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Nonrelativistic and Relativistic Coulomb Amplitude as the Matrix Element of a Rotation in $O(4,2)$†

A. O. BARUT AND W. RASMUSSEN
Institute for Theoretical Physics, University of Colorado, Boulder, Colorado 80302
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It is shown that the Coulomb amplitude is the matrix element of the group element $e^{i(\theta+\pi)J_2}$ evaluated in a continuous basis in $SO(4,2)$. The method is applied to the relativistic Kepler problem without spins, but with the recoil corrections included. The form of the amplitude for this case is believed to be new.

I. INTRODUCTION

THERE are a number of recent group-theoretical studies of the Coulomb amplitude. Zwanziger[1] and Biedenharn and Brussard[2] have used the symmetry group $SO(3,1)$ of the scattering states to determine the partial-wave amplitudes. This method does not determine the energy-dependent phase of amplitude. This phase is very important because it contains the bound-state poles of the amplitude. Finkelstein and Levy[3] formulate the scattering amplitude as a function over the group space of $O(3)$ (by relating the relative momentum $\mathbf{p}$ to a point g in group space) and then expand it in terms of the D^j functions (harmonic analysis). Finally, Fronsdal and Lundberg[4] calculate the Coulomb amplitude by using a propagator technique and sum over a complete set of intermediate states in the direct channel. Closely related to the group-theoretical methods is also the work of Schwinger,[5] who uses Green's-functions methods.

The purpose of this paper is to show that the complete Coulomb amplitude is simply the matrix element of a rotation

$$f(k,\theta)\equiv\langle \text{in}\,|\,e^{i(\theta+\pi)J_2}\,|\,\text{in}\rangle \tag{1.1}$$

in suitably defined and suitably normalized (see Sec. II) $O(4,2)$ "in" states. This method allows a direct evaluation (not via the angular momentum states) of the full amplitude including energy dependence, entirely without reference to the configuration in space or momentum space. This might be expected because the dynamical group $SO(4,2)$ contains the symmetry group $SO(3,1)$, for fixed energy, as a subgroup. We further generalize the method to the relativistic Kepler problem without spins, but with recoil corrections included. This last result is believed to be new.

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[1] D. Zwanziger, J. Math. Phys. **8**, 1858 (1967).

[2] L. C. Biedenharn and P. J. Brussard [*Coulomb Excitations* (Oxford U. P., Oxford, 1965)] suggest that Coulomb phase shifts may be obtained by analytic continuation from $O(4)$ Clebsch-Gordan coefficients.

[3] R. Finkelstein, J. Math. Phys. **8**, 443 (1967); R. Finkelstein and D. Levy, *ibid.* **8**, 2147 (1967).

[4] C. Fronsdal and L.-E. Lundberg, Phys. Rev. D **3**, 524 (1971).

[5] J. Schwinger, J. Math. Phys. **5**, 1606 (1964).

II. SCATTERING STATES AND "IN" AND "OUT" STATES IN $O(4,2)$

Basis States of $O(4,2)$ Representations

Let, as usual, $L_{ab}=-L_{ba}$ be the generators of $SO(4,2)$ $(a,b=1,2,3,4,5\equiv0,6)$; $L_{ij}(i,j=1,2,3)$ are the angular momentum operators, $L_{i4}=A_i$ the Runge-Lenz vector, $L_{i5}=M_i$ the generators of the Lorentz transformations (the generators of the Galilean transformations are given by $L_{i5}-L_{i4}$), $\Gamma_\mu=(L_{i6},L_{56})$ is a four-vector operator, and, finally, $T=L_{45}$ and $S=L_{46}$ are a rotational scalar and a Lorentz scalar operator. In this paper we shall only need the so-called parabolic coordinates defined by the following diagonal operators[6]:

$$\begin{aligned} L_{56}\,|n_1n_2m\rangle &= n\,|n_1n_2m\rangle\,,\\ L_{34}\,|n_1n_2m\rangle &= (n_1-n_2)\,|n_1n_2m\rangle\,,\\ L_{12}\,|n_1n_2m\rangle &= m\,|n_1n_2m\rangle\,,\\ n&=n_1+n_2+m+1\,. \end{aligned} \tag{2.1}$$

These quantum numbers characterize the states in the scattering experiment. The ranges of n_1, n_2, and m for stationary bound and stationary scattering states are well known.[7] It will turn out that the in and out states require values of n_1n_2m analytically continued from the stationary scattering states: Let ψ^- and ψ^+ be the wave functions which asymptotically behave as

$$\begin{aligned} \psi^- &\sim e^{ikz}+f^-(\theta)e^{-ikr}/r\,,\\ \psi^+ &\sim e^{ikz}+f^+(\theta)e^{ikr}/r\,. \end{aligned} \tag{2.2}$$

Using the wave functions in parabolic coordinates, we see that the corresponding parabolic quantum numbers are (Appendix A)

$$\begin{aligned} \psi^+&:\; n_1=-1\,,\quad n_2=n\,,\quad m=0\,,\\ \psi^-&:\; n_1=n-1\,,\quad n_2=0\,,\quad m=0\,. \end{aligned} \tag{2.3}$$

[6] For other bases and further details see, e.g., the review papers by A. O. Barut, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Gordon and Breach, New York, 1968), Vol. X B, p. 377 and in *Springer Tracts in Modern Physics* (Springer, New York, 1969), Vol. 50.

[7] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vierveg & Sohn, Braunschweig, West Germany, 1960), Vol. II; H. A. Bethe and E. E. Salpeter, *Encyclopedia of Physics* (Springer, New York, 1957), Vol. 35.

In contrast, for the stationary scattering states we have

$$n_1=-\tfrac{1}{2}(m+1)-\tfrac{1}{2}i(n'+\lambda)\,,\quad n=-in'$$
$$n_2=-\tfrac{1}{2}(m+1)-\tfrac{1}{2}i(n'-\lambda)\,,\quad \lambda=\text{real}. \tag{2.4}$$

The following relation exists between the ψ^- and ψ^+ states asymptotically (Appendix A):

$$\psi^-=R(\pi)[\psi^+]^*\,, \tag{2.5}$$

where $R(\pi)$ is a rotation about the y axis by π under which $z\to -z$.

Furthermore, two waves whose asymptotic plane-wave parts are along the z direction, and along a direction θ with respect to the z axis, respectively, are related by

$$\psi_\theta^-=R(\theta)\psi_z^-\,, \tag{2.6}$$

where $R(\theta)$ is a rotation by the scattering angle θ in the xz plane.

III. AMPLITUDE

The scattering amplitude is given by

$$M=\langle\psi_{f,\theta}^-|\psi_{i,z}^+\rangle\,, \tag{3.1}$$

where i and f refer to the initial and final states. Using first (2.6) and then (2.5), we get

$$M=\langle\psi_{f,z}^-|R^\dagger(\theta)|\psi_{i,z}^+\rangle$$
$$=\langle(\psi_{f,z}^+)^*|R^\dagger(\theta+\pi)|\psi_{i,z}^+\rangle\,. \tag{3.2}$$

In the $O(4,2)$ formulation, the states ψ^+, characterized by the quantum numbers (2.3), are the so-called tilted states,

$$|n_1n_2m\rangle_t\equiv\mathfrak{N}e^{-i\theta L_{45}}|n_1n_2m\rangle\,, \tag{3.3}$$

where $|n_1n_2m\rangle$ are normalized "parabolic" group states, θ is the tilting angle, and $\mathfrak{N}$ is a normalization constant chosen such that ${}_t\langle n_1n_2m|(\Gamma_0-\Gamma_4)|n_1n_2m\rangle_t=1$. In terms of the boson operators, the "parabolic" group states for $m\geq 0$ are given by

$$|n_1n_2m\rangle=N_{n_1n_2m}a_1^{\dagger n_2+m}a_2^{\dagger n_1}b_1^{\dagger n_1+m}b_2^{\dagger n_2}|0\rangle\,, \tag{3.4}$$

where

$$N_{n_1n_2m}=[n_1!(n_1+m)!n_2!(n_2+m)!]^{-1/2}. \tag{3.5}$$

For the relevant values (2.3) of the quantum numbers, the factor $N_{n_1n_2m}$ is a complex quantity. Taking into account the complex conjugation in (3.2), we have to write

$$M=\frac{N_{n1n2m}^*}{N_{n1n2m}}\,{}_t\langle n_1n_2m|(\Gamma_0-\Gamma_4)e^{i(\theta+\pi)J_2}|n_1n_2m\rangle_t\,.$$

Inserting a complete intermediate set of group states, using the normalization in (3.3), and the fact that $[L_{45},J_2]=0$, we obtain

$$M=\frac{N_{n1n2m}^*}{N_{n1n2m}}\langle n_1n_2m|e^{i(\theta+\pi)J_2}|n_1n_2m\rangle\,, \tag{3.6}$$

to be evaluated at the values (2.3).

Thus, aside from a phase the amplitude is simply the matrix element of a finite rotation. In Appendix B we evaluate this rotation matrix element. The final result is given in Eq. (B7). Furthermore, from (3.5) we have

$$\frac{N_{n1n2m}^*}{N_{n1n2m}}=\left[\frac{n_1!(n_2+|m|)!n_2!(n_1+|m|)!}{n_1^*!(n_2+|m|)^*!n_2^*!(n_1+|m|)^*!}\right]^{1/2}$$
$$=\frac{\Gamma(1+n)}{\Gamma(1-n)}\quad\text{for } n_1=-1,\ n_2=n,\ m=0. \tag{3.7}$$

Hence the amplitude is

$$M=\frac{\Gamma(1+n)}{\Gamma(1-n)}\sin^{-2}(\tfrac{1}{2}\theta)\exp[-n\ln\sin^2(\tfrac{1}{2}\theta)]. \tag{3.8}$$

Up to here, the states were normalized in the n_1, n_2, m basis:

$$\langle n_1'n_2'm'|n_1n_2m\rangle$$
$$=\delta_{n_1'n_1}\delta_{n_2'n_2}\delta_{m'm}\quad\text{(discrete case)}$$
$$=\delta(n_1'-n_1)\delta(n_2'-n_2)\delta_{m'm}\quad\text{(continuous case)}. \tag{3.9}$$

A more suitable basis, looking at Eq. (2.4), is where $n=n_1+n_2+m+1$, $\lambda=i(n_1-n_2)$, and m are diagonal. The absolute value of the Jacobian of the transformation is $\frac{1}{2}$. We then pass from this n scale to the k scale:

$$n=-i\alpha\mu/k\,,\quad dn=(i\alpha\mu/k^2)dk\,,$$

where α is the fine-structure constant and μ is the reduced mass. We finally have

$$M=\frac{\Gamma(1+n)}{\Gamma(1-n)}\frac{i\alpha\mu}{2k^2\sin^2(\frac{1}{2}\theta)}\exp\left(+\frac{i\alpha\mu}{k}\ln\sin^2(\tfrac{1}{2}\theta)\right) \tag{3.10}$$

or, in terms of momentum transfer and energy,

$$M=+i\frac{2\alpha\mu}{\Delta^2}\left(\frac{\Delta^2}{4k^2}\right)^{\alpha(k)}\frac{\Gamma(1-\alpha(k))}{\Gamma(1+\alpha(k))}, \tag{3.11}$$

where

$$\Delta^2=4k^2\sin^2(\tfrac{1}{2}\theta)$$

and

$$\alpha(k)=i\alpha\mu/k\,,\quad \alpha=e^2 \tag{3.12}$$

is indeed the Regge-trajectory function.

IV. RELATIVISTIC CASE

In this section we treat the relativistic Coulomb scattering problem of two spinless particles, masses m_1 and m_2. The recoil effects are included; in this sense the problem is more general than the Klein-Gordon equation with $1/r$ potential. This problem has been formulated in terms of the infinite-component wave equation,[8]

[8] A. O. Barut and A. Baiquni, Phys. Rev. 184, 1342 (1969).

and later by quasipotential and eikonal methods.[9,10] The $O(4,2)$ wave equation is

$$\left[\left(\Gamma_\mu-\frac{\alpha}{2m_2}P_\mu+\frac{1}{2m_2}P_\mu\Gamma_4\right)P^\mu +\frac{m_2{}^2-m_1{}^2}{2m_1}\Gamma_4-\alpha\frac{m_2{}^2+m_1{}^2}{2m_p}\right]\tilde{\psi}=0. \quad (4.1)$$

The mass spectrum obtained is[11]

$$m^2=(m_1{}^2+m_2{}^2)\pm 2m_1m_2[1-\alpha^2/(n^2+\alpha^2)]^{1/2}. \quad (4.2)$$

For small α—or neglecting the term $(\alpha/2m_2)P_\mu$ in (4.1)—we have

$$m^2=m_1{}^2+m_2{}^2\pm 2m_1m_2(1-\alpha^2/n^2)^{1/2}. \quad (4.3)$$

(It is actually this last case that has been formulated by quasipotential and eikonal methods.[9])

Equation (4.1) still possesses $O(4)$ symmetry, as seen by (4.2), and we can use the same parabolic states as before. It differs in this sense also from the Klein-Gordon equation. Thus the same formula (3.6) for the scattering amplitude applies. The only change is the connection between the principal quantum number n and the magnitude of momentum k, where we must use relativistic kinematics. Because the poles of the S matrix must give back the mass formula (4.3) when n is discrete, we find

$$n=-i\alpha m_1m_2/q\sqrt{s}. \quad (4.4)$$

Equation (3.10) can be written as

$$M=\frac{1}{2}\frac{dn}{dk}\sin^2(\tfrac{1}{2}\theta)[\sin^2(\tfrac{1}{2}\theta)]^{-n}\frac{\Gamma(1-n)}{\Gamma(1+n)}. \quad (4.5)$$

In the relativistic case, we use for k the magnitude of the c.m. momenta q. Then

$$\frac{dn}{dq}=i\frac{2\alpha m_1m_2}{q^2}\frac{s^{1/2}[s-(m_1{}^2+m_2{}^2)]}{s^2-(m_1{}^2-m_2{}^2)^2}. \quad (4.6)$$

Thus, identifying

$$t=-4q^2[\sin^2(\tfrac{1}{2}\theta)], \quad (4.7)$$

we can bring M into the same form as Eq. (3.11):

$$M=-i\frac{4\alpha m_1m_2}{s^{1/2}}\frac{s[s-(m_1{}^2+m_2{}^2)]}{s^2-(m_1{}^2-m_2{}^2)^2} \times\frac{1}{t}\left(-\frac{t}{4q^2}\right)^{\alpha(q)}\frac{\Gamma(1-\alpha(q))}{\Gamma(1+\alpha(q))}, \quad (4.8)$$

where

$$\alpha(q)=i\alpha m_1m_2/q\sqrt{s}.$$

In the nonrelativistic limit we recover back precisely Eq. (3.11).

[9] C. Itzykson, V. G. Kadyshevsky, and I. T. Todorov, Phys. Rev. D 1, 2823 (1970); E. Brezin, C. Itzykson, and J. Zinn-Justin, *ibid.* 1, 2349 (1970).

[10] C. Fronsdal and L.-E. Lundberg [Phys. Rev. D 1, 3247 (1970)] have also calculated the relativistic amplitude but by a quasipotential approach.

[11] A. O. Barut and A. Baiquni, Phys. Letters **30A**, 352 (1969).

APPENDIX A: PARABOLIC QUANTUM NUMBERS OF IN AND OUT STATES

The general wave function in the parabolic coordinates[7] $\xi=r+z$, $\eta=r-z$ is (in units with reduced mass μ, $\hbar=1$, $c=1$, $\alpha=e^2$)

$$\psi=e^{-(\alpha\mu/2n)(\xi+\eta)}L_{n1}\left(\frac{\alpha\mu}{n}\xi\right)L_{n2}\left(\frac{\alpha\mu}{n}\eta\right). \quad (A1)$$

The asymptotic form of (A1) is

$$\begin{aligned}&\psi\to ace^{-ikr}+ade^{-ikz}+bce^{ikz}+bde^{ikr},\\ &a=\frac{(-ik\xi)^{n_1}}{\Gamma(n_1+1)},\quad b=\frac{(ik\xi)^{-n_1-1}}{\Gamma(-n_1)},\\ &c=\frac{(-ik\eta)^{n_2}}{\Gamma(n_2+1)},\quad d=\frac{(ik\eta)^{-n_1-1}}{\Gamma(-n_2)}.\end{aligned} \quad (A2)$$

Comparing (A2) with (2.2), we obtain the assignments in Eq. (2.3). Then

$$\begin{aligned}\psi^+&\Rightarrow\frac{(-ik\eta)^n}{\Gamma(1+n)}e^{ikz}-\frac{n(ik\eta)^{-n-1}}{\Gamma(1-n)}e^{ikr},\\ \psi^-&\Rightarrow\frac{(ik\xi)^{-n}}{\Gamma(1-n)}e^{ikz}+\frac{n(-ik\xi)^{n-1}}{\Gamma(1+n)}e^{-ikr}.\end{aligned} \quad (A3)$$

In order to verify Eq. (2.5) it suffices to notice that under $R(\pi)$, $z\to -z$, $\xi\leftrightarrow\eta$, and that n is pure imaginary, $n=-i\alpha\mu/k$.

APPENDIX B: EVALUATION OF MATRIX ELEMENT

In terms of the boson operators, the parabolic states are given by[6]

$$\begin{aligned}|n_1n_2m\rangle&=N_{n_1n_2m}a_1{}^{\dagger n_2+m}a_2{}^{\dagger n_1}b_1{}^{\dagger n_1+m}b_2{}^{\dagger n_2}|0\rangle,\quad m\geq 0\\ &=N_{n_1n_2m}a_1{}^{\dagger n_2}a_2{}^{\dagger n_1+|m|}b_1{}^{\dagger n_1}b_2{}^{\dagger n_2+|m|}|0\rangle,\\ &\qquad m\leq 0.\end{aligned} \quad (B1)$$

This holds for the discrete spectrum. The operator J_i in this representation is

$$J_i=\tfrac{1}{2}(a^\dagger\sigma_ia+b^\dagger\sigma_ib). \quad (B2)$$

Since the a's and b's commute, the matrix element (3.6) is simply a product of the rotation matrices on the a's and b's separately. Taking the case of $m\geq 0$, we rewrite (B1) as

$$\begin{aligned}|n_1n_2m\rangle=&N_a(\varphi,M)a_1{}^{\dagger\varphi+M}a_2{}^{\dagger\varphi-M}|0\rangle\\ &\otimes N_b(\varphi,M')b_1{}^{\dagger\varphi+M'}b_2{}^{\dagger\varphi-M'}|0\rangle,\end{aligned} \quad (B3)$$

where we can immediately identify from (B1)

$$\varphi=\tfrac{1}{2}(n_1+n_2+m)=\tfrac{1}{2}(n-1)\,,\qquad M=\tfrac{1}{2}(n_2-n_1+m)\,,\quad M'=\tfrac{1}{2}(n_1-n_2+m) \qquad \text{(B4)}$$

or, in our case, with $m=0$,

$$\varphi=\tfrac{1}{2}(n-1)\,,\qquad M=-M'=\tfrac{1}{2}(n_2-n_1)\,. \qquad \text{(B4')}$$

Then the matrix element in (3.6) is given by

$$D=\langle n_1n_20|e^{i(\pi+\theta)L_2}|n_1n_20\rangle =\langle\varphi,M|e^{i(\pi+\theta)L_2{}^a}|\varphi,M\rangle\otimes\langle\varphi,-M|e^{i(\pi+\theta)L_2{}^b}|\varphi,-M\rangle.$$

Now each factor is a rotation matrix element with "spin" φ. Hence

$$D=D_{M,M'}{}^{\varphi}(\theta+\pi)D_{-M,-M'}{}^{\varphi}(\theta+\pi) \qquad \text{(B5)}$$

or, in terms of the hypergeometric functions,

$$D=F(-\varphi-M,\,1+\varphi-M,\,1,\,\sin^2[\tfrac{1}{2}(\theta+\pi)])\times F(-\varphi+M,\,1+\varphi+M,\,1,\,\sin^2[\tfrac{1}{2}(\theta+\pi)])\,. \qquad \text{(B6)}$$

Finally, we continue this expression to our continuum values,

$$\varphi=\tfrac{1}{2}(n-1)\,,\qquad M=\tfrac{1}{2}(n+1)\,,$$

and obtain

$$D=F(-n,\,0,\,1,\,\sin^2(\tfrac{1}{2}\theta+\tfrac{1}{2}\pi))\times F(1,\,1+n,\,1,\,\sin^2(\tfrac{1}{2}\theta+\tfrac{1}{2}\pi)) =[1-\sin^2(\tfrac{1}{2}\theta+\tfrac{1}{2}\pi)]^{-1-n}=[\cos^2(\tfrac{1}{2}\theta+\tfrac{1}{2}\pi)]^{-n-1} =[\sin^2(\tfrac{1}{2}\theta)]^{-n-1}. \qquad \text{(B7)}$$

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Scalar Mesons, Hadron Masses, and Approximate Scale Invariance*

P. Carruthers
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 16 September 1970)

The possible role of a scalar-meson nonet in determining hadron masses by means of generalized Goldberger-Treiman formulas is studied. Existing data on scalar-pseudoscalar meson couplings are analyzed, assuming SU_3-invariant couplings for the unmixed mesons. The d/f ratio for the coupling of octet scalars to the baryon octet is predicted to be the same as in the Gell-Mann–Okubo mass formula. The strength of the transition vacuum → scalar meson (induced by the energy-momentum tensor $\theta_{\mu\nu}$) is measured by the constant F_σ. Denoting the SU_3 singlet scalar meson by σ_0 and the octet η-like scalar by σ_8, we find F_{σ_0} to be much larger than F_{σ_8}, provided that the σ_0BB coupling is not considerably smaller than the σ_8BB coupling. Then F_{σ_0} is comparable to the usual pion decay constant F_π. The pseudoscalar octet dispersion relation is not scalar dominated. This result is also suggested by analysis of partial conservation of axial-vector current for the scalar-pseudoscalar system, and consideration of the relation to the underlying scale-invariant limit. Implications for the underlying dynamics are discussed.

I. INTRODUCTION

THE existence and properties of scalar mesons have been the subject[1] of much theoretical speculation.[1,2] The difficulty of obtaining unequivocal experimental information has prevented a decisive clarification of the situation. It is also possible that some of the resonances in question are too broad to be described as particles. However, it seems reasonable to assume the existence of a nonet (with average mass around 1 GeV) of 0^+ mesons. Recent studies[3–6] of broken scale invariance have attributed a more fundamental significance to the scalar mesons, i.e., that they should dominate matrix elements of the trace of the energy-momentum tensor $\theta_{\mu\nu}$ and thereby determine the masses of the hadrons. Variants of this idea have been expressed previously.[7–9] This paper is primarily concerned with this question.

In Sec. II the couplings of the scalar and pseudoscalar nonets are analyzed, assuming that SU_3 symmetry is maintained in the vertices except for mixing effects. This view is not universally maintained; for

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[1] H. Harari, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), reviews meson spectroscopy from the point of view of the quark model. Some further references are listed in this paper.

[2] B. Dutta-Roy and I. Lapidus, Phys. Rev. **169**, 1357 (1968). These authors discuss many phenomena in which scalar mesons seem to be required to give a theoretical interpretation of the data.

[3] M. Gell-Mann, lectures at the Summer School of Theoretical Physics, University of Hawaii, 1969 [Caltech Report No. CALT-68-244 (unpublished)].

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