we then form an equal-time commutator, we are studying a product of δ functions; one cannot get too much more singular. Of course, the order of integrations is important, by analogy with the problem of translating the loop momenta in the superficially divergent triangle graph in momentum space. We will unambiguously decide the order of integrations by considering which variable can get pinched in a given commutator term.

Let us consider without further ado the naive Ward identity

$$\frac{1}{i}\frac{\partial}{\partial z^\rho}T^{*\mu\nu\rho}=\frac{1}{i}\delta(y^0-z^0)\langle T^*\{A^\mu(x)[V^\nu(y),V^0(z)]\}\rangle +(1/i)\delta(x^0-z^0)\langle T^*\{V^\nu(y)[V^0(z),A^\mu(x)]\}\rangle . \quad (38)$$

The anomalous behavior occurs for x between y and z in the first term, and for y between x and z in the second term. We will define two "anomalous T^* products," integrating over d^4x for one and over d^4y for the other. As it turns out,

$$\int d^4x\, T^{*\mu\nu\rho}=\int d^4y\, T^{*\mu\nu\rho}=\int d^4z\, T^{*\mu\nu\rho}=0, \quad (39)$$

indicating that the coefficient of the δ function in any variable is zero. We thus look for a first derivative of a δ function; in particular, we define

$$\tilde{T}_{(x)}{}^{*\mu\nu\rho;\sigma}=\int d^4x\, x^\sigma T^{*\mu\nu\rho}(x,y,z), \quad (40)$$

with a similar definition for $T_{(y)}{}^{*\mu\nu\rho;\sigma}$. Since these two functions represent coefficients of derivatives of δ functions, we must decide what these δ functions are. That is, does $T_{(x)}{}^*$ give the coefficient of $\partial_\sigma\delta^4(x-y)$ or $\partial_\sigma\delta^4(x-z)$? Since $\int d^4x\, T^*=0$, it is obvious that

$$\int d^4x(x-z)^\sigma T^{*\mu\nu\rho}=\int d^4x(x-y)^\sigma T^{*\mu\nu\rho}=\int d^4x\, x^\sigma T^{*\mu\nu\rho}.$$

We require that the covariant anomalous Ward identities agree; this means that we must be considering the same δ function in both $\tilde{T}^*$'s, in particular, $\partial_\sigma\delta^4(x-y)$. We can now write the actual anomalous T^* products, omitting the details:

$$\tilde{T}_{(x)}{}^{*\mu\nu\rho}=-\left[\int d^4x\, x^\sigma T^{*\mu\nu\rho}\right]\frac{\partial}{\partial x^\sigma}\delta^4(x-y)$$
$$=-\frac{c}{(2\pi)^4}\frac{\partial}{\partial z^\alpha}\left[\frac{(z-y)^\nu}{(z-y)^4}\epsilon^{\mu\sigma\rho\alpha}-\frac{(z-y)^\rho}{(z-y)^4}\epsilon^{\mu\nu\sigma\alpha}\right]\times\frac{\partial}{\partial x^\sigma}\delta^4(x-y), \quad (41)$$

$$\tilde{T}_{(y)}{}^{*\mu\nu\rho}=-\frac{c}{(2\pi)^4}\frac{\partial}{\partial z^\alpha}\left[\frac{(z-x)^\rho}{(z-x)^4}\epsilon^{\mu\nu\sigma\alpha}-\frac{(z-x)^\mu}{(z-x)^4}\epsilon^{\sigma\nu\rho\alpha}\right]\times\frac{\partial}{\partial y^\sigma}\delta^4(y-x).$$

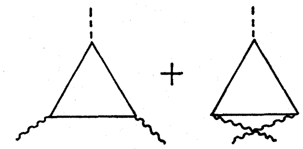

FIG. 1. Sum of direct and crossed axial-vector triangles.

It is straightforward to calculate the covariant anomalous Ward identity now. Noting that $\epsilon^{\mu\sigma\rho\alpha}\times\partial_\rho\partial_\alpha f(z)=0$ by the antisymmetry of ϵ, and that $\partial_\rho[(z-y)^\rho/(z-y)^4]=2\pi^2 i\delta^4(z-y)$, we find that

$$\frac{1}{i}\frac{\partial}{\partial z^\rho}\tilde{T}_{(x)}{}^{*\mu\nu\rho}=\frac{2\pi^2 c}{(2\pi)^4}\epsilon^{\mu\nu\sigma\alpha}\frac{\partial}{\partial x^\sigma}\delta^4(x-y)\frac{\partial}{\partial z^\alpha}\delta^4(z-y),$$
$$\frac{1}{i}\frac{\partial}{\partial z^\rho}\tilde{T}_{(y)}{}^{*\mu\nu\rho}=-\frac{2\pi^2 c}{(2\pi)^4}\epsilon^{\mu\nu\sigma\alpha}\frac{\partial}{\partial y^\sigma}\delta^4(y-x)\frac{\partial}{\partial z^\alpha}\delta^4(z-x). \quad (42)$$

It is trivial to show that these two expressions are equivalent. This result for the anomalous Ward identity may be compared with that found by Gerstein and Jackiw,[1] who found ambiguities due to the superficial linear divergence of the triangle diagram. In particular, their result corresponds to setting $c=2$ in our calculation for either $\partial/\partial y^\nu$ or $(\partial/\partial z_\rho)T^{*\mu\nu\rho}$. This is due to the fact that their calculation sums both the direct and the crossed diagrams as shown in Fig. 1. Each of these contributes the same amount to the anomaly. The difference between their results and the above calculation lies in the third Ward identity: $(\partial/\partial x^\mu)T^{*\mu\nu\rho}$. They obtained zero for this anomaly, whereas we have found a complete symmetry between the three identities. The contradiction is not due to the crossed diagrams cancelling; the same situation holds for the three-axial-vector case. What occurs is that their treatment, while making use of the ambiguity of the divergent loop integration, has still singled out one vertex as different from the others in assigning momenta; crossing symmetry is only imposed on two of

[5] R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969). See especially the note added in proof, Sec. IV.

TABLE I. Schwinger terms from axial-vector three-point function.

$\mu\nu$	$S^{\mu\nu}$
00	0
$0j$	$\frac{4\pi^2 c}{(2\pi)^4}\epsilon^{jlk}\frac{\partial}{\partial y^k}\delta^4(y-z)\frac{\partial}{\partial x^l}\delta^4(x-y)$
$j0$	$-\frac{4\pi^2 c}{(2\pi)^4}\epsilon^{jlk}\frac{\partial}{\partial y^k}\delta^4(y-z)\frac{\partial}{\partial x^l}\delta^4(x-y)$
ij	$-\frac{2\pi^2 c}{(2\pi)^4}\epsilon^{ijk}\left[\frac{\partial}{\partial x^0}\delta^4(x-y)\frac{\partial}{\partial y^k}\delta^4(y-z)-\frac{\partial}{\partial y^0}\delta^4(y-z)\frac{\partial}{\partial x^k}\delta^4(x-y)\right]$

the currents. Our result is manifestly symmetric in all the variables. The differing results may just be considered as due to different definitions of the ambiguity.

We are now ready to compute the equal-time limits from the anomalous terms, using $\tilde{T}_{(x)}^*$ for the $y-z$ commutator and $\tilde{T}_{(y)}^*$ for the $x-z$ commutator. Going to a Euclidean metric as in the naive case, we first form

$$\tilde{T}_{(x)}{}^{*\mu\nu\rho}(y^4=z^4+\epsilon)-\tilde{T}_{(x)}{}^{*\mu\nu\rho}(y^4=z^4-\epsilon)\,.$$

Using the standard integral and limiting method as before, we find a derivative of a spatial δ function in $y-z$ in the equal-time limit:

$$\begin{aligned}(1/i)\delta(y^0-z^0)\langle\tilde{T}\{A^\mu(x)[V^\rho(z),V^\nu(y)]\}\rangle\\ =-\left[\frac{2\pi^2c}{(2\pi)^4}\frac{\partial}{\partial y^k}\delta^4(y-z)\frac{\partial}{\partial x^\sigma}\delta^4(x-y)\right]\\ \times[\epsilon^{\mu\sigma\rho\alpha}(g_0{}^\nu g_\alpha{}^k-g^0{}_\alpha g^{\nu k})\\ -\epsilon^{\mu\nu\sigma\alpha}(g_0{}^\rho g_\alpha{}^k-g^0{}_\alpha g^{\rho k})],\end{aligned}\quad(43)$$

where k indicates only a spatial index. The term which occurs in the $\partial_\rho\tilde{T}^{*\mu\nu\rho}$ Ward identity of course requires $\rho=0$; the above commutator becomes

$$\frac{-2\pi^2c}{(2\pi)^4}\frac{\partial}{\partial y^k}\delta^4(y-z)\frac{\partial}{\partial x^\sigma}\delta^4(x-y)(\epsilon^{\mu\sigma 0k}g_0{}^\nu-\epsilon^{\mu\nu\sigma k})\,.\quad(44)$$

Similarly, we have for the other commutator

$$\begin{aligned}(1/i)\delta(x^0-z^0)\langle\tilde{T}\{[A^\mu(x),V^0(z)]V^\nu(y)\}\rangle\\ =\frac{2\pi^2c}{(2\pi)^4}\frac{\partial}{\partial x^k}\delta^4(x-z)\frac{\partial}{\partial y^\sigma}\delta^4(y-x)(\epsilon^{\mu\nu\sigma k}-\epsilon^{\sigma\nu 0k}g_0{}^\mu)\,.\end{aligned}\quad(45)$$

It may be noticed that each of the $\epsilon^{\mu\nu\sigma k}$ terms is plus or minus the covariant Ward-identity anomaly. Adding the two commutator terms, these two covariant contributions cancel, and we finally get for the anomalous Ward identity of the T product

$$\frac{1}{i}\frac{\partial}{\partial z^\rho}\tilde{T}^{\mu\nu\rho}=-\frac{2\pi^2c}{(2\pi)^4}\frac{\partial}{\partial y^k}\delta^4(y-z)\frac{\partial}{\partial x^\sigma}\times\delta^4(x-y)(g_0{}^\nu\epsilon^{\mu\sigma 0k}-g_0{}^\mu\epsilon^{\nu\sigma 0k})\,.\quad(46)$$

We see that there are only contributions for $(\mu,\nu)=(0,j)$ or $(j,0)$, where j is a spatial component. In particular, we can write these contributions as

$$\frac{1}{i}\frac{\partial}{\partial z^\rho}\tilde{T}^{\mu\nu\rho}=-\frac{2\pi^2c}{(2\pi)^4}\frac{\partial}{\partial y^k}\delta^4(y-z)\frac{\partial}{\partial x^\sigma}\times\delta^4(x-y)\epsilon^{0j\sigma k}(\delta_0{}^\nu\delta_j{}^\mu-\delta_0{}^\mu\delta_j{}^\nu)\,.\quad(47)$$

We now recall the covariant anomalous Ward identity, along with its evaluation for different choices of (μ,ν):

$$\begin{aligned}&\frac{1}{i}\frac{\partial}{\partial z^\rho}\tilde{T}^{*\mu\nu\rho}\\ &=-\frac{2\pi^2c}{(2\pi)^4}\epsilon^{\mu\nu\sigma\alpha}\frac{\partial}{\partial x^\sigma}\delta^4(x-y)\frac{\partial}{\partial y^\alpha}\delta^4(y-z)\\ &=-\frac{2\pi^2c}{(2\pi)^4}\epsilon^{jk0l}\left[\frac{\partial}{\partial x^0}\delta^4(x-y)\frac{\partial}{\partial y^l}\delta^4(y-z)-\frac{\partial}{\partial y^0}\delta^4(y-z)\frac{\partial}{\partial x^l}\delta^4(x-y)\right],\quad \mu,\nu=j,k\\ &=-\frac{2\pi^2c}{(2\pi)^4}\epsilon^{0j\sigma k}\frac{\partial}{\partial x^\sigma}\delta^4(x-y)\frac{\partial}{\partial y^k}\delta^4(y-z),\quad \mu,\nu=0,j\\ &=-\frac{2\pi^2c}{(2\pi)^4}\epsilon^{j0\sigma k}\frac{\partial}{\partial x^\sigma}\delta^4(x-y)\frac{\partial}{\partial y^k}\delta^4(y-z),\quad \mu,\nu=j,0\\ &=0,\quad \mu,\nu=0,0\,.\end{aligned}\quad(48)$$

We can finally evaluate the Schwinger terms by comparing the covariant and "noncovariant" Ward identities to obtain the anomalous Schwinger terms

$$S^{\mu\nu}=\frac{1}{i}\frac{\partial}{\partial z^\rho}(T^{*\mu\nu\rho}-T^{\mu\nu\rho})\,,\quad(49)$$

which we list in Table I.

IV. SUMMARY

By invoking conformal invariance or, more specifically, inversion symmetry, we have been able to derive certain facts about the three-point function. The vector and axial-vector functions have been determined to within two and one arbitrary constants, respectively. The covariant inversion-symmetric structure of possible ambiguities in these functions was also easily obtained, making possible both a comparison with a momentum-space study of anomalous triangle graphs, and a determination that in the vector case the anomaly breaks conformal symmetry. Finally, using our expression for the axial-vector three-point function, we obtained general forms for both the Ward-Takahashi identity anomalies and the Schwinger terms.

An interesting extension of this work would be a more detailed study of the vector three-point function, since the conformally invariant vector terms do not reproduce lowest-order perturbation theory as trivially as does the axial-vector function.

ACKNOWLEDGMENT

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APPENDIX

We will demonstrate the limiting procedure by calculating

$$\delta(x^0-y^0)\langle T\{[A^k(x),V^l(y)]V^0(z)\}\rangle\,.$$

We have chosen for definiteness k and l as spatial indices. We set x equal to zero for simplicity and we assume $z\neq 0$, $y\neq z$ since we are not dealing here with any anomalies. We thus exhibit the relevant T^* product.

$$\begin{aligned}T^{*kl0}(0,y,z)&\\ &=c\,\mathrm{Tr}\{\gamma^5\gamma^k S(-y)\gamma^l S(y-z)\gamma^0 S(z)\}\\ &=-\frac{c}{(2\pi^2)^3y^4z^4(y-z)^4}\mathrm{Tr}\{\gamma^5\gamma^k\gamma\cdot y\gamma^l\gamma\cdot(y-z)\gamma^0\gamma\cdot z\}\\ &=\frac{-4ic}{(2\pi^2)^3y^4z^4(y-z)^4}[\epsilon^{k\alpha 0\beta}z_\beta(y^2g_\alpha{}^l-2y_\alpha y^l)\\ &\qquad-\epsilon^{k\beta l\alpha}y_\beta(z^2g_\alpha{}^0-2z_\alpha z^0)]\,.\end{aligned}\tag{A1}$$

At this point, we find the terms odd in y^0, for it is only these which will contribute to

$$[T^*(y^0=i\epsilon)-T^*(y^0=-i\epsilon)].$$

We recognize that the commutator δ function will come from a "singularity" when both $\bar{y}$ and ϵ go to zero; thus we can let $y^0=iy_4$ go to zero right away in the $(y-z)^2$ denominator. Because of the antisymmetry of the ϵ function, it becomes obvious that the only term odd in y^0 is

$$\frac{-4ic}{(2\pi^2)^3y^4z^4(y-z)^4}2z^0z_\alpha y_0\epsilon^{k0l\alpha}.\tag{A2}$$

Now going to a Euclidean metric $y^0=iy_4$, $z^0=iz_4$, we obtain

$$\frac{icz_4z_\alpha y_4\epsilon^{k0l\alpha}}{\pi^6z^4(z^2+\bar{y}^2-2\bar{y}\cdot\bar{z})^2(\bar{y}^2+y_4{}^2)^2}.\tag{A3}$$

We have explicitly written out the relevant denominators, except that we have already set $y_4=0$ in the $(y-z)^2$ term. We may now find the coefficient of the spatial δ function in $\bar{y}$ by forming the equal-time commutator and integrating over d^3y:

$$\begin{aligned}&\lim_{\epsilon\to 0}\int d^3y[T^{*kl0}(y_4=\epsilon)-T^{*kl0}(y_4=-\epsilon)]\\ &=\lim_{\epsilon\to 0}\frac{2icz_4z_\alpha\epsilon^{k0l\alpha}}{\pi^6z^4}\int d^3y\frac{\epsilon}{(\bar{y}^2+\epsilon^2)^2(z^2+\bar{y}^2-2\bar{y}\cdot\bar{z})^2}.\end{aligned}\tag{A4}$$

The angular integration is done first, yielding

$$\int d\Omega_y\frac{1}{(z^2+\bar{y}^2-2\bar{y}\bar{z}\cos\theta)^2}=\frac{4\pi}{(z^2+\bar{y}^2)^2-4\bar{y}^2\bar{z}^2}.\tag{A5}$$

We thus have left the integral

$$4\pi\epsilon\int_0^\infty\bar{y}^2d\bar{y}\frac{1}{(\bar{y}^2+\epsilon^2)^2[(z^2+\bar{y}^2)^2-4\bar{y}^2\bar{z}^2]}.\tag{A6}$$

Letting $\mu=\bar{y}/\epsilon$, we have

$$4\pi\int_0^\infty u^2du\frac{1}{(u^2+1)^2[(z^2+u^2\epsilon^2)^2-4u^2\epsilon^2\bar{z}^2]}.\tag{A7}$$

Rather than to explicitly do this integral, we now carry the $\lim_{\epsilon\to 0}$ through the integral to obtain

$$\frac{4\pi}{z^4}\int_0^\infty\frac{u^2du}{(u^2+1)^2}=\frac{\pi^2}{z^4}.\tag{A8}$$

Our justification for the exchange of integral and limit is given by the dominated-convergence theorem. We state this theorem for reference.

Theorem: Given $\lim_{\epsilon\to 0}\int dx\,f(x,\epsilon)$ with $\lim_{\epsilon\to 0}f(x,\epsilon)=F(x)$, if we can find a $g(x)\ni|f(x,\epsilon)|<g(x)$ and $\int g(x)<\infty$ for ϵ in a neighborhood of zero, then

$$\lim_{\epsilon\to 0}\int dx\,f(x,\epsilon)=\int dx\,F(x).$$

In applying this theorem, we note that the denominator is always nonzero:

$$D=(z^2+u^2\epsilon^2)^2-4u^2\epsilon^2\bar{z}^2=u^4\epsilon^4+2u^2\epsilon^2(z^2-2\bar{z}^2)+z^4.$$

[The discriminant of D is $4(z^2-2\bar{z}^2)^2-4z^4=-16z^2\bar{z}^2<0$ since we are in a Euclidean metric with $z\neq 0$. Thus, there are no real roots in the variable $(u\epsilon)$.] Thus D^{-1} is bounded: $1/D<K$ for all $(u\epsilon)$. Then certainly the integrand is bounded by an integrable function:

$$u^2/(u^2+1)D<Ku^2/(u^2+1)^2.\tag{A9}$$

The above theorem certainly holds. We note there that this theorem is applicable to all the limit-integral exchanges done in this paper. The integrands always consist of an integrable function along with either the denominator D above or a term like $(1/u\epsilon)\ln D$ which is also bounded.[6]

Getting back to our commutator, we have found

$$\lim_{\epsilon\to 0}\int d^3y[T^{*kl0}(y_4=\epsilon)-T^{*kl0}(y_4=-\epsilon)]$$
$$=\frac{2ic}{\pi^4}\frac{z_4z_\alpha}{z^8}\epsilon^{k0l\alpha}$$
$$=\frac{2c}{\pi^4}\frac{z^0z_\alpha}{(z^2)^4}\epsilon^{k0l\alpha}. \quad \text{(A10)}$$

[6] Uniform convergence does not hold for some of the integrals, necessitating the weaker dominated-convergence theorem.

This means that the equal-time commutator has a $\delta^3(y)$ with the above coefficient. In particular,

$$(1/i)\delta(x^0-y^0)\langle T\{[A^k(x),V^l(y)]V^0(z)\}\rangle$$
$$=-\frac{2ic}{\pi^4}\epsilon^{0kl\alpha}\frac{(z-x)_\alpha(z-x)^0}{(z-x)^8}\delta^4(x-y), \quad \text{(A11)}$$

where we have reinserted x. This is seen to be a particular term of

$$\frac{ic}{\pi^4}\epsilon^{0\mu\nu\alpha}[(z-x)^2g_\alpha{}^\rho-2(z-x)_\alpha(z-x)^\rho]$$
$$\times\frac{1}{(z-x)^8}\delta^4(x-y), \quad \text{(A12)}$$

as quoted in the text. We will not bother to illustrate the veracity of the above expression for other choices of $\mu\nu\rho$.

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Group-Theoretical Construction of Dual Amplitudes

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We present general group-theoretical requirements for the construction of factorizable, dual N-point functions. Dual amplitudes for the scattering of arbitrary numbers of spinning particles are built as an example of this approach.

Although the $SU(1,1)$ invariance of the Veneziano N-point function has long been recognized,[1] it has not yet been systematically exploited in the construction of new dual amplitudes. In the following, we would like to generalize the group-theoretic structure of the multi-Veneziano function, discussed earlier by the authors,[2] to include external particles of different types. As an example of our techniques, we give a simple, factorizable dual amplitude for the absorption of arbitrary numbers of high-spin particles.

For the purpose of this paper, we regard duality as a purely group-theoretical concept, implying $SU(1,1)$ invariance but not necessarily any particular asymptotic behavior (e.g., Regge). Basic to our description are the three generators of $SU(1,1)$, namely, L_0, L_+, and L_-, which satisfy the algebra

$$[L_0,L_\pm]=\pm L_\pm\,,\quad [L_+,L_-]=-L_0\,. \quad (1)$$

We present the following minimal set of group-theoretical conditions for the construction of dual amplitudes.

(i) Associate with the absorption of a particle with momentum k_μ, spin j, j_3, internal quantum numbers $\{\lambda\}$, a vertex operator $V(k_\mu,j,j_3,\{\lambda\};z)$, where $z=e^{-i\tau}$ is a complex variable on the unit circle.

(ii) Require that $V(k_\mu,j,j_3,\{\lambda\};z)$ transforms under $SU(1,1)$ as a spin J_S representation, that is to say,

$$[L_0,V]=-z\frac{d}{dz}V\,, \quad \text{(2a)}$$

$$[L_\pm,V]=\frac{-1}{\sqrt{2}}z^{\pm 1}\left(z\frac{d}{dz}\mp J_s\right)V\,, \quad \text{(2b)}$$

where we take J_S to be in general a function of the Casimir operators of the Lorentz and internal symmetry groups

$$J_S=J_S(m^2,j,c^\lambda)\,. \quad (3)$$

(iii) Under the Lorentz and internal symmetry groups, V is required to transform in the same way as the field of the absorbed particle. This ensures the correct selection rules at each vertex.

[1] Z. Koba and H. B. Nielsen, Nucl. Phys. B12, 517 (1969).
[2] L. Clavelli and P. Ramond, Phys. Rev. D 2, 973 (1970).