on the scalar-meson loop, for which we use the following relation to split each diagram into two:

$$k^{(1)}{}_{\mu_1}(\tfrac{1}{2}g)\epsilon_{\alpha_3\alpha_2\alpha_1}(k^{(2)}-k^{(3)})_{\mu_3}=\tfrac{1}{2}g\epsilon_{\alpha_1\alpha_2\alpha_3}(k^{(2)})^2+\tfrac{1}{2}g\epsilon_{\alpha_1\alpha_3\alpha_2}(k^{(3)})^2, \quad \text{(B5)}$$

which is represented diagrammatically in Fig. 15. The diagrams left can be grouped into triplets of the type shown in Fig. 16 which are to be matched with the diagrams of the type B in Fig. 14. With the assignments of particle indices shown by the small numerals, the parts of Fig. 14(b) and Fig. 16 which appear differently contribute the following amounts.

For Fig. 14(b),

$$(-ig^2)(\epsilon_{\alpha_1\alpha_3\alpha_5}k^{(3)}{}_{\mu_2}\epsilon_{\alpha_4\alpha_5\alpha_2}+\epsilon_{\alpha_1\alpha_4\alpha_5}k^{(4)}{}_{\mu_2}\epsilon_{\alpha_3\alpha_5\alpha_2})$$
$$=(ig^2)(\epsilon_{\alpha_5\alpha_1\alpha_3}\epsilon_{\alpha_5\alpha_4\alpha_2}k^{(3)}{}_{\mu_2}+\epsilon_{\alpha_5\alpha_1\alpha_4}\epsilon_{\alpha_5\alpha_3\alpha_2}k^{(4)}{}_{\mu_2}). \quad \text{(B6)}$$

For Fig. 16,

$$(-ig)^2[\tfrac{1}{4}\epsilon_{\alpha_1\alpha_5\alpha_3}\epsilon_{\alpha_5\alpha_4\alpha_2}(k^{(4)}-k^{(1)}-k^{(3)})_{\mu_2}+\tfrac{1}{2}\epsilon_{\alpha_1\alpha_5\alpha_2}\epsilon_{\alpha_3\alpha_4\alpha_5}(k^{(4)}-k^{(3)})_{\mu_2}+\tfrac{1}{4}\epsilon_{\alpha_1\alpha_5\alpha_4}\epsilon_{\alpha_5\alpha_4\alpha_2}(k^{(3)}-k^{(1)}-k^{(4)})_{\mu_2}]$$
$$=\tfrac{1}{2}(ig^2)(\epsilon_{\alpha_5\alpha_1\alpha_3}\epsilon_{\alpha_5\alpha_4\alpha_2}k^{(3)}{}_{\mu_2}+\epsilon_{\alpha_5\alpha_1\alpha_4}\epsilon_{\alpha_5\alpha_3\alpha_2}k^{(4)}{}_{\mu_2})-\tfrac{1}{4}(ig^2)(\epsilon_{\alpha_5\alpha_1\alpha_3}\epsilon_{\alpha_5\alpha_4\alpha_2}+\epsilon_{\alpha_5\alpha_1\alpha_4}\epsilon_{\alpha_5\alpha_2\alpha_3})k^{(2)}{}_{\mu_2}, \quad \text{(B7)}$$

using the conservation of momentum $k^{(1)}+k^{(2)}+k^{(3)}+k^{(4)}=0$ and the identity $\epsilon_{\alpha\alpha_1\alpha_2}\epsilon_{\alpha\alpha_3\alpha_4}+\epsilon_{\alpha\alpha_1\alpha_3}\epsilon_{\alpha\alpha_4\alpha_2}+\epsilon_{\alpha\alpha_1\alpha_4}\epsilon_{\alpha\alpha_2\alpha_3}=0$. The term proportional to $k^{(2)}{}_{\mu_2}$ in (B7) can be dropped since it is multiplied into a complete set of trees. It is now obvious that diagrams with a scalar meson should be assigned a weight factor 2 in order that (5.2) will hold. This completes the proof of (5.2).

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Realizations of Representations of the Poincaré Group and Associated Similarity Transformations

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Recent contributions on canonical and manifestly covariant realizations of the representations of the Poincaré group are discussed. A $2(2j+1)$-dimensional operator for arbitrary spin j is constructed, permitting simple interpretations and discussions of similarity transformations currently used in the literature.

I. INTRODUCTION

Recently two papers[1,2] (quoted hereafter as PSP and PS, respectively) have appeared, which deal with canonical and manifestly covariant representations of the Poincaré group P and with the associated descriptions of particles of nonzero rest mass and arbitrary spin. The authors show, through the use of the general Gel'fand equations[3] and of the corresponding representations of P, their reducible character and their interrelation with the general "canonical form" proposed by Foldy[4] as well as with the canonical representations of P. This reduction is performed by a transformation presented as a generalized Foldy-Wouthuysen transformation.[5]

These developments refer to important fields which have already been explored by many authors during the last decade. On one hand, let us simply recall, besides the fundamental paper of Wigner[6] on unitary representations of P, the main contributions of Bargmann and Wigner,[7] Shirokov,[8] Wightman,[9] Joos,[10] Shaw,[11] Weinberg,[12] and Moussa and Stora.[13–16] On the other hand, let us also mention papers dealing especially with covariant equations for particles of nonzero rest mass and arbitrary spin, such as those of Bargmann and Wigner,[7] Foldy,[4] Fronsdal,[17] Weaver, Hammer, and

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[12] S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[13] P. Moussa and R. Stora, in *Boulder Lectures in Theoretical Physics*, edited by W. E. Brittin and A. O. Barut (Colorado U. P., Boulder, 1965), Vol. VIIA, p. 37.

[14] We apologize to the authors of the many papers not cited here. We refer for example to the recent review papers of Winternitz (Ref. 15) and Balachandran (Ref. 16) for further developments and references.

[15] P. Winternitz, Rutherford Laboratory Report No. RPP/T/3, 1969 (unpublished).

[16] A. P. Balachandran, Syracuse University Report No. NYO-3399-272, SU-7206-272, 1969 (unpublished).

[17] C. Fronsdal, Phys. Rev. **113**, 1367 (1959).

Good,[18] Pürsey,[19] and Tung.[20,14] Therefore, the PSP-PS contributions should be reinterpreted in order to take account of the above-mentioned works.

The aim of the present note is twofold. Firstly, we want to construct a $2(2j+1)$-dimensional[21] transformation for arbitrary spin j with a view to interpreting different similarity transformations currently used in the literature. This is based on group-theoretical methods and especially on the works of Wigner,[6] Shaw,[11] and Weinberg.[12] Secondly, we want to draw attention to a set of papers dealing directly with manifestly covariant equations and their characteristic operators (signs of the energy, position operators, etc.), i.e., the contributions of Chakrabarti,[22] Sesma,[23] De Azcarraga and Boya,[24] and De Azcarraga and Oliver.[25] The combination of both points of view will show *the origin* and *the physical meaning* of the PSP transformation and its connection with other similarity transformations.

The main results contained in Shaw's paper[11] as well as some relations of Weinberg[12] are collected in Sec. II. The $2(2j+1)$-dimensional transformation is constructed in Sec. III, and Sec. IV is devoted to the discussion of our transformation in connection with the Chakrabarti and PSP operators.

Our units are such that $c=1$, $\hbar=1$, and our metric is $g^{\mu\nu}=0$ ($\mu\neq\nu$; $\mu,\nu=0,1,2,3$), $g^{00}=-g^{ii}=1$ ($i=1,2,3$). Throughout this paper we limit ourselves to the case of nonzero-rest-mass particles with timelike momenta.

II. RESULTS OF SHAW AND WEINBERG

The so-called $[m,j]$-irreducible unitary representations of P corresponding to the case of spin j and nonzero-rest-mass particles and of timelike momenta have been studied by Shaw.[11] He proposed a realization of these representations which, like the Bargmann-Wigner one,[7] has simple transformation properties and, furthermore, has no superfluous spin components. In fact, the Shaw realization, which allows one to show the equivalences

$$[m,0]\otimes D^{(j,0)}\sim[m,j]\sim[m,0]\otimes D^{(0,j)}, \qquad (2.1)$$

is characterized [for the first part of (2.1)] by the transformation law

$$[U(a,\Lambda)\chi_{j\sigma}](p)=e^{ip\cdot a}Q_{\sigma\sigma'}{}^{(j,0)}(\Lambda)\chi_{j\sigma'}(\Lambda^{-1}p). \qquad (2.2)$$

These representations are then unitary with respect to the inner product

$$(\chi^{(1)},\chi^{(2)})=\int\frac{d\mathbf{p}}{p^0}\chi^{(1)}(p)^\dagger[L_p{}^{(j,0)}]^{-2}\chi^{(2)}(p), \qquad (2.3)$$

where $L_p{}^{(j,0)}$ (for arbitrary j) are the matrices

$$L_p{}^{(j,0)}=L^{(j,0)}(\mathbf{p})=L^{(0,j)}(-\mathbf{p}) \qquad (2.4)$$

$$=\exp\lambda_p(\hat{\mathbf{p}}\cdot\mathbf{S}), \qquad (2.5)$$

with $\lambda_p=\operatorname{arctanh}(|\mathbf{p}|/p^0)$, $\hat{\mathbf{p}}$ being the unit vector in the direction of $\mathbf{p}$, and $\mathbf{S}$ the usual $(2j+1)\times(2j+1)$ spin matrices. The $L_p{}^{(j,0)}$ can also be written as polynomials of degree $2j$ in $(\mathbf{p}\cdot\mathbf{S})$ and are of great interest in many respects. Let us remember here that the corresponding $[L_p{}^{(j,0)}]^{-2}$ occurring in Eq. (2.3) are directly related to the π matrices of Weinberg,[12,10] generalized by Pürsey[19] and Tung.[20] In particular, for $j=\frac{1}{2}$ ($\mathbf{S}=\frac{1}{2}\boldsymbol{\sigma}$), we have

$$L_p{}^{(1/2,0)}=\frac{p^0+m+\mathbf{p}\cdot\boldsymbol{\sigma}}{[2m(p^0+m)]^{1/2}},\qquad [L_p{}^{(1/2,0)}]^{-1}=\frac{p^0+m-\mathbf{p}\cdot\boldsymbol{\sigma}}{[2m(p^0+m)]^{1/2}}, \qquad (2.6)$$

and

$$[L_p{}^{(1/2,0)}]^{-2}=\frac{p^0-\mathbf{p}\cdot\boldsymbol{\sigma}}{m}. \qquad (2.7)$$

Beside the equivalences (2.1), Shaw has established the reducible character of the representations $[m,0]\otimes D^{(j,j')}$:

$$[m,0]\otimes D^{(j,j')}\sim\sum_{s=|j-j'|}^{j+j'}\oplus|m,s| \qquad (2.8)$$

so that their analysis reduces effectively to (2.1). Furthermore, if we recall that the canonical representations[6] are characterized by the transformation law

$$[U(a,\Lambda)\phi]_{j\sigma}(p)=e^{ip\cdot a}D_{\sigma\sigma'}{}^{(j)}[R(p,\Lambda)]\phi_{j\sigma'}(\Lambda^{-1}p) \qquad (2.9)$$

and are unitary with respect to the scalar product

$$(\phi_1(p),\phi_2(p))=\int\frac{d\mathbf{p}}{p^0}\phi_1{}^\dagger(p)\phi_2(p), \qquad (2.10)$$

the connection between Wigner's (ϕ) and Shaw's (χ) realizations is simply[11]

$$\phi_{j\sigma}(p)=[L_p{}^{(j,0)}]^{-1}\chi_{j\sigma}(p) \qquad (2.11)$$

for the first part of the equivalences (2.1). Let us remember that in (2.9) $R(p,\Lambda)$ is the Wigner rotation $L^{-1}(p)\Lambda L(\Lambda^{-1}p)$ belonging to the little group of $p^{(0)}=(m,0,0,0)$, so that $D^{(j)}[R(p,\Lambda)]$ is nothing but the familiar $(2j+1)$-dimensional unitary matrix representation of the rotation group, $L(p)$ being the "boost" which takes the particle of mass m from rest to a state of momentum $\mathbf{p}$ [$L(p)p^{(0)}=p\equiv(p^0,\mathbf{p})$].

Now, from Weinberg's developments,[12] let us pick out the construction of $2(2j+1)$-component fields trans-

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forming according to the $(j,0)\oplus(0,j)$ representation. After Weinberg, if we denote the $(2j+1)$-dimensional matrices representing a finite Lorentz transformation Λ by $D^{(j)}(\Lambda)$ and $\bar{D}^{(j)}(\Lambda)$ in the $(j,0)$ and $(0,j)$ representations, respectively, we can unite these two matrices into a single $2(2j+1)$-dimensional one

$$\mathfrak{D}^{(j)}(\Lambda)=\begin{pmatrix} D^{(j)}(\Lambda) & 0 \\ 0 & \bar{D}^{(j)}(\Lambda)\end{pmatrix}, \tag{2.12}$$

the $D^{(j)}$ and $\bar{D}^{(j)}$ being characterized as usual by

$$\mathbf{J}\to\mathbf{J}^{(j)},\quad \mathbf{K}\to -i\mathbf{J}^{(j)}\quad \text{for}\quad (j,0)$$

and (2.13)

$$\mathbf{J}\to\mathbf{J}^{(j)},\quad \mathbf{K}\to +i\mathbf{J}^{(j)}\quad \text{for}\quad (0,j),$$

where $\mathbf{J}$ and $\mathbf{K}$ generate rotations and boosts, respectively. This construction will be quoted hereafter as "Weinberg's doubling." Finally, let us note that the transformation law of the corresponding $2(2j+1)$-component field has been discussed in great detail by Weinberg and led to the "Weinberg equations" (which reduce to the Dirac equations in the case of spin $\frac{1}{2}$).

III. TRANSFORMATION MATRIX FOR ARBITRARY j

There is an evident correspondence between Shaw's notations $L_p^{(j,0)}$ $(L_p^{(0,j)})$ and Weinberg's $D^{(j)}(\Lambda)$ $(\bar{D}^{(j)}(\Lambda))$ when Λ is the boost $L(p)$, so that we can directly construct the $2(2j+1)$-dimensional matrix

$$\mathfrak{D}^{(j)}[L(p)]\equiv L_p{}^{\oplus,j}=\begin{pmatrix} L_p{}^{(j,0)} & 0 \\ 0 & L_p{}^{(0,j)}\end{pmatrix} \tag{3.1}$$

and its inverse $[L_p{}^{\oplus,j}]^{-1}$.

For simplicity, let us consider the $j=\frac{1}{2}$ case (on the basis of Shaw's results,[26] the generalization for arbitrary j is straightforward). For the $(\frac{1}{2},0)$ representation, Eq. (2.11) becomes, with (2.6),

$$\phi_{\frac{1}{2}\sigma}(p)=\frac{p^0+m-\boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(p^0+m)]^{1/2}}\chi_{\frac{1}{2}\sigma}(p) \tag{3.2}$$

and for the $(0,\frac{1}{2})$ case, we have

$$\bar{\phi}_{\frac{1}{2}\sigma}(p)=[L^{(\frac{1}{2},0)}(-\mathbf{p})]^{-1}\bar{\chi}_{\frac{1}{2}\sigma}(p)=\frac{p^0+m+\boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(p^0+m)]^{1/2}}\bar{\chi}_{\frac{1}{2}\sigma}(p). \tag{3.3}$$

Following Weinberg's doubling, we construct the four-component fields

$$\Phi_{\frac{1}{2}\sigma}=\begin{pmatrix}\phi_{\frac{1}{2}\sigma}\\ \bar{\phi}_{\frac{1}{2}\sigma}\end{pmatrix}\quad\text{and}\quad \mathrm{X}_{\frac{1}{2}\sigma}=\begin{pmatrix}\chi_{\frac{1}{2}\sigma}\\ \bar{\chi}_{\frac{1}{2}\sigma}\end{pmatrix} \tag{3.4}$$

[26] See especially the first Appendix of Ref. 11.

and we obtain, in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation,

$$\Phi_{\frac{1}{2}\sigma}(p)=[L_p{}^{\oplus,\frac{1}{2}}]^{-1}\mathrm{X}_{\frac{1}{2}\sigma}(p), \tag{3.5}$$

where

$$[L_p{}^{\oplus,\frac{1}{2}}]^{-1}=\frac{p^0+m-\boldsymbol{\alpha}\cdot\mathbf{p}}{[2m(p^0+m)]^{1/2}},\quad \boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix}. \tag{3.6}$$

As follows from Weinberg's discussion, the fields $\mathrm{X}_{(\frac{1}{2})}$ and $\Phi_{(\frac{1}{2})}$ satisfy not only the Klein-Gordon equation but also the (expected) Dirac equation. At this stage, it is thus important to take into account the sign of the energy[27] $\epsilon=p^0/|p^0|$, as also in the generalized Wigner-Shaw developments. In fact, it is easy to show that in correspondence with the Shaw representations $[m,0]_\epsilon\otimes D^{(j,0)}$ and $[m,0]_\epsilon\otimes D^{(0,j)}$, we have

$$[L_{p,\epsilon}{}^{(\frac{1}{2},0)}]^{-1}=\frac{p^0+m-\epsilon\boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(p^0+m)]^{1/2}} \tag{3.7}$$

and

$$[L_{p,\epsilon}{}^{(0,\frac{1}{2})}]^{-1}=\frac{p^0+m+\epsilon\boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(p^0+m)]^{1/2}} \tag{3.8}$$

so that

$$[L_{p,\epsilon}{}^{\oplus,\frac{1}{2}}]^{-1}=\frac{p^0+m-\epsilon\boldsymbol{\alpha}\cdot\mathbf{p}}{[2m(p^0+m)]^{1/2}}. \tag{3.9}$$

For arbitrary spin j, the generalization of Eq. (3.5), i.e.,

$$\Phi_{j,\sigma}(p,\epsilon)=[L_{p,\epsilon}{}^{\oplus,j}]^{-1}\mathrm{X}_{j,\sigma}(p,\epsilon), \tag{3.10}$$

gives the connection between canonical and manifestly covariant realizations of P, the latter being always reduced finally to $2(2j+1)$ components (as in Dirac theory or, through subsidiary conditions, as in Gel'fand or Bargmann-Wigner equations, for example).

IV. CONNECTIONS WITH CHAKRABARTI AND PSP OPERATORS

Let us now consider the contributions of Chakrabarti,[22] Sesma,[23] and De Azcarraga and Boya.[24] Essentially, they give a similarity transformation—the Chakrabarti transformation—leading to a canonical form which is to be compared with that of Foldy.[4] Chakrabarti has particularly studied the spin-$\frac{1}{2}$ case. Sesma has formally extended the Chakrabarti transformation to the cases of spins 0 and 1 through Hamiltonian formulations, and De Azcarraga and Boya have considered the case of arbitrary spin through the Bargmann-Wigner equations. Let us also remark that consequently, through these results, different authors[22,25] have been led to establish comparisons between the Chakrabarti and Foldy-Wouthuysen operators and also between the corresponding usual observables.

If we limit ourselves to the case of spin $\frac{1}{2}$, the Chakrabarti operator is

$$Q_{\mathrm{op}}=[\gamma^0(\gamma\cdot p)+M][2M(p^0+M)]^{-1/2}, \tag{4.1}$$

[27] See, for example, the very clear Sec. III of Ref. 19.

where the matrices γ^μ ($\mu=0, 1, 2, 3$) are such that

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}, \quad (\gamma^0)^\dagger=\gamma^0, \quad (\gamma^i)^\dagger=-\gamma^i \quad (i=1, 2, 3) \tag{4.2}$$

and

$$M\equiv M_{\rm op}\equiv\epsilon(p^\nu p_\nu)^{1/2}, \quad \epsilon=p^0/|p^0|=\pm1. \tag{4.3}$$

Since, when the similarity transformation (4.1) has been applied, the Dirac equation in momentum space takes the form

$$(\gamma^0 M-m)\Psi=0 \tag{4.4}$$

(m being the rest mass), therefore its solutions belong to the eigenvalues of γ^0 according as ϵ. So, separating the cases of positive and negative energies, the canonical form proposed by Chakrabarti is finally

$$[(p^\nu p_\nu)^{1/2}-m]\Psi=0. \tag{4.5}$$

We can thus compare the Chakrabarti operator (4.1) with our transformation matrix (3.9). It follows directly that

$$[L_{p,\epsilon}{}^{\oplus,\frac{1}{2}}]^{-1}\equiv\epsilon Q_{\rm op} \tag{4.6}$$

and, therefore, Eq. (3.5) shows clearly the physical meaning of these operators on group-theoretical grounds. Furthermore, the connection of these operators with the Foldy-Wouthuysen transformation can be directly obtained through the Chakrabarti development.[28]

Now, the PSP-PS contributions[29] can be directly inserted in this context. As already mentioned, the realizations discussed by PSP are precisely the canonical and the manifestly covariant ones, since if the former are those discussed by Wigner of the type (2.9) and (2.10), the latter are essentially those of Shaw, (2.2) and (2.3), expressed on the rotation group basis, the two signs of the energy being obviously always considered. Let us recall that for each sign of the energy, the Shaw representations correspond *in general* to the reducible case (2.8) so that their analysis reduces effectively to (2.1). Then it is very natural to make the correspondences of PSP's equations [for example, their Eqs. (2.3) and (2.8)] with ours [correspondingly, Eqs. (3.7) and (3.6)] and evidently with the Chakrabarti ones. With their notation, we have exactly

$$D[L^{-1}(p)]=[L_{p,\epsilon}{}^{\oplus,\frac{1}{2}}]^{-1} \tag{4.7}$$

in the spin-$\frac{1}{2}$ case.

Following Eqs. (4.6) and (4.7) in the Dirac case, we can compare the generalizations of the Chakrabarti operator proposed by Sesma[23] and De Azcarraga and Boya,[24] with the applications considered in PSP. The three examples of PSP correspond to the Chakrabarti and Sesma cases and reduce to $2(2j+1)$-component relations as when Bargmann-Wigner equations are used by De Azcarraga and Boya.

Thus, apart from the fact that PSP deals with Gel'fand equations as manifestly covariant descriptions, their transformation is essentially of the Chakrabarti type for arbitrary spin. The construction given above (Sec. III) explains how the transformation works when the descriptions become free of redundant components.

Let us also note that in PS (§7), special emphasis on position operators is given while other well-known papers[30,31] of Chakrabarti already contain the corresponding discussion (which has also been extended by Sesma[23] and by De Azcarraga and Oliver[25]). More precisely, PS defines a position operator $\mathbf{q}$ which, in the momentum representation, is[31]

$$(q^i\Phi)(p)=\left(i\frac{\partial}{\partial p^i}-\frac{p^i}{2|p^0|^2}\right)\Phi(p).$$

After the application of their similarity transformation, the corresponding operator of PS should be compared with the Chakrabarti one [cf. Eq. (2.21) of Ref. 30] (if, for example, the spin-$\frac{1}{2}$ case is considered). For arbitrary spin, we refer to the Sesma[23] and De Azcarraga-Oliver[25] contributions. The only modifications arise from the ϵ factor whose presence is made clear by Eqs. (4.6) and (4.7).

As a last conclusion in connection with related works, let us point out that the comparisons between the Lorentz and Foldy-Wouthuysen transformations established by Bollini and Giambiagi[32,33] and Good and Rose,[34] as also their relationship given by Jehle and Parke,[35] find here a group-theoretical explanation. Furthermore, the $2(2j+1)$-component Lorentz-covariant description proposed by Weaver, Hammer, and Good[18] *is* thus a generalized Shaw realization.

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[28] We refer to Appendix B of Ref. 22 where some features of the Chakrabarti transformation are compared with the corresponding ones of the Foldy-Wouthuysen transformation.

[29] These authors seem to be unaware of an important set of references (Refs. 22, 23, 24, and 30).

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